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Homology cylinders: an enlargement of the mapping class group

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Abstract We consider a *homological* enlargement of the mapping class group, defined by *homology cylinders* over a closed oriented surface (up to homology cobordism). These are important model objects in the recent Goussarov-Habiro theory of finite-type invariants of 3-manifolds. We study the structure of this group from several directions: the relative weight filtration of Dennis Johnson, the finite-type filtration of Goussarov-Habiro, and the relation to string link concordance.

We also consider a new *Lagrangian* filtration of both the mapping class group and the group of homology cylinders.

AMS Classification 57N10; 57M25

Keywords Homology cylinder, mapping class group, clasper, finite-type invariant

1 Introduction

The mapping class group Γ_g is the group of diffeotopy classes of orientation preserving diffeomorphisms of the closed oriented surface Σ_g of genus g. There has been a great deal of work aimed at the determination of the algebraic structure of this group. For example, some years ago D. Johnson defined a filtration on Γ_g (the *relative weight filtration*) and observed that the associated graded group is a Lie subalgebra of a Lie algebra $\mathsf{D}(H)$ constructed explicitly from $H = H_1(\Sigma_g)$. Johnson, Morita and others (see [J],[Mo],[M]) have investigated this Lie subalgebra but its precise determination is still open. R. Hain [Hn] has studied the lower central series filtration of the Torelli group (the subgroup of Γ_g whose elements are homologically trivial) and found a simple explicit presentation over $\mathbb{Q}$. We also mention the work of Oda (see [O],[L]) relating the pure braid group to Γ_g.

One reason for interest in Γ_g is that it is related in an obvious way to the structure of 3-manifolds via the Heegard construction. From this viewpoint, the subgroup Γ_g^B consisting of diffeomorphisms which extend over the handlebody T_g of genus g, and the coset space Γ_g/Γ_g^B, are of obvious interest.

In this note we propose a "homological" generalization of these groups, where we replace Γ_g by a group $\mathcal{H}_g$ of homology bordism classes of homology cylinders over Σ_g. Homology cylinders have appeared and been studied in recent work of Goussarov [Go] and Habiro [H], as important model objects for their new theory of finite-type invariants of general 3-manifolds. We will see that Γ_g is a subgroup of $\mathcal{H}_g$. Furthermore the notion of Heegard construction translates to the more general context of homology cylinders — the relevant subgroup is now $\mathcal{H}_g^B$, consisting of those homology cylinders which extend to homology cylinders over T_g.

It turns out that the structure of $\mathcal{H}_g$ presents different problems than Γ_g. The relative weight filtration extends to a filtration of $\mathcal{H}_g$ but now the associated graded group is all of $\mathsf{D}(H)$. On the other hand the residue of the filtration (i.e. the intersection of the subgroups of the filtration) is non-trivial (it is trivial for Γ_g). There are some approaches to the study of $\mathcal{H}_g$ which don't seem to have useful analogs for Γ_g. Using the recent work of Goussarov [Go] and Habiro [H], there is a notion of finite-type invariants for homology cylinders. We will show, using results *announced* by Habiro [H], that this is entirely captured by the relative weight filtration except for one piece in degree 1, which is a $\mathbb{Z}/2$-vector space defined by a natural generalization of the Birman-Craggs homomorphisms. In a different direction we will exhibit a close relationship between $\mathcal{H}_g$ and a *framed string link concordance group* $\mathcal{S}_g^{\mathrm{fr}}$ — extending the natural map from the pure braid group into Γ_g defined by Oda. We will see that $\mathcal{S}_g^{\mathrm{fr}}$ maps into $\mathcal{H}_g$, inducing a *bijection* of $\mathcal{S}_g^{\mathrm{fr}}$ with the coset space $\tilde{\mathcal{H}}_g/\mathcal{H}_g^B$, where $\tilde{\mathcal{H}}_g$ is a natural subset of $\mathcal{H}_g$ containing the "Torelli" subgroup — it is interesting to note that there is no analogous result for Γ_g.

The inclusion $\Gamma_g^B \subseteq \Gamma_g$ can be illuminated by the introduction of a new filtration of Γ_g which we call the *Lagrangian filtration*. The residue of this filtration is exactly Γ_g^B and the associated graded group imbeds in a Lie algebra $\mathsf{D}(L)$ constructed from the Lagrangian subgroup $L = \mathrm{Ker}\{H_1(\Sigma_g) \to H_1(T_g)\}$. Determination of the image seems as difficult as the analogous problem for the relative weight filtration. As in the case of the relative weight filtration, the associated graded group for the Lagrangian filtration of $\mathcal{H}_g$ is isomorphic to $\mathsf{D}(L)$. However the residue of the filtration turns out to be *larger* than $\mathcal{H}_g^B$.

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2 Homology cylinders

2.1 Preliminaries

For the usual technical reasons it will be easier to work with the punctured surface. Let $\Sigma_{g,1}$ denote the compact orientable surface of genus g with one boundary component. A *homology cylinder* over $\Sigma_{g,1}$ is a compact orientable 3-manifold M equipped with two imbeddings $i^-, i^+ : \Sigma_{g,1} \to \partial M$ so that i^+ is orientation-preserving and i^- is orientation-reversing and if we denote $\partial^\pm = \operatorname{Im} i^\pm(\Sigma_{g,1})$, then $\partial M = \partial^+ \cup \partial^-$ and $\partial^+ \cap \partial^- = \partial\partial^+ = \partial\partial^-$. We also require that $i^\pm$ be homology isomorphisms. This notion is introduced in [H], using the terminology *homology cobordism*. We can multiply two homology cylinders by identifying ∂^- in the first with ∂^+ in the second via the appropriate $i^\pm$. Thus C_g, the set of orientation-preserving diffeomorphism classes of homology cylinders over $\Sigma_{g,1}$ is a semi-group whose identity is the product $I \times \Sigma_{g,1}$, with $\partial^- = 0 \times \Sigma_{g,1}, \partial^+ = 1 \times \Sigma_{g,1}$, with their collars stretched half-way along $I \times \partial\Sigma_{g,1}$. C_g is denoted $C(\Sigma_g)$ in [H].

There is a canonical homomorphism $\Gamma_{g,1} \to C_g$ that sends h to $(M = I \times \Sigma_{g,1}, i^- = 0 \times \mathrm{id}, i^+ = 1 \times h)$. Nielsen [N] showed that the natural map $\rho : \Gamma_{g,1} \to \operatorname{Aut}_0(F)$ is an isomorphism, where F is the free group on $2g$ generators $\{x_i, y_i\}$, identified with the fundamental group of $\Sigma_{g,1}$ (with base-point on $\partial\Sigma_{g,1}$) — see Figure 1 — and $\operatorname{Aut}_0(F)$ is the group of automorphisms of F which fix the element $\omega_g = (y_1 \cdots y_g)^{-1}(x_1y_1x_1^{-1} \cdots x_gy_gx_g^{-1})$, representing the boundary of $\Sigma_{g,1}$.

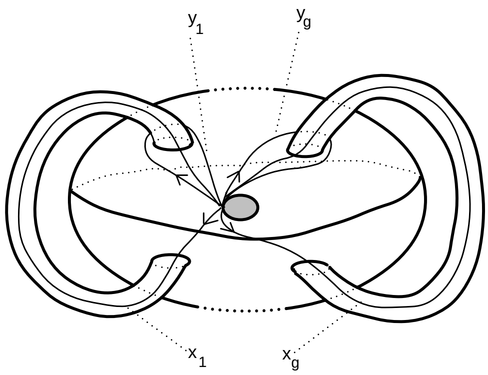


Figure 1: Generators of $\pi_1(\Sigma_{g,1})$

We can convert C_g into a group H_g by considering *homology bordism classes* of homology cylinders. If M, N are homology cylinders, we can construct a closed manifold $W = M \cup (-N)$, where $\partial_- M$ of M is attached to $\partial_- N$ of N via their identifications with $\Sigma_{g,1}$. A *homology bordism* between M and N is a manifold X such that $\partial X = W$ and the inclusions $M \subset X, N \subset X$ are homology equivalences — we say that M and N are *homology bordant*. This is an equivalence relation, since we can paste two homology bordisms together to create a third. Furthermore the multiplication of homology cylinders preserves homology bordism classes — see Figure 2. For any homology cylinder M we can

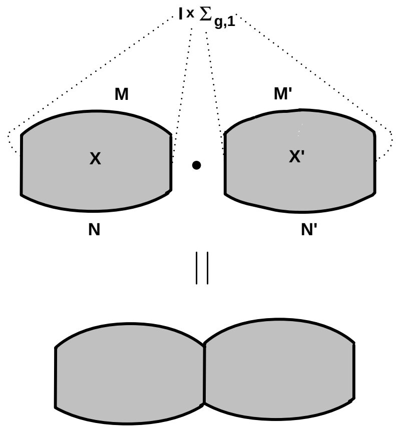


Figure 2: Homology bordism invariance of multiplication of homology cylinders

also consider $-M$ as a homology cylinder with the roles of $\partial_\pm$ reversed. Then $I \times M$ is a homology bordism between the product $M \cdot (-M) = \partial I \times M \cup I \times \partial_-$ and $I \times \Sigma_{g,1} = I \times \partial_+$, the identity element of $\mathcal{C}_g$. Thus $\mathcal{H}_g$ is a group.

For any group G, let G_q denote the subgroup generated by commutators of order q. In [GL] the isomorphism $\phi : \Gamma_{g,1} \cong \mathrm{Aut}_0(F)$ is extended to a sequence of maps

$$\phi_k : \mathcal{H}_g \to \mathrm{Aut}_0(F/F_{k+1})$$

for $k \geq 1$, which are consistent in the obvious sense. $\mathrm{Aut}_0(F/F_q)$ consists of all automorphisms of F/F_q which satisfy the equation

$$[h(x_1), h(y_1)] \cdots [h(x_g), h(y_g)] = [x_1, y_1] \cdots [x_g, y_g] \mod F_{q+1}$$

For example $\mathrm{Aut}_0(F/F_2)$ is just the symplectic group $\mathrm{Sp}(H)$, where $H = H_1(\Sigma_{g,1})$.

For the reader's convenience we recall the definition of ϕ_k. Given $(M, i^+, i^-) \in \mathcal{H}_g$ consider the homomorphisms $i_\pm : F \to \pi_1(M)$, where the base-point is taken in $\partial\Omega^+ = \partial\Omega^-$. In general, $i_\pm$ are not isomorphisms — however, since

$i^{\pm}$ are homology isomorphisms, it follows from Stallings [St] that they induce isomorphisms $i^{\pm}_n : F/F_n \to \pi_1(M)/\pi_1(M)_n$ for any n. We then define $\sigma_k(M, i^{\pm}) = (i^-_{k+1})^{-1} \circ i^+_{k+1}$. It is easy to see that $\sigma_k(M, i^{\pm}) \in \mathrm{Aut}_0(F/F_{k+1})$.

One consequence of the existence of these maps is that the map $\Gamma_{g,1} \to \mathcal{H}_g$ is injective. Furthermore, the following theorem is proved in [GL].

Theorem 1 *Every* σ_k *is* onto.

We now define a filtration of $\mathcal{H}_g$ by setting $F^w_k(\mathcal{H}_g) = \mathrm{Ker}\,\sigma_k$. Then $F^w_k(\mathcal{H}_g) \cap \Gamma_{g,1} = \Gamma_{g,1}[k]$ is the standard relative weight filtration of $\Gamma_{g,1}$ (this is denoted $\mathcal{M}(k+1)$ in [Mo]). Let $G^w_k(\mathcal{H}_g) = F^w_k(\mathcal{H}_g)/F^w_{k+1}(\mathcal{H}_g)$. There is, for every $k \geq 1$, a short exact sequence

$$1 \to \mathsf{D}_k(H) \to \mathrm{Aut}_0(F/F_{k+2}) \to \mathrm{Aut}_0(F/F_{k+1}) \to 1$$

where $\mathsf{D}_k(H)$ is the kernel of the bracketing map $H \otimes \mathsf{L}_{k+1}(H) \to \mathsf{L}_{k+2}(H)$. $\mathsf{L}_q(H)$ is the degree q part of the free Lie algebra over H (see [GL]). Theorem 1 has, as an immediate consequence:

Corollary 2.1 σ_{k+1} *induces an isomorphism* $J^H_k : G^w_k(\mathcal{H}_g) \cong \mathsf{D}_k(H)$ *for* $k \geq 1$.

Note that the induced monomorphism $J_k : \Gamma_{g,1}[k]/\Gamma_{g,1}[k+1] \to \mathsf{D}_k(H)$ is *not* generally onto — computing its image is a fundamental problem in the study of the mapping class group (see [J],[Mo],[M]). We will see that $F^w_\infty(\mathcal{H}_g) = \bigcap_k F^w_k(\mathcal{H}_g)$ is non-trivial whereas it follows from Nielsen's theorem that $\Gamma_{g,1}[\infty] = \{1\}$.

2.2 Filtrations of the Torelli group

Since $\Gamma_{g,1}[1] = \mathcal{T}_g$ is the classical Torelli group, we will refer to $F^w_1(\mathcal{H}_g) = \mathcal{T}\mathcal{H}_g$ as the *homology Torelli group*.

It is pointed out in [Mo] that

$$[\Gamma_{g,1}[k], \Gamma_{g,1}[l]] \subset \Gamma_{g,1}[k+l]$$

Thus $(\mathcal{T}_g)_k \subset \Gamma_{g,1}[k]$. Furthermore the associated graded groups $G^l(\mathcal{T}_g)$ and $G^w(\Gamma_{g,1})$, where $G^l_k(\mathcal{T}_g) = (\mathcal{T}_g)_k/(\mathcal{T}_g)_{k+1}$ and $G^w_k(\Gamma_{g,1}) = \Gamma_{g,1}[k]/\Gamma_{g,1}[k+1]$, are graded Lie algebras with bracket defined by the commutator and we have a Lie algebra homomorphism $j : G^l(\mathcal{T}_g) \to G^w(\Gamma_{g,1})$ induced by the inclusions. Also $D(H)$ is a Lie algebra, as described in [M], and the inclusion $J : G^w(\Gamma_{g,1}) \hookrightarrow$

$D(H)$ is a Lie algebra homomorphism. Johnson [J] shows that, after $\otimes\,\mathbb{Q}$, both j and J are isomorphisms at the degree 1 level.

$$G^l_1(\mathcal{T}_g) \otimes \mathbb{Q} \xrightarrow{\cong} G^w_1(\Gamma_{g;1}) \otimes \mathbb{Q} \xrightarrow{\cong} D_1(H) \otimes \mathbb{Q}$$

Thus $\operatorname{Im} j \otimes \mathbb{Q}$ and $\operatorname{Im} J \otimes \mathbb{Q}$ are the Lie subalgebras of $D(H) \otimes \mathbb{Q}$ generated by elements of degree 1.

The exact same considerations apply to the lower central series filtration of $\mathcal{H}_g$ and we obtain a Lie algebra homomorphism

$$j^H : G^l(\mathcal{T}\mathcal{H}_g) \to G^w(\mathcal{H}_g) = D(H)$$

The image of $j^H \otimes \mathbb{Q}$ is again the subalgebra generated by degree 1 elements, which is the same as $\operatorname{Im} J \otimes \mathbb{Q}$, and so we conclude that j^H is not onto.

2.3 Finite-type invariants of $\mathcal{H}_g$

There have been several proposals for a theory of finite-type invariants for general 3-manifolds, extending Ohtsuki's theory [Oh] for homology 3-spheres — perhaps the first given in Cochran-Melvin [CM]. We will use, however, the particular version proposed independently by M. Goussarov [Go],[GGP] and K. Habiro [H]. We will show that ${}_k$ and a homomorphism $: \mathcal{T}\mathcal{H}_g \to V$ defined by Birman-Craggs homomorphisms [BC], where V is a vector space over $Z/2$, make up the *universal multiplicative homology bordism invariant of type k* on the class of homology cylinders for any $k \geq 1$. It is to be emphasized that *we make use of results announced in [H] but, for which no proofs have yet appeared.*

We give a brief summary of the Goussarov-Habiro theory, and refer the reader to [GGP],[H],[Ha] for details. Let G be a unitrivalent graph whose trivalent vertices are equipped with a cyclic ordering of its incident edges and whose univalent vertices are decorated with an element of an abelian group H. We also insist that each component of G have at least one trivalent vertex. We refer to such a graph as an *H-graph* and define the *degree* of G to be the number of trivalent vertices. If M is a 3-manifold and $H = H_1(M)$, then a *clasper* (using the terminology of [H] — *clover* in the terminology of [GGP]) in M associated to G is a framed link C in M obtained in the following way. Associate to each trivalent vertex of G a copy of the Borromean rings, in disjoint balls of M, each component associated to an end of an edge incident to that vertex (even if two of those ends are just opposite ends of the same edge) and given the 0-framing. Associate to each univalent vertex a framed knot in M, disjoint from the balls, and representing the element of H labeling that vertex. These are the *leaves* of the clasper. Finally for each edge of G introduce a simple clasp between the knots associated to the two ends of that edge. The construction of a clasper C from an H-graph G involves a choice of framed imbedding into M of the graph obtained from G by attaching circles to the univalent vertices. See Figure 3 for a typical example. If G has degree n we say C is

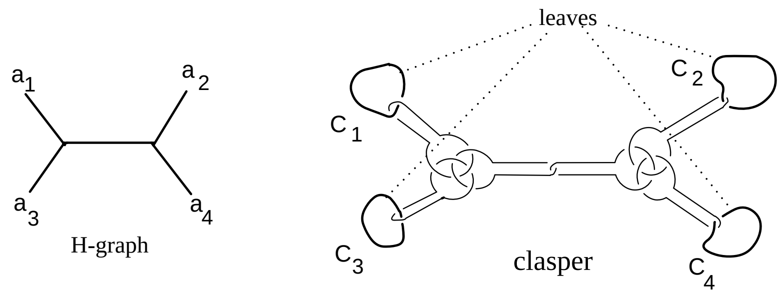


Figure 3: An H-graph and an associated clasper, where C_i represents a_i

an *n-clasper*. Surgery on a clasper produces a new manifold $M' = M_C$ with isomorphic homology and torsion pairing. We will refer to surgery on an n-clasper as an *n-surgery*. Thus if M is a homology cylinder of genus g so is M'. It is easy to see that the associated automorphisms of $H_1(\Sigma_{g,1})$ associated to M and M' are the same. Matveev [Ma] has proved the converse: If M, M' have isomorphic homology and torsion pairing then there is a clasper C in M such that $M' = M_C$. We define the relation of *A_k-equivalence* (in the terminology of [H]) to be generated by the following elementary move: $M \sim_k M'$ if M' is diffeomorphic to M_C, for some connected k-clasper C. According to [H], the set of A_k-equivalence classes in $\mathcal{C}_g$ is a group (finitely-generated and nilpotent) under the stacking multiplication. Let $F_k(\mathcal{C}_g)$ denote the subsemigroup of $\mathcal{C}_g$ consisting of homology cylinders A_k-equivalent to the trivial one $I \times \Sigma_{g,1}$ and let $G_k(\mathcal{C}_g) = F_k(\mathcal{C}_g)/A_{k+1}$

Habiro defines $A_k(H)$, for $k > 1$, to be the abelian group generated by connected H-graphs of degree k with an extra structure of a total ordering of the univalent vertices, subject to the AS and IHX relations, multilinearity of the labels and an STU-like relation(see [H]). $A_1(H)$ is defined explicitly to be $\Lambda^3 H \oplus V$, where $V = \Lambda^2 H_2 \oplus H_2 \otimes \mathbb{Z}/2$, where $H_2 = H \otimes \mathbb{Z}/2$. Habiro constructs an *epimorphism* $A_k(H) \to G_k(\mathcal{C}_g)$, which can be defined for $k > 1$ by clasper surgery on $I \times \Sigma_{g,1}$, in which the ordering of the univalent vertices is used to stack the imbeddings (with horizontal framings) of the leaves representing the elements of $H = H_1(\Sigma_{g,1})$ decorating those vertices. Habiro conjectures that this epimorphism is an isomorphism and claims that it is so for $k = 1$ and, for $k > 1$, induces an isomorphism $A_k(H) \otimes \mathbb{Q} \cong G_k(C_g) \otimes \mathbb{Q}$. See [Ha] for an alternative construction for homology handlebodies.

Let $A_k(H)$ denote, for $k > 1$, the subgroup of $A_k(H)$ generated by connected H-graphs with non-zero first Betti number, i.e. with a non-trivial cycle. The quotient $A^t_k(H) = A_k(H)/A_k(H)$ is generated by H*-trees*. Note that the ordering of the univalent vertices in $A^t_k(H)$ does not matter, because of the STU-relation, and so this structure can be ignored. Set $A^t_1(H) = A_1(H)$.

We recall a fact known to the experts (see also [Ha]).

Proposition 2.2 *$A^t_k(H) \cong D_k(H)$ for $k > 1$. $A^t_1(H) \twoheadrightarrow D_1(H) = \Lambda^3 H$.*

Outline of proof. If $k = 1$, this was already pointed out by Johnson [J]. Now suppose $k > 1$. Let T be an H-tree of degree k. If we choose any univalent vertex to be the root of T, then we can associate to this rooted H-tree an element in $H \otimes L_{k+1}(H)$. For example, if T is the H-graph in Figure 3, with the counter-clockwise orientation at each vertex and root at a_1, then the associated element would be $a_1 \otimes [\gamma_3, [\gamma_4, \gamma_3]]$, where γ_i is a lift of a_i into $\pi_1(\Sigma_{g,1})$. Now we associate to T the sum of the associated elements of $H \otimes L_{k+1}(H)$ as the roots range over all the univalent vertices of T. It is not hard to see that this element actually lies in $D_k(H)$ and that, in fact, $D_k(H)$ is generated by the elements associated to all possible H-trees of degree k. □

To bring homology bordism into the picture we need the following theorem.

Theorem 2 *Let M be a 3-manifold and G a connected H-graph, where $H = H_1(M)$, with at least one non-trivial cycle. If C is any clasper associated to G, then M_C is homology bordant to M.*

Remark 2.3 As an immediate consequence of this theorem we conclude that if $L \subseteq M$ is a link in a 3-manifold and C is such a clasper in $M - L$ then the links $(M; L)$ and $(M_C; L)$ are *homology concordant*, i.e. there is a homology bordism V between M and M_C and a proper imbedding $I \times L \subseteq V$ such that $\partial_0 L = L \subseteq M$ and $\partial_1 L = L \subseteq M_C$. Compare this to Theorem 2.9 of [Ha] which shows that, in the case of M a homology ball, the Milnor μ-invariants of $L \subseteq M$ and $L \subseteq M_C$ coincide.

Remark 2.4 A more delicate argument will strengthen Remark 2.3 as follows. If the clasper C is *strict* in the sense of [H], i.e. the leaves of C bound disks which are disjoint from each other and the rest of C except for a single intersection point with the companion component of C (but will generally intersect L), then $(M_C; L)$ is *concordant* to $(M; L)$, i.e. there is an imbedding $I \times L \subset I \times M$ so that $(0 \times M; 0 \times L)$ is diffeomorphic to $(M; L)$ and $(1 \times M; 1 \times L)$ is diffeomorphic to $(M_C; L)$.

To prove Theorem 2 we need the following Lemma.

Lemma 2.5 *Let M be a 3-manifold and $L \subset M$ a framed link. Suppose that $L = L' \cup L''$, where L' and L'' have the same number of components and*

(a) *L' is a trivial link bounding disjoint disks D,*

(b) *L' is 0-framed,*

(c) *the matrix of intersection numbers of the components of D with those of L'' is non-singular.*

Then M_L is homology bordant to M.

Proof We construct a manifold V from $I \times M$ by adjoining handles to $0 \times M$ along the framed link L'' and by removing tubular neighborhoods of properly imbedded disjoint disks D' in $I \times M$ obtained by pushing $0 \times \operatorname{Int} D$ into the interior of $I \times M$. Then $\partial V = M_L - (1 \times M)$, so it only remains to observe that the pair $(V; 1 \times M)$ is acyclic.

Let $W = (I \times M) - D'$. It is easy to see that the only non-zero homology group of $(W; 1 \times M)$ is $H_1(W; 1 \times M)$, which is freely generated by the meridians of D'. By considering the triple $(V; W; 1 \times M)$ we find there is an exact sequence:

$$0 \to H_2(V; 1 \times M) \to H_2(V; W) \to H_1(W; 1 \times M) \to H_1(V; 1 \times M) \to 0$$

Now $H_2(V; W)$ is the only non-zero homology group of $(V; W)$ and it is freely generated by the disks adjoined along L''. Since the homomorphism $H_2(V; W) \to H_1(W; 1 \times M)$ is represented by the matrix of intersection numbers of the components of D' with those of $I \times L''$, it follows that $(V; 1 \times M)$ is acyclic if and only if (c) is satisfied. □

Proof of Theorem 2 Recall that for each edge of G there are two components of C, one at each end of the edge (see Figure 3). We will call them *companion* components. We will construct C' and C'' by assigning, for each edge of G, one of the associated companions to C' and the other to C''. This choice can be represented by an orientation of the edge pointing from the end associated to the companion in C' toward the end associated to C''. Our aim will be to make these choices satisfy:

(i) An edge with a univalent vertex (a *leaf* of G) is oriented toward the leaf, i.e. *outward* (thus the leaves of C will all belong to C''),

(ii) No trivalent vertex is a source, i.e. not all the incident edges are oriented away from the vertex.

We will see that these conditions can be satisfied if and only if G is not a tree. But for now note that if these conditions are satisfied then the decomposition of C will satisfy conditions (a)-(c) of Lemma 2.5. Since the three components of C associated to any trivalent vertex are a Borromean rings, any two of the components bound disjoint disks. Thus we can choose disjoint disks D bounded by each component of C', the disks from components associated to different vertices will be disjoint. The only problem would be if these components were associated to the same edge, but this is ruled out. Now each disk from a component of C' will intersect the companion component of C once — the only other intersections will be with one of the components of C'' which is associated to the same trivalent vertex, but the intersection number will be 0. Thus the intersection matrix of the components of D with those of C'' will be the identity matrix. See Figure 4 for an example.

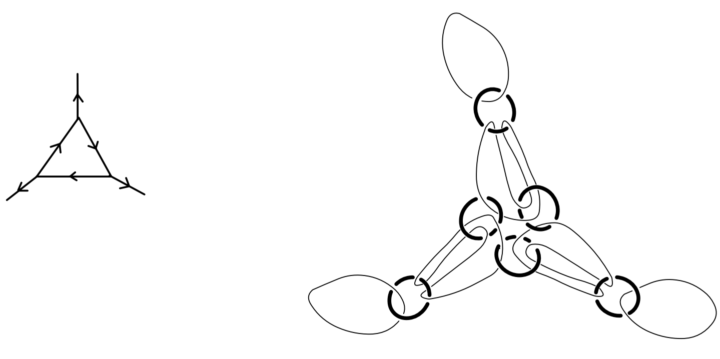


Figure 4: An oriented graph on the left and a corresponding clasper C on the right. The bold components define C'.

Lemma 2.6 *Suppose that T is a unitrivalent tree, with an orientation prescribed for each leaf edge, not all outward. Then we can extend this to an orientation of all the edges of T which satisfies (ii).*

Proof A similar fact is proved in [Ha]. Choose one of the leaves e of T which is oriented inward. Now orient every edge of T which is not a leaf so that it points *away* from e, i.e. if we travel along any non-singular edge path which begins at e and ends at a non-leaf, then the orientations of all the edges in the path point in the direction of travel. Then it is clear that any trivalent vertex will have at least one of its incident edges oriented toward that vertex. Thus (ii) will be satisfied whatever the orientations of the other leaves. See Figure 5 for an example. □

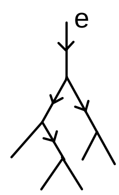


Figure 5

We can now complete the proof of Theorem 2. Since G has a cycle we can make one or more cuts in edges of G to create a tree. Each edge of G which is cut will create two new leaves in T. We now choose arbitrary orientations of each cut edge of G, which will induce orientations of the new leaf edges of T. Note that one of each pair of new leaves will be oriented inwards. Thus the outward orientations of the leaves of G together with these orientations of the new leaves of T provides orientations of all the leaves of T which satisfies the hypothesis of Lemma 2.6. Applying this lemma gives an orientation of T satisfying (ii). But now we can glue the cut edges back together and we get an orientation of G satisfying (i) and (ii), thus proving the Theorem. See Figure 6 for an example. □

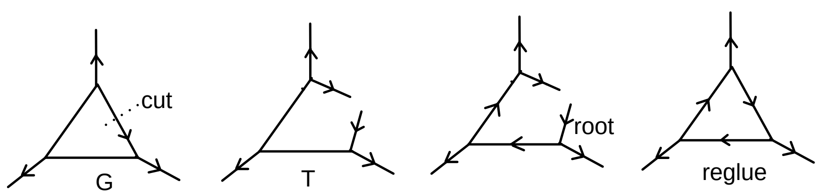


Figure 6

Let's define a filtration of H_g by $F_k^Y(H_g) = \mathrm{Im}(F_k(C_g) \to H_g)$, where we consider the restriction of the quotient map $C_g \to H_g$. By [Ma], $F_1^Y(H_g) = F_1^w(H_g) = T H_g$.

Theorem 3 *$F_k^Y(H_g) \subseteq F_k^w(H_g)$, thus inducing a map $G_k^Y(H_g) \to G_k^w(H_g)$*

and a commutative diagram:

$$\begin{array}{ccccccc} A_k(H) & \longrightarrow & \mathsf{G}_k(C_g) & & & & \\ \downarrow & & \downarrow & & & & \\ A^t_k(H) & \xrightarrow{\ \ k\ \ } & \mathsf{G}^Y_k(H_g) & \longrightarrow & \mathsf{G}^w_k(H_g) & \xrightarrow[=]{J^H_k} & \mathsf{D}_k(H) \end{array} \qquad (1)$$

$_k$ *is an isomorphism for all* k *and the composition* $A^t_k(H)$! $\mathsf{G}^Y_k(H_g)$! $\mathsf{G}^w_k(H_g)$! $\mathsf{D}_k(H)$ *is the epimorphism (isomorphism if* $k > 1$*) of Proposition 2.2.*

Proof First note that it follows from Theorem 2 that the composition $A_k(H)$! $\mathsf{G}_k(C_g)$! $\mathsf{G}^Y_k(H_g)$ factors through $A^t_k(H)$. This yields the commutative diagram (3), assuming for the moment that $F^Y_k(H_g)$ $F^w_k(H_g)$.

To see that $F^Y_k(H_g)$ $F^w_k(H_g)$ for $k > 1$, and, at the same time, identify the composition of the maps in the bottom line of the diagram, we rst consider a k-clasper C in I $_{g;1}$ associated to a connected H-tree T. If we choose a root of T and the element of H $L_{k+1}(H)$ associated to this rooted H-tree is y, then, by a sequence of Kirby moves, we can convert C into a 2-component link $(l; l^0)$, where l is the leaf of C corresponding to the root, and so represents $2 = {}_2$, and l^0 represents $y\ 2$ $_{k+1}$ mod $_{k+2}$. For example, Figure 7 explains this for a rooted H-tree of degree 1.

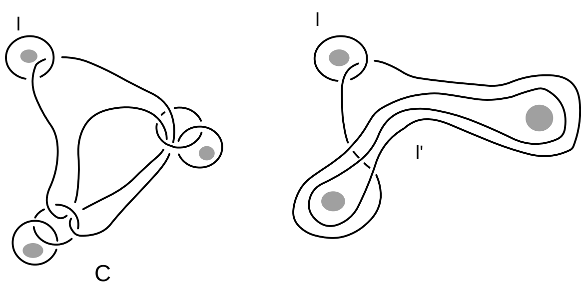


Figure 7: A clasper C on the left, and the Kirby-equivalent link on the right

Now suppose $M = (I$ $_{g;1})_C$ and 2 $_1($ $_+)$. To compute $_k(M)$ we push a curve K representing down I $_{g;1}$ and observe how it changes every time we cross C. When it crosses the leaf l, the e ect may be computed by using the Kirby-equivalent link l [l^0 instead of C. But then it is not hard to see that the e ect is to add a copy of l^0 to K. Thus the contribution of all the crossings of K with the leaf l is $<\ ;\ >y$, where $<;>$ denotes the homological intersection number in $\Sigma_{g,1}$. Under the canonical isomorphism $\mathrm{Hom}(H, L_{k+1}(H)) = H \otimes L_{k+1}(H)$ this corresponds to $x \otimes y$. The total change in η is then the sum of these contributions over all the univalent vertices of T, which is exactly the element of $D_k(H)$ corresponding to T. This shows that $F_k^Y(\mathcal{H}_g) \subset F_k^w(\mathcal{H}_g)$, at least for the elements represented by trees in $F_k^Y(\mathcal{H}_g)$, and that the composition $\mathcal{A}_k(H) \to \mathcal{G}_k(\mathcal{C}_g) \to \mathcal{G}_k^Y(\mathcal{H}_g) \to \mathcal{G}_k^w(\mathcal{H}_g) \to D_k(H)$ associates to any H-tree the corresponding element of $D_k(H)$ under the isomorphism of Proposition 2.2. The proof of the commutativity of diagram (3) is completed by noting that if G is any H-graph with a cycle (and an ordering of its univalent vertices), then we can cut open some edges to create a tree T, where the new vertices are labeled by $0 \in H$. Now a clasper representing G is also a clasper representing T and the above argument applies. Since some of the labels are 0, the image in $D_k(H)$ is 0.

To prove that η_k is an isomorphism first note that the map $\mathcal{A}_k^t(H) \to \mathcal{G}_k^Y(\mathcal{H}_g)$ is onto, since $\mathcal{A}_k(H) \to \mathcal{G}_k(\mathcal{C}_g)$ is onto. So if $k > 1$ it must also be one-one, since the composition to $D_k(H)$ is an isomorphism. Thus all the maps in the bottom row must be isomorphisms. ◻

To deal more completely with $k = 1$ we must consider some additional $\mathbb{Z}/2$-invariants on $\mathcal{H}_g$ — or rather on $\mathcal{T}\mathcal{H}_g = F_1^Y(\mathcal{H}_g) = F_1^w(\mathcal{H}_g)$. Recall the Birman-Craggs homomorphisms on the classic Torelli group $\mathcal{T}_g$ from [BC]. To define these first choose an imbedding $i: \Sigma_g \subset S^3$. Then if $h \in \mathcal{T}_g$ we can cut S^3 along $i(\Sigma_g)$ and reglue, using h, to obtain a homology sphere Σ_h. We then define $\rho_i(h)$ to be the Rochlin invariant of Σ_h. It is clear that this makes perfectly good sense if we substitute a homology cylinder in $\mathcal{T}\mathcal{H}_g$ for h and the result depends only on its homology bordism class. Thus we obtain homomorphisms $\rho_i: \mathcal{T}\mathcal{H}_g \to \mathbb{Z}/2$.

Johnson shows that one can choose a number of $\{\rho_i\}$ exactly equal to the dimension of the vector space V, thereby defining a homomorphism $\sigma: \mathcal{T}_g \to V$, which determine all the Birman-Craggs homomorphisms, and he then shows that the combined homomorphism $H_1(\mathcal{T}_g) \to \Lambda^3 H \oplus V$ is an isomorphism.

Now consider the corresponding $\sigma: \mathcal{T}\mathcal{H}_g \to V$. We claim that $\sigma(F_2^Y(\mathcal{H}_g)) = 0$ and so we get an induced $\sigma: \mathcal{G}_1^Y(\mathcal{H}_g) \to V$. As is pointed out in [H] and [GGP], surgery on a clasper of degree n is the same as cutting and pasting along some imbedded surface using an element of the n-th lower central series term $(\mathcal{T}_g)_n$ of the Torelli group. Thus a Birman-Craggs homomorphism on a homology cylinder in $F_n^Y(\mathcal{H}_g)$ takes the same value as some Birman-Craggs homomorphism on some element of $(\mathcal{T}_g)_n$, which is always zero if $n \geq 2$.

Let us now consider the combined homomorphism $G_1^Y(H_g) \to G_1^w(H_g) \oplus V = \Lambda^3 H \oplus V$, which must be an epimorphism since it is when restricted to T_g. Since we also have Habiro's epimorphism $\Lambda^3 H \oplus V = A_1^t(H) \to G_1^Y(H_g)$, it follows that both of these epimorphisms are actually isomorphisms.

To summarize, we have:

Corollary 2.7 (1) $G_1^Y(H_g) \cong G_1^w(H_g) \oplus V \cong \Lambda^3 H \oplus V$,

(2) $G_k^Y(H_g) \cong G_k^w(H_g)$ *for* $k > 1$,

(3) $F_k^Y(H_g) = F_k^w(H_g) \cap \operatorname{Ker}\Phi$ *for* $k > 1$.

Proof (1) and (2) are proved above. To prove (3) first consider $k = 2$. Suppose that $M \in F_2^w(H_g) \subset F_1^w(H_g) = F_1^Y(H_g)$ and $\Phi(M) = 0$. Then $M \to 0 \in G_1^w(H_g) \oplus V \cong G_1^Y(H_g)$ and so $M \in F_2^Y(H_g)$. For $k \geq 3$ we proceed by induction, using the following commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_k^Y(H_g) & \longrightarrow & F_k^w(H_g) & \longrightarrow & V \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & F_{k-1}^Y(H_g) & \longrightarrow & F_{k-1}^w(H_g) & \longrightarrow & V \\
 & & \downarrow & & \downarrow & & \\
 & & G_{k-1}^Y(H_g) & \xrightarrow{\cong} & G_{k-1}^w(H_g) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The columns are exact and, by induction, the middle row is exact. It then follows that the top row is exact. □

Suppose that Θ is a multiplicative invariant of homology cylinders of genus g, i.e. a homomorphism $\Theta : C_g \to G$ for some group G. We will say that Θ is of *finite type* if $\Theta(F_{k+1}(C_g)) = 1$ for some k, and is of type k for the minimum such value. This is actually a *weaker* definition than the usual notion of finite type.

For example $_k$ is an invariant of type k, as is , although it is only defined on $\mathcal{T}H_g$. The following corollary asserts that $_k$ and are the *universal* homology bordism invariant of type k.

Corollary 2.8 *Suppose is a multiplicative homology bordism invariant of type k. If M, N are two homology cylinders such that $_k(M) = {}_k(N)$ and $(M \cdot N^{-1}) = 0$, then $(M) = (N)$.*

Proof By Corollary 2.7 $M \cdot N^{-1} \in F^Y_{k+1}(H_g)$ and so $(M \cdot N^{-1}) = 1$. □

3 String links and homology cylinders

3.1 String links

Let P_g denote the group of pure braids with g strands. Let S_g denote the group of concordance classes of string links with g strands in a homology 3-ball whose boundary is identified with $\partial(I \times D^2)$. Two string links S_1, S_2 in homology 3-balls B_1, B_2, respectively, are *concordant* if there is a homology 4-ball B whose boundary is identified with $B_1 \cup B_2$ (with their boundaries identified), and a concordance between S_1 and S_2 imbedded in B. There is an obvious homomorphism $P_g \to S_g$. Recall the theorem of Artin which states that the map $P_g \to \mathrm{Aut}(F')$ — where F' is the free group with basis $\{y_1, \ldots, y_g\}$ identified with $_1(D_g)$, and D_g is the 2-disk with g holes — is injective with image $\mathrm{Aut}_1(F')$. $\mathrm{Aut}_1(F')$ is the subgroup of $\mathrm{Aut}(F')$ consisting of all automorphisms h such that $h(y_i) = {}_i^{-1} y_i {}_i$, for some choice of $_i$ and $h(y_1 \cdots y_g) = y_1 \cdots y_g$. Note that $_i$ is uniquely determined if we specify that the exponent sum of y_i in $_i$ is 0.

The Milnor -invariants can be formulated as a sequence of homomorphisms $_k \colon S_g \to \mathrm{Aut}_1(F'/F'_{k+2})$, $k \geq 1$, where $\mathrm{Aut}_1(F'/F'_q)$ consists of all automorphisms h satisfying $h(y_i) = {}_i^{-1} y_i {}_i$ for some $_i \in F'/F'_{q-1}$ and satisfying the equation

$$h(y_1 \cdots y_g) = y_1 \cdots y_g \tag{2}$$

(see [HL]). Note now that $_i$ is uniquely determined by h up to left multiplication by a power of y_i. One consequence of the existence of the $_k$ is that the map $P_g \to S_g$ is injective. It is known that $_k$ is onto (see e.g. [HL]). If we define $S_g[k]$ = Ker $_k$ and $S_g[0] = S_g$, then we have an isomorphism $S_g[k]/S_g[k+1] = D_k(H^0)$, for k 0, where $H^0 = H_1(D_g)$. This isomorphism is defined by $\to \sum_i y_i \ {}_i$, for $\in S_g[k]$, where $_i \in F_{k+1}/F_{k+2}$ are determined by $_{k+1}$. Note that $S_g[\infty] = \bigcap_k S_g[k]$ is non-trivial since, for example, it contains the knot concordance group.

If S_g^0 denotes the standard string link concordance group, i.e. string links in $I \ D^2$ and concordances in $I \ (I \ D^2)$, then we have natural maps $P_g \ S_g^0 \to S_g$.

Question 1 Is $S_g^0 \to S_g$ injective? Since the realization theorem of [HL] produces standard string links the induced map $S_g^0/S_g^0[\infty] \to S_g/S_g[\infty]$ is an isomorphism.

If denotes the group of homology bordism classes of closed homology 3-spheres, then we have an obvious injection $\ S_g$, defined by connected sum with the trivial string link in $I \ D^2$, whose image is a central subgroup of S_g, lying in $S_g[\infty]$, and a retraction $S_g \to$.

Question 2 Is the combined map $\ S_g^0 \to S_g$ an isomorphism?

We will need to consider the *framed* versions. Let P_g^{fr} and S_g^{fr} denote the groups of framed pure braids on g strands and concordance classes of framed string links with g strands. Note that $P_g^{\mathrm{fr}} = P_g \ \mathbb{Z}^g$ and $S_g^{\mathrm{fr}} = S_g \ \mathbb{Z}^g$. We have an isomorphism $: P_g^{\mathrm{fr}} \to A_1(F^0)$ and homomorphisms $_k\colon S_g^{\mathrm{fr}} \to A_1(F^0/F^0_{k+1})$ for k 1. $A_1(F^0)$ (resp. $A_1(F^0/F^0_q)$) consists of sequences $= (_1, \ldots, _g)$ of elements $_i \in F^0$ (resp. F^0/F^0_q) which satisfy the following

(i) The map $y_i \to \ _i^{-1} y_i \ _i$ defines an *automorphism* of F^0 (resp. F^0/F^0_{q+1})

(ii) $(y_i \ y_g) = y_1 \ y_g$.

The multiplication in $A_1(F^0); A_1(F^0/F^0_q)$ is defined by $(\ )_i = \ _i \ (\ _i)$. Then $\to$ defines epimorphisms with kernel $\mathbb{Z}^g$. Note that we have the commutative diagram

$$\begin{array}{ccc} S_g^{\mathrm{fr}} & \longrightarrow & S_g \\ \big\downarrow {\scriptstyle k} & & \big\downarrow {\scriptstyle k} \\ A_1(F^0/F^0_{k+1}) & \longrightarrow & \mathrm{Aut}_1(F^0/F^0_{k+2}) \end{array}$$

We define the filtration $S_g^{\mathrm{fr}}[k]$ = Ker $_k$.

3.2 Relating string links and homology cylinders

Recall the imbedding : $P_g^{\mathrm{fr}} \to \Gamma_{g,1}$, defined by [O] and studied in [L, Section 2.2]. There is a commutative diagram

$$\begin{array}{ccc} P_g^{\mathrm{fr}} & \longrightarrow & \Gamma_{g,1} \\ \big\downarrow & & \big\downarrow \\ A_1(F') & \longrightarrow & \mathrm{Aut}_0(F) \end{array}$$

where is defined by

$$\begin{aligned} ((\ _i))\ \ y_i &= \ _i^{-1} y_i\ \ _i \\ ((\ _i))\ \ x_i &= x_i\ \ _i \end{aligned} \tag{3}$$

Since and are isomorphisms and is injective, it follows that is injective.

We now extend to an imbedding $\hat{}: S_g^{\mathrm{fr}} \to H_g$. Choose an imbedding $D_g \quad _{g,1}$ where the meridians of the holes in D_g, which correspond to the generators $y_i \in F'$, are mapped to the meridians of the handles of $_{g,1}$ which correspond to the generators $y_i \in F$ — see Figure 1. Suppose S is a framed string link with g strands in a homology 3-ball B and C is the complement of an open tubular neighborhood of S in B. Then there is a canonical identification of ∂C with $\partial(I \quad D_g)$. Now we create a homology cylinder by taking $I \quad _{g,1}$, with $^- = 0 \quad _{g,1}; \quad ^+ = 1 \quad _{g,1}$, removing $I \quad D_g$ and replacing it with C, using the canonical identification of the boundaries. We take this homology cylinder to represent $\hat{}(S)$.

It is clear that $\hat{}$ is a well-defined homomorphism and extends (see the definition of in [L]). By examining Figure 1 we can see that the following diagram commutes:

$$\begin{array}{ccc} S_g^{\mathrm{fr}} & \xrightarrow{\ \ \hat{}\ \ } & H_g \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ A_1(F'/F'_{k+1}) & \xrightarrow{\ \hat{}\ } & \mathrm{Aut}_0(F/F_{k+1}) \end{array}$$

where $\hat{}$ is the injection defined by

$$\begin{aligned} \hat{}((\ _i))\ \ y_i &\quad \ _i^{-1} y_i\ \ _i \mod F_k \\ \hat{}((\ _i))\ \ x_i &\quad x_i\ \ _i \mod F_k \end{aligned} \tag{4}$$

As a consequence we see that $\hat{}$ preserves the weight filtrations and the induced map

$$S_g^{\mathrm{fr}}[k]/S_g^{\mathrm{fr}}[k+1] \longrightarrow F_k^w(\mathcal{H}_g)/F_{k+1}^w(\mathcal{H}_g) = \mathsf{G}_k^w(\mathcal{H}_g)$$

corresponds to the *monomorphism* $\mathsf{D}_k(H') \to \mathsf{D}_k(H)$ induced by inclusion $H' \subseteq H$.

Theorem 4 $\hat{}$ *is injective.*

Proof Let $\mathcal{H}_g$ denote the subset (it is not a subgroup) of $\mathcal{H}_g$ consisting of all M such that $p\,{}_1(M)i$ is an isomorphism, where $i\colon H' \to H$ is the inclusion and $p\colon H \to H'$ is the projection with $p(x_i) = 1;\ p(y_i) = y_i$. Note that $\hat{}(S_g^{\mathrm{fr}}) \subseteq \mathcal{H}_g$. We will define a "retraction" $\colon \mathcal{H}_g \to S_g^{\mathrm{fr}}$, which is not a homomorphism, but will satisfy $\hat{} = \mathrm{id}$ and so prove the theorem.

We identify $I \times D_g$ with the complement of the trivial framed string link. Now consider an imbedding of D_g into $\frac{1}{2} \times D_g$ defined by removing a thin collar of the entire boundary of $\frac{1}{2} \times D_g$. Thus we obtain an imbedding of D_g into the interior of $I \times D_g$. Now let H_g denote the solid handlebody of genus g whose boundary is ${}_g \cup {}_{g;1}$. We will make this identification so that the y_i represent a basis for ${}_1(H_g)$, which we identify with F', and the x_i correspond to the meridians of the handles of H_g. We can imbed H_g into $I \times D_g$ as a thickening of the imbedded D_g and so that $D_g \cong {}_{g;1} \subseteq H_g$ is the imbedding which we used to define $\hat{}$.

Now suppose M represents an element of $\mathcal{H}_g$. Then we can cut open $I \times D_g$ along the imbedded ${}_{g;1}$ and insert a copy of M so that $\bar{} \subseteq H_g$. If we identify $I \times D_g$ with the complement of the trivial framed string link then our newly constructed manifold is identified with the complement of some string link which lies in a homology 3-ball precisely when $M \in \mathcal{H}_g$. We take this to represent (M). It is not hard to see that $\hat{} = \mathrm{id}$. □

Remark 3.1 Note that $\mathcal{H}_g = {}_1^{-1}(A)$, where A is the subset of $\mathrm{Sp}(H)$ consisting of all h such that $p\,h\,i$ is an isomorphism. It is not hard to see that $A = P \cdot Q$, where $P;Q$ are subgroups of $\mathrm{Sp}(H)$ defined as follows: $h \in P$ if and only if $h(L) = L$, where $L = \mathrm{Ker}\,p$, and $h \in Q$ if and only if $h|H' = \mathrm{id}$. The bijection is defined by multiplication $(h_1; h_2) \mapsto h_1 \cdot h_2$. This product decomposition of A lifts to a product decomposition of $\mathcal{H}_g$ — see Corollary 3.3.

Remark 3.2 Since is a retraction, $(\mathcal{H}_g \setminus \Gamma_{g;1})$ certainly contains P_g^{fr}, but also contains, for example, the Whitehead string link, Figure 8.

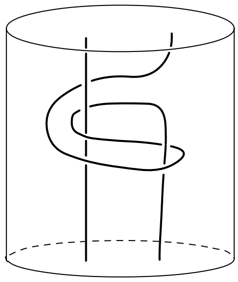


Figure 8

Generally $\in (\mathcal{H}_g \cap \Gamma_{g,1})$ if and only if is represented by a framed string link whose complement is diffeomorphic to H_g (i.e. the complement of the trivial string link).

Question 3 What is $\wedge^{-1}(\Gamma_{g,1})$? Clearly $P_g^{\mathrm{fr}} \quad \wedge^{-1}(\Gamma_{g,1}) \quad (\mathcal{H}_g \cap \Gamma_{g,1})$.

3.3 Boundary homology cylinders

Let M be a homology cylinder over $_{g,1}$ and so $\partial M = {}^+ \cup {}^-$. We can also write ${}^+ \cup {}^- = \partial H^+ \natural H^-$, where H are two copies of H_g and $\natural$ denotes boundary connected sum. Let V be the closed manifold $M \cup (H^+ \natural H^-)$. We will say that M is a *boundary homology cylinder* if V bounds a compact orientable manifold W such that the inclusions $H \quad W$ are homology equivalences. the boundary homology cylinders define a subgroup $\mathcal{H}_g^B \quad \mathcal{H}_g$. Note that $_1(\mathcal{H}_g^B) \quad P$ and so $\mathcal{H}_g^B \quad \mathcal{H}_g$. In fact, since $P \quad A \quad A$, we have $\mathcal{H}_g^B \quad \mathcal{H}_g \quad \mathcal{H}_g$, and so the right coset space $\mathcal{H}_g/\mathcal{H}_g^B$ is defined.

Theorem 5 (a) $\mathcal{H}_g^B \cap \Gamma_{g,1} = \Gamma_{g,1}^B$ *the subgroup of* $\Gamma_{g,1}$ *consisting of diffeomorphisms which extend to diffeomorphisms of* H_g.

(b) $\mathcal{H}_g^B = {}^{-1}(0)$, *where* 0 *denotes the trivial string link.*

Proof (a) Suppose $h \in \mathcal{H}_g^B \cap \Gamma_{g,1}$. Then $h\,(x_i) \in \mathrm{Ker}\{\,_1(\,_{g,1}) \to \,_1(H_g)\}$ and, therefore, by Dehn's lemma, if D_i is the meridian disk of H_g corresponding to x_i, $h(D_i)$ bounds a disk $D_i' \quad H_g$ By standard cut and paste techniques we can assume that the D_i' are disjoint. We can now extend h over each D_i by mapping it onto D_i'. Since the complement of $\cup_i D_i$ is a 3-ball, as is the complement of $\cup_i D_i'$, we can extend over H_g.

(b) Suppose $M \in \mathcal{H}_g^B$. Let V be a homology bordism from H_g to itself, with $\partial V = H^+ \cup M \cup H^-$, where H are two copies of H_g. Let $W = I \quad (I \quad D_g) \cup V$, where V is attached to $(1 \quad H_g) \quad (1 \quad (I \quad D_g))$ along H^+. Then W is a homology bordism between $0 \quad (I \quad D_g)$, the complement of the trivial string link (*triv* in Figure 9), and $C =$ complement of S, where S is the string link constructed as the representative of (M). Thus we can fill in W with product strings to yield a concordance from the trivial string link to S. See Figure 9

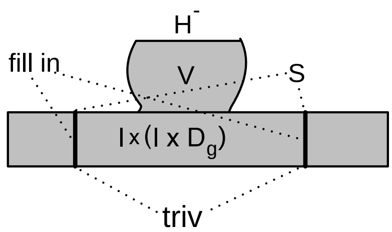


Figure 9: $(\mathcal{H}_g^B) = 0$

Conversely suppose $M \in \mathcal{H}_g$ and $(M) = 0$. Let W be the complement of a concordance between the trivial string link and S, the constructed representative of (M). If C is the complement of S, then we can decompose $C = I \quad _{g;1} \cup M \cup H_g$. Thus

$$\partial W = H_g \cup C = H^+ \cup M \cup H^-$$

In this way we can see that W is a homology bordism from H_g to itself which exhibits M as an element of $\mathcal{H}_g^B$. □

Corollary 3.3 *The monomorphisms* $\mathcal{H}_g^B \quad \mathcal{H}_g$ *and* $\hat{}: \mathcal{S}_g^{fr} \quad \mathcal{H}_g$ *define a bijection*

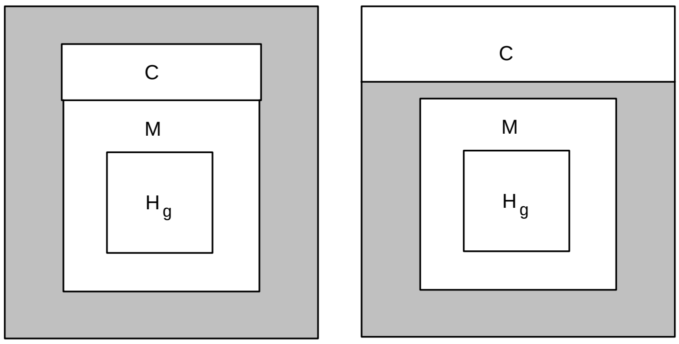
$$: \mathcal{H}_g^B \quad \mathcal{S}_g^{fr} \to \mathcal{H}_g$$

In particular $\hat{}$ *induces a bijection* $\mathcal{S}_g^{fr} = \mathcal{H}_g / \mathcal{H}_g^B$.

Proof is defined by $(M; S) = M \quad \hat{}(S) \in \mathcal{H}_g$ (see Remark 3.1). To see that is one-one we only need note that $\mathcal{H}_g^B \cap \hat{}(\mathcal{S}_g^{\mathrm{fr}}) = 1$, which follows from Theorem 5(b) and that $\hat{} = 1$. To prove that is onto we need:

Lemma 3.4 *If* $M \in \mathcal{H}_g; S \in \mathcal{S}_g^{fr}$, *then* $(M \quad \hat{}(S)) = (M) \quad S$.

Proof The idea is given by the schematic pictures below. The rst picture shows the complement of $(M \ {}^{\wedge}(S))$. C is the complement of the string link S and the shaded region is a product $I \ \ _g$. Note that $M \ {}^{\wedge}(S)$ can be constructed by stacking C on M along $D_g \ \ ^+$. The second picture gives an alternative view of the rst picture which can then be recognized as the complement of $(M) \ S$. Filling in the strings completes the proof.



□

Now suppose M 2 H_g. Then $(M \ {}^{\wedge}((\ (M))^{-1}) = 0$, by Lemma 3.4, and $(M \ {}^{\wedge}((\ (M))^{-1}; \ (M)) = M$. □

Remark 3.5 In a recent work of N. Habegger [Ha] a bijection is constructed between the \Torelli" subsemigroup of C_g and the semigroup of $2g$-strand string links in homology balls with vanishing linking numbers, which although not multiplicative, induces isomorphisms between the associated graded groups of the relative weight ltrations.

The homomorphisms $_k$: H_g ! $\mathrm{Aut}_0(F{=}F_{k+1})$, induce homomorphisms ^B_k: H^B_g ! $\mathrm{Aut}(F^0{=}F^0_{k+1})$.

Proposition 3.6 *^B_k is onto, for all k 1.*

Proof If h 2 $\mathrm{Aut}(F^0{=}F^0_{k+1})$, then we can lift h to an endomorphism h of F^0 and we can realize h by an imbedding $H_g \ \ H_g$. If H^0_g denotes the imbedded copy of H_g, then de ne $V = (I \ \ H^0_g) \ [\ (I \ \ H_g)$, attaching $1 \ \ H^0_g$ to $0 \ \ H_g$. V is a homology bordism between $0 \ \ H^0_g$ and $1 \ \ H_g$ and clearly maps to h under ^B_k. □

Remark 3.7 The restriction of ^B_k to $\Gamma^B_{g;1}$ is induced by a map ^{B}: $\Gamma^B_{g;1}$! $\mathrm{Aut}\,F^0$ and it is known (see [G]) that ^{B} is onto.

To see that Ker ρ_k^B is large we consider another imbedding of S_g^{fr} into $\mathcal{H}_g$, $\hat{\Psi}\colon S_g^{\mathrm{fr}} \to \mathcal{H}_g$, "dual" to $\hat{\ }$. This is defined in the same way as $\hat{\ }$ — from an imbedding $D_g \subset \Sigma_{g,1}$ — except that we now choose the imbedding so that the meridians of the holes in D_g map onto the meridians of the handles corresponding to $\{x_i\} \subset F$. If F'' is the free group with basis $\{x_i\}$, then we have an injection $\hat{\ }\colon A_1(F''/F''_q) \to \mathrm{Aut}_0(F/F_q)$, defined by

$$\begin{aligned} \hat{\ }((\ _i))\ x_i &\equiv \ _i^{-1} x_i \ _i \mod F_q \\ \hat{\ }((\ _i))\ y_i &\equiv \ _i y_i \mod F_q \end{aligned} \tag{5}$$

There is a commutative diagram

$$\begin{array}{ccc} S_g^{\mathrm{fr}} & \xrightarrow{\hat{\Psi}} & \mathcal{H}_g \\ \Big\downarrow{\scriptstyle \ _k^0} & & \Big\downarrow{\scriptstyle \ _k} \\ A_1(F''/F''_{k+1}) & \xrightarrow{\hat{\ }} & \mathrm{Aut}_0(F/F_{k+1}) \end{array} \tag{6}$$

Theorem 6 (a) $\hat{\Psi}(S_g^{fr}) \subset \mathcal{H}_g^B$.

(b) $\rho_k^B \circ \hat{\Psi} = 0$ *for all* k.

Proof (a) Write $\mathcal{H}_g = D^3 \cup H$, where H represents the g disjoint handles. Then the imbedding $D_g \subset \Sigma_{g,1}$ which defines $\hat{\Psi}$ can be regarded as the restriction of an identification of D^2 with the upper hemisphere D^2_+ of D^3 — the holes in D^2 which are removed to obtain D_g are the disks along which one end of each handle of H is attached to D^3. Now consider $I \times D^2 \subset I \times D^3$ and a framed string link $S \subset I \times D^2$. We create a manifold V by attaching $I \times H$ to $I \times D^3$ by attaching one end of each handle to S and the other to the trivial framed string link in $I \times D^2_-$. It is not difficult to see that this is a homology bordism from $\mathcal{H}_g$ to itself which extends $\hat{\Psi}(S)$.

(b) This follows directly from Equation (3.3) and diagram (3.3), since $\ _i \in F''$. ☐

Question 4 Describe $\bigcap_k$ Ker ρ_k^B. It is known (see [Lu]) that the kernel of ρ^B is generated by so-called *twist automorphisms*, i.e. Dehn twists along properly imbedded 2-disks in $\mathcal{H}_g$.

4 A Lagrangian filtration of the mapping class group

4.1 Definition of the filtration

Let H_g and $\Sigma_{g,1}$ be as above. We define a new filtration of $\Gamma_{g,1}$. Let $p\colon F \to F'$ the epimorphism induced by the inclusion $\Sigma_{g,1} \subset H_g$, i.e. $p(x_i) = 1, p(y_i) = y_i$. We define $L_g[k] \subset \Gamma_{g,1}$, for $k \geq 1$, by the condition that $h \in L_g[k]$ if and only if $p \circ h(x_i) \in F'_{k+1}$ for all i. It is not hard to see that these are subgroups. Set $L_g[\infty] = \bigcap_k L_g[k] = \{h | \operatorname{Ker} p \subset \operatorname{Ker} p \circ h\}$. Note that $h \in L_g[k]$ $(k \geq 1)$ induces an automorphism of F'/F'_{k+1} and, therefore, of $H' = H_1(F')$. This defines a homomorphism $\rho_k\colon L_g[k] \to \operatorname{Auto}(H')$. We define $\hat{L}_g[k] = \operatorname{Ker} \rho_k$.

We identify some of these groups. Let $L = \operatorname{Ker}\{H \to H'\}$, a Lagrangian subgroup of the symplectic space H. Let $\operatorname{Sp}(H)$ denote the group of symplectic automorphisms of H. Let $\sigma\colon \Gamma_{g,1} \to \operatorname{Sp}(H)$ denote the standard epimorphism. Let $P(L) \subset \operatorname{Sp}(H)$ be the subgroup of all ϕ such that $\phi(L) = L$ and $P_0(L) \subset P(L)$ those ϕ such that $\phi|L = \mathrm{id}_L$. then $L_g[1] = \sigma^{-1}(P(L))$ and $\hat{L}_g[1] = \sigma^{-1}(P_0(L))$, which was denoted L_g^L in [GL1].

Proposition 4.1 (a) *ρ_k is onto for all $k \geq 1$.*

(b) *$\hat{L}_g[2]$ is the subgroup generated by Dehn twists on simple closed curves representing elements of L. This is the subgroup denoted L_g^L in [GL1].*

(c) *$L_g[\infty] = \Gamma^B_{g,1}$.*

Proof (a) Since $\Gamma^B_{g,1} \subset L_g[k]$ for all k, this follows immediately from the fact that ρ^B is onto [G].

(b) Note that $h \in \hat{L}_g[2]$ iff $h(x_i)x_i^{-1} \in (F_3 \Lambda) \cap F_2$ for all i, where $\Lambda = \operatorname{Ker} p$, the normal closure of $\{x_i\}$. To see this first note that $h \in L_g[2]$ iff $h(x_i)x_i^{-1} \in F_3 \Lambda$ for every i. Secondly note that $h \in \operatorname{Ker} \rho_k$ iff $h(y_i) \equiv y_i \mod F_2$ for all i — but this is equivalent to $h(x_i) \equiv x_i \mod F_2$ for all i, since, if a symplectic matrix has the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, it follows that $A = I$ iff $C = I$.

Now note that, as a consequence of [L, Cor. 2], $h \in L_g^L$ iff $h(x_i)x_i^{-1} \in [F, \Lambda F_2]$ for all i. Thus, since $(F_3 \Lambda) \cap F_2 = F_3 (F_2 \cap \Lambda)$ and $[F, \Lambda F_2] = F_3 [F, \Lambda]$, (b) will follow from

Lemma 4.2 $F_2 \cap \Lambda = [F, \Lambda]$.

Proof By Hopf's theorem the quotient $\dfrac{F_2 \cap}{[F; \;]} = H_2(F= \;)$. But $F= \; = F^0$, which is a free group and so $H_2(F= \;) = 0$. □

(c) If $h \in L_g[1]$, then $h\;(x_i) \in$, for all i. We can thus apply the argument in the proof of Theorem 5(a). □

Define a function $J^L_k : L_g[k] \to \mathrm{Hom}(L; \mathsf{L}_{k+1}(H^0)) = H^0 \; \mathsf{L}_{k+1}(H^0)$ by $J^L_k(h)$ $a = p(h(\;))$, where $\in$ is any lift of $a \in L$. Note that the symplectic form on H induces an identification of H^0 with the dual space of L. Furthermore if $h : H \to H$ is the induced automorphism, then $h\;(L) = L$ and so there is an induced automorphism of H^0 and of $\mathsf{L}_k(H^0)$, both of which we also denote by h .

Proposition 4.3 (a) $J^L_k(\hat{L}_g[k]) \quad \mathsf{D}_k(H^0)$.

(b) $J^L_k(h_1 \; h_2) = (h_2 \quad 1)J^L_k(h_1) + (1 \quad h_1\;)J^L_k(h_2)$. *Therefore* $J^L_k | \hat{L}_g[k]$ *is a homomorphism.*

(c) $L_g[k+1] = \mathrm{Ker}\, J^L_k$.

Proof (a) If $h \in L_g[k]$ then $J^L_k(h) = \sum_i y_i \quad ph\;(x_i)$. We will abuse notation and allow ourselves to denote the induced bases of L and H^0 by $\{x_i\}$ and $\{y_i\}$, respectively. If $h \in \hat{L}_g[k]$, then $ph\;(y_i) = y_i$ in H^0 and so

$$y^0_k J^L_k(h) = \prod_i [y_i; ph\;(x_i)] = \prod_i [ph\;(y_i); ph\;(x_i)]$$
$$= ph \prod_i [y_i; x_i] = p \prod_i [y_i; x_i] = 1$$

in $F_{k+2}=F_{k+3}$, since $h \prod_i [y_i; x_i] = \prod_i [y_i; x_i]$.

(b) Let $\in$ be a lift of $a \in L$. Then we can write $h_2(\;) = \;_1 \;^0$ for some $_1 \in$ and $^0 \in F^0 \quad F$, where the latter inclusion is some splitting of p. Since $h_2 \in L_g[k]$ we can choose $^0 \in F^0_{k+1}$. If a_1 is the homology class of $_1$ in L, then $a_1 = h_2\;(a)$ and so $ph_1(\;_1)$ represents $J^L_k(h_1)\;(h_2\; a)$. If ^{0} represents $a^0 \in F^0_{k+1}=F^0_{k+2} = \mathsf{L}_{k+1}(H^0)$, then $a^0 = J^L_k(h_2)\; a$. Thus $ph_1(\;^0)$ represents $h_1\;(J^L_k(h_2)\; a)$.

From these observations we conclude that $J^L_k(h_1 \; h_2)\; a$, which is represented by $p(h_1h_2(\;)) = ph_1(\;_1)ph_1(\;^0)$ is given by

$$J^L_k(h_1 \; h_2)\; a = J^L_k(h_1)\; h_2\, a + h_1\; J^L_k(h_2)\; a$$

(c) This is immediate. □

4.2 An estimate of $\operatorname{Im} J_k^L$

We make use of the imbedding $\theta\colon P_g^{\mathrm{fr}} \to \Gamma_{g,1}$.

Theorem 7 *θ induces imbeddings*

$$\theta_k\colon (P_g^{fr})_{k+1}/(P_g^{fr})_{k+2} \hookrightarrow \hat{\mathcal{L}}_g[k]/\hat{\mathcal{L}}_g[k+1] = \mathcal{L}_g[k]/\mathcal{L}_g[k+1] \hookrightarrow \mathsf{D}_k(H') \quad (k \geq 2)$$

where $(P_g^{fr})_q$ is the q-th lower central series subgroup of P_g^{fr}.

The composite imbedding $(P_g^{fr})_{k+1}/(P_g^{fr})_{k+2} \hookrightarrow \mathsf{D}_k(H')$ is described as follows. If $\beta \in (P_g^{fr})_{k+1}$ is specified by longitude elements $\lambda_1, \dots, \lambda_g \in F'_{k+1}$, then β maps to $\sum_i y_i \otimes [\lambda_i]$, where y_i is a basis of H' and $[\lambda_i] \in F'_{k+1}/F'_{k+2} = \mathsf{L}_{k+1}(H')$ denotes the coset of λ_i.

From this we conclude that $\operatorname{rank} \mathcal{L}_g[k]/\mathcal{L}_g[k+1] \geq r(g,k)$, where $r(g,k) = \operatorname{rank}(P_g^{\mathrm{fr}})_{k+1}/(P_g^{\mathrm{fr}})_{k+2}$ is explicitly computable (see the discussion in [L]). The gap between $r(g,k)$ and $\operatorname{Ker} \gamma_k\colon H' \otimes \mathsf{L}_{k+1}(H') \to \mathsf{L}_{k+2}(H')$ is also explicitly computable. For example if $k = 2$ it is $\frac{1}{6}(g^3 - g)$ and for $k = 3$ it is $\frac{1}{8}(g^3 - g)(g - 2)$ — for $k = 1$ it is zero.

Proof Recall the definition of θ from Equation (3.2).

It is well-known that $\beta \in (P_g^{\mathrm{fr}})_q$, the q-th term of the lower central series, if and only if every $\lambda_i \in F'_q$ — see e.g. [F]. We therefore have $\theta((P_g^{\mathrm{fr}})_{k+1}) \subseteq \mathcal{L}_g[k]$ and the induced θ_k is injective. That $\hat{\mathcal{L}}_g[k]/\hat{\mathcal{L}}_g[k+1] = \mathcal{L}_g[k]/\mathcal{L}_g[k+1]$ follows from Proposition 4.1(a).

It also follows directly from the definitions that the composite $(P_g^{\mathrm{fr}})_{k+1} \to \mathcal{L}_g[k] \to H' \otimes \mathsf{L}_{k+1}(H')$ is as claimed. □

5 Lagrangian filtration of $\mathcal{H}_g$

We can extend the Lagrangian filtration over $\mathcal{H}_g$ in a natural way, using the $\{\sigma_k\}$. Set $F_k^L(\mathcal{H}_g) = \{M \in \mathcal{H}_g \colon p \circ \sigma_k(M)|F'' = 0\}$, for $k \geq 1$, where $p\colon F/F_{k+1} \to F'/F'_{k+1}$ is the projection and $F'' \subseteq F$ via $x_i \to x_i$. These are subgroups and obviously $F_k^w(\mathcal{H}_g) \subseteq F_k^L(\mathcal{H}_g)$; $F_k^L(\mathcal{H}_g) \cap \Gamma_{g,1} = \mathcal{L}_g[k]$ and $\mathcal{H}_g^B \subseteq F_k^L(\mathcal{H}_g)$ for every k. Now σ_k induces a map $\sigma'_k\colon F_k^L(\mathcal{H}_g) \to \operatorname{Aut}(F'/F'_{k+1})$ and clearly $\sigma'_k|\mathcal{H}_g^B = \sigma_k^B$. Thus σ'_k is onto by Proposition 3.6.

We can also define $\hat{F}_k^L(\mathcal{H}_g) = F_k^L(\mathcal{H}_g) \cap \operatorname{Ker} \hat{}$, where $\hat{}\colon \mathcal{H}_g \to \operatorname{Sp}(H)$ is the obvious extension of .

The map $J_k^L : L_g[k] \to H^0 \otimes L_{k+1}(H^0)$ extends to a map $\hat{J}_k^L : F_k^L(H_g) \to H^0 \otimes L_{k+1}(H^0)$ and the obvious generalization of Proposition 4.3 holds. In particular $F_{k+1}^L(H_g) = \operatorname{Ker} \hat{J}_k^L$.

We now address the homomorphism $\hat{\ } : S_g^{\mathrm{fr}} \to H_g$. Recall the filtration $\{S_g^{\mathrm{fr}}[k]\}$ from Section 3.1.

Theorem 8 $\hat{\ }(S_g^{fr}[k]) \subset \hat{F}_k^L(H_g)$ *and induces isomorphisms*

$$S_g^{fr}[k]/S_g^{fr}[k+1] = \hat{F}_k^L(H_g)/\hat{F}_{k+1}^L(H_g) = F_k^L(H_g)/F_{k+1}^L(H_g) = \mathsf{D}_k(H^0)$$

Proof The first assertion is clear from the definitions. The rest of the theorem follows from the observation that the following diagram is commutative

$$\begin{array}{ccc} S_g^{\mathrm{fr}}[k] & \xrightarrow{\ \hat{\ }\ } & \hat{F}_k^L(H_g) \\ & {}_k\searrow \quad \swarrow \hat{J}_k^L & \\ & \mathsf{D}_k(H^0) & \end{array}$$

and the fact that ${}_k$ is onto. □

We now have $\hat{\ }(S_g^{\mathrm{fr}}[1]) \subset \hat{F}_1^L(H_g)$ and $H_g^B \subset F_1^L(H_g)$. We will see that these two subgroups are independent and generate $F_1^L(H_g)$.

Recall the bijection $\ : H_g^B \times S_g^{\mathrm{fr}} \to H_g$ from Corollary 3.3.

Theorem 9

$$(H_g^B \times S_g^{fr}[k]) = F_k^L(H_g)$$

for every $2 \le k \le 1$. *In particular* $F_k^L(H_g)$ *is generated by the independent subgroups* H_g^B *and* $\hat{\ }(S_g^{fr}[k])$.

Proof We need to show that $(H_g^B \times S_g^{\mathrm{fr}}[k]) = F_k^L(H_g)$. First note that $(H_g^B \times S_g^{\mathrm{fr}}[k]) \subset F_k^L(H_g)$ — this follows from Theorem 8. For the ontoness we can apply the ontoness argument of Corollary 3.3 with the extra fact that

$(F_k^L(H_g))\quad S_g^{\mathrm{fr}}[k]$. This follows from the observation that the following diagram is commutative

$$\begin{array}{ccc} H_g & \longrightarrow & S_g^{\mathrm{fr}} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k} \\ \mathrm{Aut}_0(F=F_{k+1}) & \xrightarrow{\;\wedge\;} & A_1(F'=F'_{k+1}) \end{array}$$

where $\wedge$ is de ned by $\wedge(h) = (ph(x_i))$. □

Note that $^\wedge(S_g^{\mathrm{fr}}[1])\quad$ Ker $'_k$ and $'_k j H_g^B$ is onto, for every k.

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