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# Addendum and correction to: Homology cylinders: an enlargement of the mapping class group

Jerome Levine

**Abstract** In a previous paper [1], a group $\mathcal{H}_g$ of *homology cylinders* over the oriented surface of genus $g$ is defined. A filtration of $\mathcal{H}_g$ is defined, using the Goussarov-Habiro notion of finite-type. It is erroneously claimed that this filtration essentially coincides with the relative weight filtration. The present note corrects this error and studies the actual relation between the two filtrations.

**AMS Classification** 57N10; 57M25

**Keywords** Homology cylinder, mapping class group

## 1 Introduction

In [1] we consider a group $\mathcal{H}_g$ consisting of homology bordism classes of *homology cylinders*, where a homology cylinder is defined as a homology bordism between two copies of $\Sigma_{g,1}$, the once punctured oriented surface of genus $g$. This bordism is equipped with an explicit identification of each end with $\Sigma_{g,1}$—see [1] for more details. In particular there is a canonical injection of the mapping class group $\Gamma_{g,1}$ into $\mathcal{H}_g$.

Two filtrations of $\mathcal{H}_g$ are considered in the first part of the paper. The first is the relative weight filtration $\mathcal{F}^w_k(\mathcal{H}_g)$, the obvious extension of the relative weight filtration $\mathcal{F}^w_k(\Gamma_{g,1})$ of $\Gamma_{g,1}$ considered by Johnson, Morita and others. The canonical injection $J_k : \mathcal{G}^w_k(\Gamma_{g,1}) \to \mathsf{D}_k(H)$, where $\mathcal{G}^w_k(\Gamma_{g,1}) = \mathcal{F}^w_k(\Gamma_{g,1})/\mathcal{F}^w_{k+1}(\Gamma_{g,1})$ is the associated graded group, lifts, in a natural way, to $\mathcal{G}^w_k(\mathcal{H}_g)$ and there becomes an *isomorphism* $J^H_k : \mathcal{G}^w_k(\mathcal{H}_g) \xrightarrow{\cong} \mathsf{D}_k(H)$. $\mathsf{D}_k(H)$ is the kernel of the bracket map $\beta_k : H \otimes L_{k+1}(H) \to L_{k+2}(H)$, where $L_k(H)$ is the degree $k$ component of the free Lie algebra on $H = H_1(\Sigma_{g,1})$.

The second filtration $\mathcal{F}^Y_k(\mathcal{H}_g)$ is defined using the Goussarov-Habiro theory of finite-type invariants of 3-manifolds. It is shown in [1] that $\mathcal{F}^Y_k(\mathcal{H}_g) \subseteq \mathcal{F}^w_k(\mathcal{H}_g)$

and so we have induced homomorphisms $\mathcal{G}^Y_k(\mathcal{H}_g) \to \mathcal{G}^w_k(\mathcal{H}_g) \cong \mathsf{D}_k(H)$. To study these homomorphisms we use results *announced* by Habiro. Habiro considers an abelian group $\mathcal{A}_k(H)$ defined by unitrivalent graphs with $k$ trivalent vertices and with univalent vertices labelled by elements of $H$, subject to anti-symmetry, the IHX relation and linearity of labels (see [1] for a more complete description). We then consider the quotient $\mathcal{A}^t_k(H)$ in which only trees are allowed. Using results of Habiro it is proved in [1] that there is a well-defined *epimorphism* $\theta_k : \mathcal{A}^t_k(H) \to \mathcal{G}^Y_k(\mathcal{H}_g)$. Furthermore there is a combinatorially defined homomorphism $\eta_k : \mathcal{A}^t_k(H) \to \mathsf{D}_k(H)$ (which can be defined for *any* abelian group $H$) which coincides with the composition:

$$\mathcal{A}^t_k(H) \xrightarrow{\theta_k} \mathcal{G}^Y_k(\mathcal{H}_g) \longrightarrow \mathcal{G}^w_k(\mathcal{H}_g) \underset{\cong}{\xrightarrow{J_k(H)}} \mathsf{D}_k(H) \tag{1}$$

Note that this is different from the map called $\eta_k$ in [1].

It is erroneously claimed in Proposition 2.2 of [1] that $\eta_k$ is an isomorphism for $k > 1$. But in fact this is FALSE. Thus the implications that all the maps in diagram (1) are isomorphisms for $k > 1$ is false. (For $k = 1$ the result $\mathcal{G}^Y_1(\mathcal{H}_g) \cong \mathcal{G}^w_1(\mathcal{H}_g) \oplus V$, where the projection $\mathcal{G}^Y_1(\mathcal{H}_g) \to V$ is defined by Birman-Craggs homomorphisms, is still true.)

It is the aim of this note to correct this error and, in particular, study the homomorphism $\eta_k$. In fact it is known, and will be reproved below, that $\eta_k$ induces an isomorphism $\mathcal{A}^t_k(H) \otimes \mathbb{Q} \cong \mathsf{D}_k(H) \otimes \mathbb{Q}$. Thus the maps in diagram (1) are isomorphisms for $k > 1$ after tensoring with $\mathbb{Q}$. To handle the more general case it will be natural to introduce a variation on the notion of Lie algebra by replacing the axiom $[x, x] = 0$ with the weaker anti-symmetry axiom $[x, y] + [y, x] = 0$ and investigate the corresponding free objects. This variation does not seem to have been studied before, even though it arises naturally from the study of oriented graphs.

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## 2 A different notion of Lie algebra

We want to discuss the map $\eta_k : \mathcal{A}^t_k(H) \to \mathsf{D}_k(H)$, for $k > 1$. For this purpose it will be more appropriate to consider a replacement for the free Lie algebra $L(H)$. Let us define a *quasi-Lie algebra* by replacing the axiom $[x, x] = 0$ with the axiom $[x, y] + [y, x] = 0$, for any $x, y$. Thus we only can conclude $2[x, x] = 0$, and so if $L$ is quasi-Lie algebra then $L \otimes \mathbb{Z}[1/2]$ is a Lie algebra.

We can now define the free quasi-Lie algebra $L'(H)$ over a free abelian group $H$ in the obvious way (using the free *magma* over $H$, for example—see [2]). There is an obvious map $\gamma : L'(H) \to L(H)$, which is a map of quasi-Lie algebras. Let $\gamma_k : L'_k(H) \to L_k(H)$ be the degree $k$ component.

**Lemma 2.1** (1) *If $k$ is odd then $\gamma_k$ is an isomorphism.*

(2) *If $k = 2l$, then there is an exact sequence of additive homomorphisms*

$$L_l(H)/2L_l(H) \longrightarrow L'_k(H) \xrightarrow{\gamma_k} L_k(H) \to 0$$

**Proof** Clearly $\gamma_k$ is onto. Furthermore the kernel $K_k(H)$ of $\gamma_k$ is generated additively by all brackets which contain a sub-bracket of the form $[\alpha, \alpha]$ for some $\alpha \in L'(H)$. In fact such a bracket will be zero in $L'(H)$ unless it is exactly of the form $[\alpha, \alpha]$. In other words for any $\alpha, \eta \in L'(H)$

$$[[\alpha, \alpha], \eta] = 0 = [\eta, [\alpha, \alpha]]$$

This follows directly from the Jacobi relation and anti-symmetry.

Thus we can define a map $L'(H) \to L'(H)$ by $\alpha \longmapsto [\alpha, \alpha]$—it is an additive homomorphism by anti-symmetry— and the image of this map is exactly the kernel of $\gamma$. Note that this map vanishes on $2L'(H)$ and on any element of the form $\alpha = [\eta, \eta]$. The assertions of the lemma follow. □

**Conjecture 1** *It is easy to see that $L_l(H)/2L_l(H) \to L'_{2l}(H)$ is a monomorphism for $l = 1$ and it is reasonable to conjecture that this is true for all $l$.*

Analogous to $\beta_k$ we can define a homomorphism $\beta'_k : H \otimes L'_{k+1}(H) \to L'_{k+2}(H)$ by $\beta'_k(h \otimes \alpha) = [h, \alpha]$. We see that $\beta'_k$ is onto by the Jacobi identity and anti-symmetry and denote the kernel by $\mathsf{D}'_k(H)$. If we apply the snake lemma to the diagram:

$$\begin{array}{ccccccc} 0 \to \mathsf{D}'_k(H) & \longrightarrow & H \otimes L'_{k+1}(H) & \xrightarrow{\beta'_k} & L'_{k+2}(H) \to 0 \\ \downarrow & & \downarrow{\scriptstyle 1\otimes\gamma_{k+1}} & & \downarrow{\scriptstyle \gamma_{k+2}} \\ 0 \to \mathsf{D}_k(H) & \longrightarrow & H \otimes L_{k+1}(H) & \xrightarrow{\beta_k} & L_{k+2}(H) \to 0 \end{array}$$

we conclude:

**Corollary 2.2** *The canonical map $\mathsf{D}'_k(H) \to \mathsf{D}_k(H)$ fits into the following exact sequences, depending on whether $k$ is odd or even.*

$$0 \to \mathsf{D}'_{2l}(H) \to \mathsf{D}_{2l}(H) \to K_{2l+2}(H) \to 0$$

$$0 \to H \otimes K_{2l}(H) \to \mathsf{D}'_{2l-1}(H) \to \mathsf{D}_{2l-1}(H) \to 0$$

## 3 $\mathcal{A}_k^t(H)$ and Lie algebras

We will refer to a univalent vertex of a tree as a *leaf*, except when the tree is *rooted*, i.e. one of the univalent vertices is designated a root. In that case only the remaining univalent vertices will be referred to as leaves.

We can graphically interpret $L'_k(H)$ as the abelian group generated by rooted binary planar trees with $k$ leaves, whose leaves are labelled by elements of $H$ modulo the anti-symmetry and IHX relations and linearity of labels. These relations correspond exactly to the axioms for a quasi-Lie algebra. The correspondence is described in [2], for the case of a free magma. Similarly we can interpret $H \otimes L'_k(H)$ as rooted binary planar trees with $k$ leaves whose leaves *and root* are labelled by elements of $H$, modulo anti-symmetry, IHX and label linearity. See Figure 1.

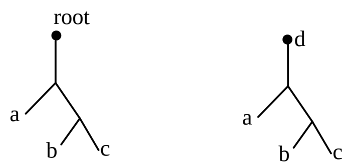


Figure 1: The left-hand tree corresponds to $[a,[b,c]]$ in $L'_3(H)$. The right-hand tree corresponds to $d \otimes [a,[b,c]]$ in $H \otimes L'_3(H)$.

We now define a map $\eta'_k : \mathcal{A}_k^t(H) \to H \otimes L'_{k+1}(H)$, in the same way that $\eta_k$ was defined, by sending each labelled binary planar tree to the sum of the rooted labelled binary planar trees, one for each leaf, obtained by designating that leaf as the (labelled) root. We want to show that $\operatorname{Im}\beta'_k = \mathsf{D}'_k(H)$, i.e. that the following sequence is exact.

$$\mathcal{A}_k^t(H) \xrightarrow{\eta'_k} H \otimes L'_{k+1}(H) \xrightarrow{\beta'_k} L'_{k+2}(H) \to 0$$

We first prove:

**Lemma 3.1** $\operatorname{Im}\eta'_k \subseteq \mathsf{D}'_k(H)$.

**Proof** Let $T$ be a labelled planar binary tree, representing $t \in \mathcal{A}_k^t(H)$. Then $\beta'_k \circ \eta'_k(t) \in L'_{k+2}(H)$ is represented by a sum $\sum_l T_l$ , over all leaves $l$ of $T$, where $T_l$ is the rooted tree obtained by adjoining to the edge of $T$ containing $l$ a rooted edge as in Figure 2. We need to show that this sum represents $0$.

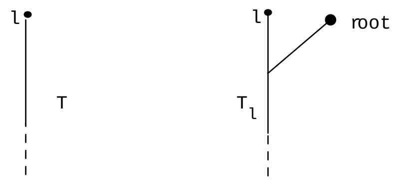


Figure 2: Define a rooted tree from a tree and one of its leaves

Consider now the sum $\sum_{(v,e)} T_{v,e}$, over all pairs $(v,e)$, where $v$ is a vertex of $T$ and $e$ an edge containing $v$. $T_{v,e}$ is the rooted tree obtained by adjoining to $e$, near $v$, a rooted edge as in Figure 3. The terms of this sum for univalent vertices $v$ is clearly just $\sum_l T_l$. The remaining terms correspond to the internal vertices and, for each internal vertex, there are three terms, whose sum will vanish by the IHX relation. Thus it suffices to prove that $\sum_{(v,e)} T_{v,e}$ represents $0$. But this is clear since, for each edge $e$, with vertices $v', v''$, we have $T_{v',e} = -T_{v'',e}$, by the anti-symmetry relation. □

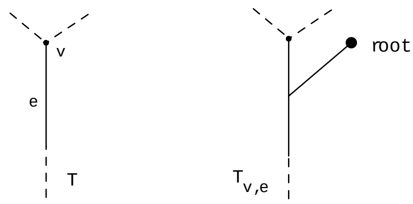


Figure 3: Define a rooted tree from a tree and an edge-vertex pair

**Theorem 1** *$\eta'_k : \mathcal{A}^t_k(H) \to \mathsf{D}'_k(H)$ is a split surjection. $\mathrm{Ker}\,\eta'_k$ is the torsion subgroup of $\mathcal{A}^t_k(H)$, if $k$ is even. It is the odd torsion subgroup if $k$ is odd. In either case*

$$(k+2)\,\mathrm{Ker}\,\eta'_k = 0.$$

**Corollary 3.2** $\mathcal{A}^t_k(H) \otimes \mathbb{Q} \cong \mathsf{D}'_k(H) \otimes \mathbb{Q} \cong \mathsf{D}_k(H) \otimes \mathbb{Q}$

**Conjecture 2** *It is reasonable to conjecture that $\eta'_k$ is an isomorphism.*

**Proof of Theorem 1** We will need some auxiliary maps. First we define

$$\rho_k : H \otimes L'_{k+1}(H) \to \mathcal{A}^t_k(H).$$

Let $\alpha$ be a generator of $H \otimes L'_{k+1}(H)$ represented by a rooted tree with labels on all leaves and root. Define $\rho_k(\alpha)$ to be the same labelled tree obtained by just forgetting which vertex is the root. This obviously preserves the anti-symmetry and IHX relations and label linearity, and so gives a well-defined additive homomorphism. The important property to observe is

$$\rho_k \circ \eta'_k = \text{multiplication by } k+2.$$

This shows that $(k+2)\,\mathrm{Ker}\,\eta'_k = 0$.

When $k$ is even, $\mathsf{D}'_k(H)$ is torsion-free, by Corollary 2.2. This shows that $\mathrm{Ker}\,\eta'_k$ is the torsion subgroup of $\mathcal{A}^t_k(H)$. If $k$ is odd, then Corollary 2.2 shows that all the torsion in $\mathsf{D}'_k(H)$ is of order 2, since $L(H)$ is torsion-free and $K_{2l}(H)$ is 2-torsion by Lemma 2.1. Since $k+2$ is odd, we conclude that $\mathrm{Ker}\,\eta'_k$ is the odd torsion subgroup of $\mathcal{A}^t_k(H)$. From this it follows that $\eta'_k$ splits.

It remains only to show that $\eta'_k$ is onto. For this we will construct a map

$$\tau_k : L'_{k+2}(H) \to H \otimes L'_{k+1}(H)/\eta'_k(\mathcal{A}^t_k(H)).$$

Consider a generator $\alpha$ of $L'_{k+2}(H)$ represented by a rooted tree $T$ with labelled leaves as in Figure 4. Here $v$ is the trivalent vertex adjacent to the root and $A, B$ the two subtrees (rooted, with labelled leaves) with $v$ as their common root. We then form a labelled tree $T'$ from $A$ and $B$ by eliminating the root of $T$ and making $v$ the midpoint of an edge connecting $A$ to $B$—see Figure 4

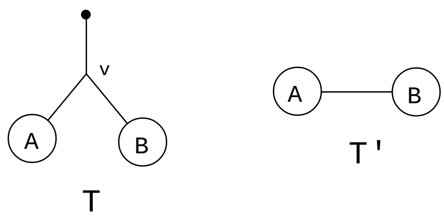


Figure 4

Now for each leaf $w$ of $T'$ we can create a rooted tree $T_w$ by making $w$ the root. Then $T_w$ represents an element of $H \otimes L'(H)$. Recall that we defined $\eta'_k(T')$ to be the sum $\Sigma_w T_w$ over all leaves of $T'$. We now define $\tau_k(\alpha)$ to be the class represented by the sum $\Sigma_w T_w$ over all leaves $w$ of $A$. We need to check that this is well-defined modulo $\eta'_k(\mathcal{A}^t_k(H))$.

If we consider an anti-symmetry relation in $T$ at a trivalent vertex in $A$ or $B$ then the image is clearly an anti-symmetry relation in every $T_w$. The anti-symmetry relation at the vertex $v$ is easily seen to map to precisely $\eta'_k(T')$.

Now consider an IHX relation at an internal edge $e$ of $T$. If $e$ is an internal edge of $A$ or $B$ then it induces an IHX relation in every $T_w$. Suppose, on the other hand that $e$ contains $v$. If the other vertex of $e$ is in $A$, for example, then we can represent the IHX relation as in Figure 5.

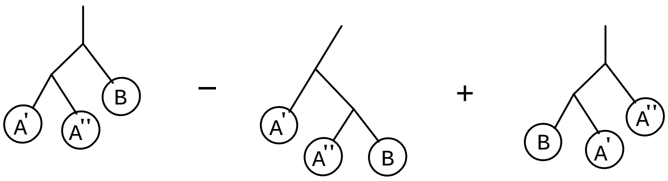


Figure 5: Graphical representation of the IHX relation

Here we have split $A$ into two subtree pieces $A'$ and $A''$. The image of this IHX relation is pictured in Figure 6, where we take the sum over all leaves $w$ in each subtree.

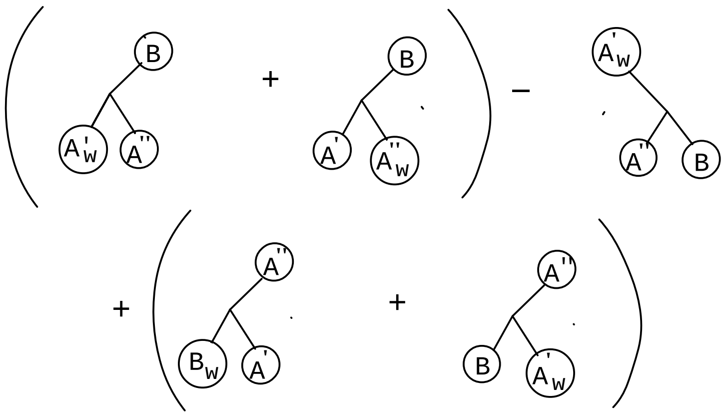


Figure 6: The image of the IHX relation in Figure 5

Then we can see that the first and third terms cancel while the second, fourth and fifth terms add up to exactly $\eta'_k(T')$.

Now it is easy to see that the composition

$$\tau_k \circ \beta'_k : H \otimes L'_{k+1}(H) \to H \otimes L'_{k+1}(H)/\eta'_k(\mathcal{A}^t_k(H))$$

is just the canonical projection. From this it follows immediately that $\operatorname{Ker} \beta'_k = \operatorname{Im} \eta'_k$. □

## 4 Relation between $\mathcal{G}_k^Y(\mathcal{H}_g)$ and $\mathcal{G}_k^w(\mathcal{H}_g)$

Finally we can draw some conclusions about the natural map $\mathcal{G}_k^Y(\mathcal{H}_g) \to \mathcal{G}_k^w(\mathcal{H}_g)$.

**Corollary 4.1** (1) *For all $k$*

$$\mathcal{A}_k^t(H) \otimes \mathbb{Q} \cong \mathcal{G}_k^Y(\mathcal{H}_g) \otimes \mathbb{Q} \cong \mathcal{G}_k^w(\mathcal{H}_g) \otimes \mathbb{Q}$$

(2) *For $k = 1$ we have $\mathcal{G}_1^Y(\mathcal{H}_g) \cong V \oplus \mathcal{G}_1^w(\mathcal{H}_g)$.*

(3) *If $k$ is even, there is an exact sequence*

$$\mathcal{G}_k^Y(\mathcal{H}_g) \to \mathcal{G}_k^w(\mathcal{H}_g) \to K_{k+2}(H) \to 0$$

(4) *If $k > 1$ is odd, then $\mathcal{G}_k^Y(\mathcal{H}_g) \to \mathcal{G}_k^w(\mathcal{H}_g)$ is onto and there is an exact sequence*

$$H \otimes K_{k+1}(H) \to \mathcal{G}_k^Y(\mathcal{H}_g) \otimes \mathbb{Z}_{(2)} \to \mathcal{G}_k^w(\mathcal{H}_g) \otimes \mathbb{Z}_{(2)} \to 0$$

*where $\mathbb{Z}_{(2)}$ is the ring of fractions with odd denominator.*

**Conjecture 3** *Taking account of the various conjectures mentioned above we can conjecture that the precise relationship between $\mathcal{G}_k^w(\mathcal{H}_g)$ and $\mathcal{G}_k^Y(\mathcal{H}_g)$, for $k > 1$ is given by the following exact sequences:*

$$0 \to \mathcal{G}_{2l}^Y(\mathcal{H}_g) \to \mathcal{G}_{2l}^w(\mathcal{H}_g) \to L_{l+1}(H)/2L_{l+1}(H) \to 0$$
$$H \otimes L_l(H)/2L_l(H) \to \mathcal{G}_{2l-1}^Y(\mathcal{H}_g) \to \mathcal{G}_{2l-1}^w(\mathcal{H}_g) \to 0 \quad (l > 1)$$

## References

[1] J. Levine, *Homology cylinders: an enlargement of the mapping class group*, Algebr. Geom. Topol. 1 (2001) 243–270, arXiv:math.GT/0010247

[2] C. Reutenauer, *Free Lie algebras*, Oxford University Press, 1993.

*Department of Mathematics, Brandeis University*
*Waltham, MA 02454-9110, USA*

Email: levine@brandeis.edu

URL: http://people.brandeis.edu/~levine/

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