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A note on the Lawrence–Krammer–Bigelow representation

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Abstract A very popular problem on braid groups has recently been solved by Bigelow and Krammer, namely, they have found a faithful linear representation for the braid group B_n. In their papers, Bigelow and Krammer suggested that their representation is the monodromy representation of a certain fibration. Our goal in this paper is to understand this monodromy representation using standard tools from the theory of hyperplane arrangements. In particular, we prove that the representation of Bigelow and Krammer is a sub-representation of the monodromy representation which we consider, but that it cannot be the whole representation.

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1 Introduction

Consider the ring $R = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of Laurent polynomials in two variables and the (abstract) free R-module:

$$V = \bigoplus_{1 \le i < j \le n} R\, e_{ij}$$

For $k \in \{1, \dots, n-1\}$ define the R-homomorphism $\sigma_k \colon V \to V$ by

$$\sigma_k(e_{ij}) = \begin{cases} x e_{i-1\,j} + (1-x) e_{ij} & \text{if } k = i-1 \\ e_{i+1\,j} - xy(x-1) e_{k\,k+1} & \text{if } k = i < j-1 \\ -x^2 y e_{k\,k+1} & \text{if } k = i = j-1 \\ e_{ij} - y(x-1)^2 e_{k\,k+1} & \text{if } i < k < j-1 \\ e_{i\,j-1} - xy(x-1) e_{k\,k+1} & \text{if } i < j-1 = k \\ x e_{i\,j+1} + (1-x) e_{ij} & \text{if } k = j \\ e_{ij} & \text{otherwise.} \end{cases}$$

The starting point of the present work is the following theorem due to Bigelow [1] and Krammer [5, 6].

Theorem 1.1 (Bigelow, [1]; Krammer, [5, 6]) *Let B_n be the braid group on n strings, and let $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators of B_n. Then the mapping $\sigma_k \mapsto \tau_k$ induces a well-defined faithful representation $\psi: B_n \to \mathrm{Aut}_R(V)$. In particular, the braid group B_n is linear.*

Let V be an R-module. A *representation* of B_n on V is a homomorphism $\psi: B_n \to \mathrm{Aut}_R(V)$. By abuse of notation, we may identify the underlying module V with the representation if no confusion is possible. Two representations ψ_1 and ψ_2 on V_1 and V_2, respectively, are called *equivalent* if there exist an automorphism $\alpha: R \to R$ and an isomorphism $f: V_1 \to V_2$ of abelian groups such that:

$f(\psi_1(b)v) = \psi_2(b)f(v)$ for all $b \in B_n$ and all $v \in V_1$;

$f(\lambda v) = \alpha(\lambda)f(v)$ for all $\lambda \in R$ and all $v \in V_1$.

An *LKB representation* is a representation of B_n equivalent to the one of Theorem 1.1 (LKB stands for Lawrence–Krammer–Bigelow).

Let $\mathbb{D}$ be a disc embedded in $\mathbb{C}$ such that $1, \dots, n$ lie in the interior of $\mathbb{D}$ (say $\mathbb{D} = \{z \in \mathbb{C} \mid |z - (n+1)/2| \le (n+1)/2\}$), and choose a basepoint P_b on the boundary of $\mathbb{D}$ (say $P_b = (n+1)(1-i)/2$). Define a *fork* to be a tree embedded in $\mathbb{D}$ with four vertices P_b, p, q, z and three edges, and such that $T \cap \partial\mathbb{D} = \{P_b\}$, $T \cap \{1, \dots, n\} = \{p, q\}$, and all three edges have z as vertex. The LKB representation V defined in [5] is the quotient of the free R-module generated by the isotopy classes of forks by certain relations. One can easily verify that these relations are invariant by the action of B_n, viewed as the mapping class group of $\mathbb{D} \setminus \{1, \dots, n\}$, thus V is naturally endowed with a B_n-action. Krammer in [5] stated that a monodromy representation of B_n on a twisted homology, $H_2(F_n; \Gamma)$, is an LKB representation and referred to Lawrence's paper [8] for the proof. The object of the present paper is the study of this monodromy representation on $H_2(F_n; \Gamma)$. Let $R \to \mathbb{C}$ be an embedding. The representation considered by Lawrence [7, 8] is isomorphic to $V \otimes \mathbb{C}$, but her geometric construction is slightly different from the construction suggested by Krammer [5]. In his proof of the linearity of braid groups, Bigelow [1] associated to each fork T an element $S(T)$ of $H_2(F_n; \Gamma)$, and used this correspondence to compute the action of B_n on $H_2(F_n; \Gamma)$. A consequence of his calculation is that $H_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$ is isomorphic to $V \otimes \mathbb{Q}(x, y)$.

In [3] and [2], Digne, Cohen and Wales introduced a new conceptual approach to the LKB representations based on the theory of root systems, and extended the results of [6] to all spherical type Artin groups. Using the same approach, linear representations have been defined for all Artin groups [9], but it is not known whether the resulting representations are faithful in the non-spherical case.

The formulae in our definition of the LKB representations are those of [3]; the formulae of [5], [1] and [7] can be obtained by a change of basis which will be given in Section 5. We choose this basis because it is the most natural basis in our construction and, as pointed out before, it has an interpretation in terms of root systems which can be extended to all Artin groups.

Our goal in this paper is to understand the monodromy action on $H_2(F_n;\Gamma)$ using standard tools from the theory of hyperplane arrangements, essentially the so-called Salvetti complexes. These tools are especially interesting in the sense that they are less specific to the case "braid groups" than the tools of Lawrence, Krammer and Bigelow, and we hope they will be used in the future for constructing linear representations of other groups like Artin groups. The main result of the paper is the following:

Theorem 1.2 *There is a sub-representation V of $H_2(F_n;\Gamma)$ such that:*

(i) *V is an LKB representation;*

(ii) *$V \neq H_2(F_n;\Gamma)$ if $n \geq 3$;*

(iii) *if V' is a sub-representation of $H_2(F_n;\Gamma)$ and V' is an LKB representation, then $V' \subset V$, if $n \geq 4$;*

(iv) *$V \otimes \mathbb{Q}(x,y) = H_2(F_n;\Gamma) \otimes \mathbb{Q}(x,y)$.*

We also prove that $H_2(F_n;\Gamma)$ is a free R-module of rank $n(n-1)/2$ and give a basis for $H_2(F_n;\Gamma)$. Note that (ii) and (iii) imply that $H_2(F_n;\Gamma)$ is not an LKB representation if $n \geq 4$. This fact is still true if $n = 3$ but, in this case, one has two minimal LKB representations in $H_2(F_n;\Gamma)$. The proof of this fact is left to the reader. Note also that the equality $V \otimes \mathbb{Q}(x,y) = H_2(F_n;\Gamma) \otimes \mathbb{Q}(x,y)$ is already known and can be found in [1].

We end this section with a detailed description of the monodromy representation $H_2(F_n;\Gamma)$.

For $1 \leq i < j \leq n$, let H_{ij} be the hyperplane of $\mathbb{C}^n$ with equation $z_i = z_j$, and let $M'_n = \mathbb{C}^n \setminus (\bigcup_{1 \leq i<j \leq n} H_{ij})$ denote the complement of these hyperplanes. The symmetric group Σ_n acts freely on M'_n and B_n is the fundamental group of $M'_n/\Sigma_n = M_n$. By [4], the map $p' \colon M'_{n+2} \to M'_n$ which sends $(z_1,\dots,z_{n+2})$ to $(z_1,\dots,z_n)$ is a locally trivial fibration. Let

$$L_{1t} = \{z \in \mathbb{C}^2 \mid z_1 = t\},\ L_{2t} = \{z \in \mathbb{C}^2 \mid z_2 = t\}, \quad t = 1,\dots,n,$$

$$L_3 = \{z \in \mathbb{C}^2 \mid z_1 = z_2\}.$$

The fibre of p' at $(1, \dots, n)$ is the complement of the above $2n+1$ complex lines:

$$F'_n = \mathbb{C}^2 \setminus \left(\bigcup_{t=1}^{n} L_{1t} \cup \bigcup_{t=1}^{n} L_{2t} \cup L_3\right)$$

Let $N_n = M'_{n+2}/(\Sigma_n \times \Sigma_2)$. Then $p' \colon M'_{n+2} \to M'_n$ induces a locally trivial fibration $p \colon N_n \to M_n$ whose fibre is $F_n = F'_n/\Sigma_2$.

Write $\|z\| = \max\{|z_i| \mid i = 1, \dots, n\}$ for $z \in \mathbb{C}^n$. The map $s' \colon M'_n \to M'_{n+2}$ given by

$$s'(z) = \begin{cases} (z, n+1, n+2) & \text{if } \|z\| \le n \\ (z, \frac{n+1}{n}\|z\|, \frac{n+2}{n}\|z\|) & \text{if } \|z\| \ge n \end{cases}$$

is a well-defined section of p' and moreover induces a section $s \colon M_n \to N_n$ of p. So, by the homotopy long exact sequence of p, the group $\pi_1(N_n)$ can be written as a semi-direct product $\pi_1(F_n) \rtimes B_n$.

To construct the monodromy representation we need the following two propositions whose proofs will be given in Sections 3 and 4, respectively.

Proposition 1.3 *$H_1(F_n)$ is a free $\mathbb{Z}$-module of rank $n+1$.*

In fact, we shall see that $H_1(F_n)$ has a natural basis $\{[a_1], \dots, [a_n], [c_1]\}$. Let H be the free abelian group freely generated by $\{x, y\}$, let $\alpha_0 \colon H_1(F_n) \to H$ be the homomorphism which sends $[a_i]$ to x for $i = 1, \dots, n$, and $[c_1]$ to y, and let $\alpha \colon \pi_1(F_n) \to H_1(F_n) \to H$ be the composition of the natural projection $\pi_1(F_n) \to H_1(F_n)$ with α_0.

Proposition 1.4

(i) *The kernel of α is invariant for the action of B_n. In particular, the action of B_n on $\pi_1(F_n)$ induces an action of B_n on H.*

(ii) *The action of B_n on H is trivial.*

Let $\tilde{F}_n \to F_n$ be the regular covering space associated to α. One has $\pi_1(\tilde{F}_n) = \ker \alpha$, H acts freely and discontinuously on $\tilde{F}_n$, and $\tilde{F}_n/H = F_n$. The action of H on $\tilde{F}_n$ endows $H_*(\tilde{F}_n)$ with a structure of $\mathbb{Z}[H]$-module. This homology group is called *the homology of F_n with local coefficients* associated to α, and is denoted by $H_*(F_n; \Gamma_\alpha)$.

Now, Proposition 1.4 implies that the fibration $p \colon N_n \to M_n$ induces a representation $\rho_* \colon \pi_1(M_n) = B_n \to \mathrm{Aut}_{\mathbb{Z}[H]}(H_*(F_n; \Gamma_\alpha))$, called *monodromy representation* on $H_*(F_n; \Gamma_\alpha)$. In this paper, we shall consider the monodromy representation $\rho \colon \pi_1(M_n) = B_n \to \mathrm{Aut}_{\mathbb{Z}[H]}(H_2(F_n; \Gamma_\alpha))$ which is the one referred to by Krammer and Bigelow.

2 The Salvetti complex

An *arrangement of lines* in $\mathbb{R}^2$ is a finite family $\mathcal{A}$ of affine lines in $\mathbb{R}^2$. The complexification of a line L is the complex line $L_{\mathbb{C}}$ in $\mathbb{C}^2$ with the same equation as L. The *complement of the complexification* of $\mathcal{A}$ is

$$M(\mathcal{A}) = \mathbb{C}^2 \setminus \Big(\bigcup_{L \in \mathcal{A}} L_{\mathbb{C}} \Big).$$

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{R}^2$. Then $\mathcal{A}$ subdivides $\mathbb{R}^2$ into *facets*. We denote by $\mathcal{F}(\mathcal{A})$ the set of facets and, for $h = 0, 1, 2$, we denote by $\mathcal{F}_h(\mathcal{A})$ the set of facets of dimension h. A *vertex* is a facet of dimension 0, an *edge* is a facet of dimension 1, and a *chamber* is a facet of dimension 2. We partially order $\mathcal{F}(\mathcal{A})$ with the relation $F < G$ if $F \subset \overline{G}$, where $\overline{G}$ denotes the closure of G.

We now define a CW-complex of dimension 2, called the *Salvetti complex* of $\mathcal{A}$, and denoted by $Sal(\mathcal{A})$. This complex has been introduced by Salvetti in [10] in the more general setting of hyperplane arrangements in $\mathbb{R}^n$, n being any positive integer, and Theorem 2.1, stated below for the case $n = 2$, is proved in [10] for any n.

To every chamber $C \in \mathcal{F}_2(\mathcal{A})$ we associate a vertex w_C of $Sal(\mathcal{A})$. The 0-skeleton of $Sal(\mathcal{A})$ is $Sal_0(\mathcal{A}) = \{w_C \mid C \in \mathcal{F}_2(\mathcal{A})\}$.

Let $F \in \mathcal{F}_1(\mathcal{A})$. There exist exactly two chambers $C, D \in \mathcal{F}_2(\mathcal{A})$ satisfying $C, D > F$. We associate to F two oriented 1-cells of $Sal(\mathcal{A})$: $a(F, C)$ and $a(F, D)$. The source of $a(F, C)$ is w_C and its target is w_D while the source of $a(F, D)$ is w_D and its target is w_C (see Figure 1). The 1-skeleton of $Sal(\mathcal{A})$ is the union of the $a(F, C)$'s, where $F \in \mathcal{F}_1(\mathcal{A})$, $C \in \mathcal{F}_2(\mathcal{A})$ and $F < C$.

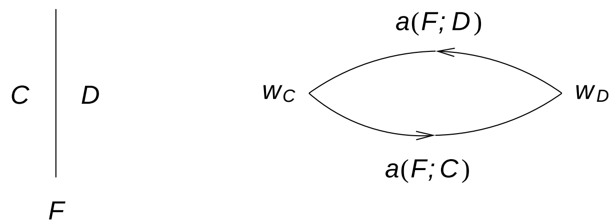

Figure 1: Edges in $Sal(\mathcal{A})$

Let $P \in \mathcal{F}_0(\mathcal{A})$ and let $\mathcal{F}_P(\mathcal{A})$ be the set of chambers $C \in \mathcal{F}_2(\mathcal{A})$ such that $P < C$. Fix some $C \in \mathcal{F}_P(\mathcal{A})$ and write $\mathcal{F}_P(\mathcal{A}) = \{C, C_1, \dots, C_{n-1}, D, D_{n-1}, \dots, D_1\}$ (see Figure 2). The set $\mathcal{F}_P(\mathcal{A})$ has a natural cyclic ordering induced by the orientation of $\mathbb{R}^2$, so we shall assume the list given above to be cyclically ordered in this way. Write $C = C_0 = D_0$ and $D = C_n = D_n$. For all $i = 1, \dots, n$, there is a unique edge a_i of $Sal_1(\mathcal{A})$ with source $w_{C_{i-1}}$ and target w_{C_i} and a unique edge b_i of $Sal_1(\mathcal{A})$ with source $w_{D_{i-1}}$ and target w_{D_i}. We associate to the pair (P, C) an oriented 2-cell $A(P, C)$ of $Sal(\mathcal{A})$ whose boundary is

$$\partial A(P, C) = a_1 a_2 \dots a_n b_n^{-1} \dots b_2^{-1} b_1^{-1}.$$

The 2-skeleton of $Sal(\mathcal{A})$ is the union of the $A(P, C)$'s, where $P \in \mathcal{F}_0(\mathcal{A})$ and $C \in \mathcal{F}_P(\mathcal{A})$.

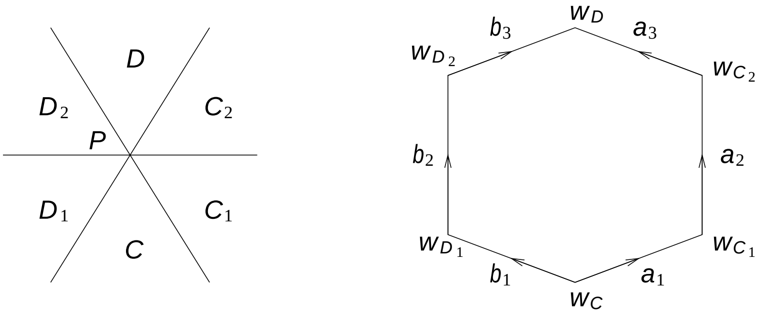

Figure 2: A 2-cell in $Sal(\mathcal{A})$

Theorem 2.1 (Salvetti, [10]) *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{R}^2$. There exists an embedding $: Sal(\mathcal{A}) \to M(\mathcal{A})$ which is a homotopy equivalence.*

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{R}^2$, and let G be a finite subgroup of Aff$(\mathbb{R}^2)$ which satisfies:

- $g(\mathcal{A}) = \mathcal{A}$ for all $g \in G$;
- G acts freely on $\mathbb{R}^2 \setminus (\bigcup_{L \in \mathcal{A}} L)$.

Then G acts freely on $Sal(\mathcal{A})$ and acts freely on $M(\mathcal{A})$, and the embedding $: Sal(\mathcal{A}) \to M(\mathcal{A})$ can be chosen to be equivariant with respect to these actions. Such an equivariant construction can be found in [11] for the particular case where G is a Coxeter group, and can be carried out in the same way for any group G which satisfies the above two conditions. So, $: Sal(\mathcal{A}) \to M(\mathcal{A})$ induces a homotopy equivalence $: Sal(\mathcal{A})/G \to M(\mathcal{A})/G$.

Recall now the spaces F_n and F_n' defined in Section 1. Let

$$L_{1t} = \{x \in \mathbb{R}^2 \mid x_1 = t\},\ L_{2t} = \{x \in \mathbb{R}^2 \mid x_2 = t\},\ t = 1, \dots, n,$$

$$L_3 = \{x \in \mathbb{R}^2 \mid x_1 = x_2\},$$

$$\mathcal{A}_n = \{L_{1\,1}, \dots, L_{1\,n}, L_{2\,1}, \dots, L_{2\,n}, L_3\}.$$

Then $F_n' = M(\mathcal{A}_n)$ and $F_n = M(\mathcal{A}_n)/\Sigma_2$. The action of Σ_2 on $\mathbb{R}^2$ satisfies:

$g(\mathcal{A}_n) = \mathcal{A}_n$ for all $g \in \Sigma_2$;

Σ_2 acts freely on $\mathbb{R}^2 \setminus (\bigcup_{L \in \mathcal{A}_n} L)$.

It follows that the embedding $\iota : Sal(\mathcal{A}_n) \to M(\mathcal{A}_n)$ induces a homotopy equivalence $\bar\iota : Sal(\mathcal{A}_n)/\Sigma_2 \to M(\mathcal{A}_n)/\Sigma_2 = F_n$.

We now define a new CW-complex, denoted by $Sal(F_n)$, obtained from the complex $Sal(\mathcal{A}_n)/\Sigma_2$ by collapsing cells, and having the same homotopy type as F_n. Most of our calculations in Sections 3 and 4 will be based on the description of this complex.

The complex $Sal(\mathcal{A}_n)/\Sigma_2$ can be formally described as follows (see Figure 3):

The set of vertices of $Sal(\mathcal{A}_n)/\Sigma_2$ is

$$\{P_{ij} \mid 1 \le i \le j \le n+1\}$$

The set of edges of $Sal(\mathcal{A}_n)/\Sigma_2$ is

$$\{c_i \mid 1 \le i \le n+1\} \cup \{a_{ij}, a_{ij} \mid 1 \le i \le j \le n\} \cup \{b_{ij}, b_{ij} \mid 1 \le i \le j \le n\}.$$

One has:

source(a_{ij}) = target(a_{ij}) = P_{ij} source(b_{ij}) = target(b_{ij}) = $P_{i+1\,j+1}$
source(a_{ij}) = target(a_{ij}) = $P_{i\,j+1}$ source(b_{ij}) = target(b_{ij}) = $P_{i\,j+1}$
source(c_i) = target(c_i) = P_{ii}

The set of 2-cells of $Sal(\mathcal{A}_n)/\Sigma_2$ is

$$\{A_{ijr} \mid 1 \le i < j \le n \text{ and } 1 \le r \le 4\} \cup \{B_{ir} \mid 1 \le i \le n \text{ and } 1 \le r \le 3\}.$$

One has:

$$\partial A_{ij\,1} = (b_{i\,j-1} a_{ij})(a_{i+1\,j} b_{ij})^{-1} \qquad \partial B_{i\,1} = (a_{ii} b_{ii} c_{i+1})(c_i a_{ii} b_{ii})^{-1}$$
$$\partial A_{ij\,2} = (a_{i+1\,j} b_{i\,j-1})(b_{ij} a_{ij})^{-1} \qquad \partial B_{i\,2} = (b_{ii} c_{i+1} b_{ii})(a_{ii} c_i a_{ii})^{-1}$$
$$\partial A_{ij\,3} = (a_{ij} b_{ij})(b_{i\,j-1} a_{i+1\,j})^{-1} \qquad \partial B_{i\,3} = (c_{i+1} b_{ii} a_{ii})(b_{ii} a_{ii} c_i)^{-1}$$
$$\partial A_{ij\,4} = (b_{ij} a_{i+1\,j})(a_{ij} b_{i\,j-1})^{-1}$$

Let K be the union of all the $A_{ij\,4}$'s. The set K is a subcomplex of $Sal(\mathcal{A}_n)/\Sigma_2$ which contains all the vertices and all the edges of $\{a_{ij}, b_{ij} \mid 1 \le i \le j \le n\}$, and which is homeomorphic to a disc. Collapsing K to a single point, we obtain a new CW-complex denoted by $Sal'(F_n)$. The complex $Sal'(F_n)$ has a unique vertex, its set of edges is

$$\{c_i \mid 1 \le i \le n+1\} \cup \{a_{ij}, b_{ij} \mid 1 \le i \le j \le n\},$$

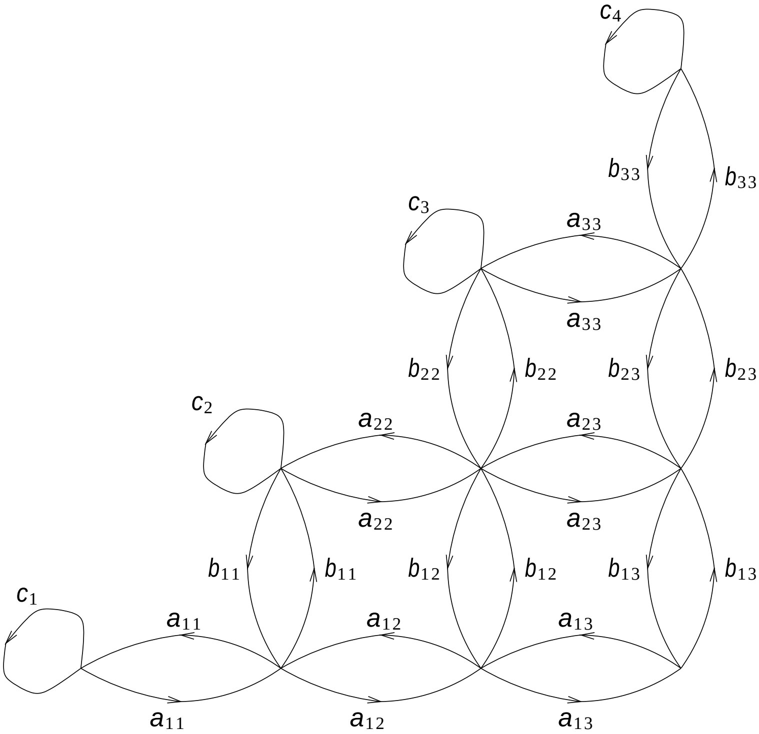

Figure 3: 1-skeleton of $Sal(A_3)/\mathbb{Z}_2$

and its set of 2-cells is

$$\{A_{ijr} \mid 1 \le i < j \le n \text{ and } 1 \le r \le 3\} \cup \{B_{ir} \mid 1 \le i \le n \text{ and } 1 \le r \le 3\}.$$

Note that, in $Sal^0(F_n)$, the cell $A_{ij\,2}$ is a bigon with boundary $\partial A_{ij\,2} = b_{i\,j-1}b_{ij}^{-1}$, and $A_{ij\,3}$ is a bigon with boundary $\partial A_{ij\,3} = a_{ij}a_{i+1\,j}^{-1}$. The complex $Sal(F_n)$ is obtained from $Sal^0(F_n)$ by collapsing all the $A_{ij\,2}$'s for $j = i+1, \dots, n$ to a single edge, $b_i = b_{ii}$, and by collapsing all the $A_{ij\,3}$'s for $i = 1, \dots, j-1$ to a single edge, $a_j = a_{jj}$. The complex $Sal(F_n)$ has a unique vertex, its set of edges is

$$\{c_i \mid 1 \le i \le n+1\} \cup \{a_i, b_i \mid 1 \le i \le n\},$$

and its set of 2-cells is

$$\{A_{ij} = A_{ij\,1} \mid 1 \le i < j \le n\} \cup \{B_{ir} \mid 1 \le i \le n \text{ and } 1 \le r \le 3\}.$$

One has:

$$\partial A_{ij} = (b_i a_j)(a_j b_i)^{-1} \qquad \partial B_{i\,2} = (c_{i+1} b_i)(c_i a_i)^{-1}$$
$$\partial B_{i\,1} = (a_i c_{i+1})(c_i a_i)^{-1} \qquad \partial B_{i\,3} = (c_{i+1} b_i)(b_i c_i)^{-1}$$

3 Computing the homology

For a loop γ in $Sal_1(F_n)$, we denote by $[\gamma]$ the element of $H_1(Sal(F_n))$ represented by γ. Now, standard methods in homology of CW-complexes immediately show:

Proposition 3.1 *$H_1(F_n) = H_1(Sal(F_n))$ is the free abelian group with basis $\{[a_1], \dots, [a_n], [c_1]\}$.*

Remark 3.2 *We also have the equalities:*

$$[c_i] = [c_1] \quad \text{for } 1 \le i \le n+1$$
$$[b_i] = [a_i] \quad \text{for } 1 \le i \le n$$

Recall that H denotes the free abelian group generated by $\{x, y\}$. Define $\psi_0 \colon H_1(F_n) \to H$ to be the homomorphism which sends $[a_i]$ to x for $i = 1, \dots, n$, and sends $[c_1]$ to y, and let $\psi \colon \pi_1(F_n) \to H_1(F_n) \to H$ be the composition.

In the following we shall describe a chain complex $C_*(F_n; \Gamma)$ whose homology is $H_*(F_n; \Gamma)$, define a family $\{E_{ij} \mid 1 \le i < j \le n\}$ in $H_2(F_n; \Gamma)$, and prove that $\{E_{ij} \mid 1 \le i < j \le n\}$ is a basis for $H_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$. The sub-module generated by this family will be the LKB representation V of the statement of 1.2. We shall end Section 3 by showing that $H_2(F_n; \Gamma)$ is a free $\mathbb{Z}[H]$-module.

For $h = 0, 1, 2$, let C_h be the set of h-cells in $Sal(F_n)$, and let $C_h(F_n; \Gamma)$ be the free $\mathbb{Z}[H]$-module with basis C_h. Define the differential $d \colon C_2(F_n; \Gamma) \to C_1(F_n; \Gamma)$ as follows. Let $D \in C_2$. Write $\partial D = \gamma_1^{\varepsilon_1} \cdots \gamma_l^{\varepsilon_l}$, where γ_i is an (oriented) 1-cell and $\varepsilon_i \in \{\pm 1\}$. Set

$$\psi^{(i)}(D) = \begin{cases} \psi(\gamma_1^{\varepsilon_1} \cdots \gamma_{i-1}^{\varepsilon_{i-1}}) & \text{if } \varepsilon_i = 1 \\ \psi(\gamma_1^{\varepsilon_1} \cdots \gamma_{i-1}^{\varepsilon_{i-1}} \gamma_i^{-1}) & \text{if } \varepsilon_i = -1. \end{cases}$$

Then

$$dD = \sum_{i=1}^{l} \varepsilon_i \, \psi^{(i)}(D) \, \gamma_i.$$

The following lemma is a straightforward consequence of this construction.

Lemma 3.3

(i) $\ker d = H_2(F_n; \Gamma)$.

(ii) *Let $d_{\mathbb{Q}} = d \otimes \mathbb{Q}(x, y) \colon C_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y) \to C_1(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$. Then* $\ker d_{\mathbb{Q}} = H_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$.

It is easy to obtain the following formulae:

$$\begin{aligned} dA_{ij} &= (x-1)(a_j - b_i) \\ dB_{i\,1} &= (1-y)a_i - c_i + xc_{i+1} \\ dB_{i\,2} &= -ya_i + yb_i - c_i + c_{i+1} \\ dB_{i\,3} &= (y-1)b_i - xc_i + c_{i+1} \end{aligned}$$

Now, we define the family $\{E_{ij} \mid 1 \le i < j \le n\}$. For $1 \le i \le n$ set:

$$\begin{aligned} V_{i\,b} &= -xyB_{i\,1} + x(y-1)B_{i\,2} + B_{i\,3} \\ V_{i\,a} &= B_{i\,1} + x(y-1)B_{i\,2} - xyB_{i\,3} \\ V_{i\,0} &= -yB_{i\,1} + (y-1)B_{i\,2} - yB_{i\,3} \end{aligned}$$

For $1 \le i < j \le n$ set

$$E_{ij} = (y-1)(xy+1)A_{ij} + (x-1)V_{i\,b} + (x-1)V_{j\,a} + \sum_{k=i+1}^{j-1} (x-1)^2 V_{k\,0}.$$

The chains $V_{i\,b}$, $V_{i\,a}$ and $V_{i\,0}$ have been found with algebraic manipulations. Their interest lies in the fact that the support of each of them is $\{B_{i\,1}, B_{i\,2}, B_{i\,3}\}$, the boundary of $V_{i\,b}$ is a multiple of c_{i+1} minus a multiple of b_i, the boundary of $V_{i\,a}$ is a multiple of c_i minus a multiple of a_i, and the boundary of $V_{i\,0}$ is a multiple of $c_i - c_{i+1}$. More precisely, one has:

$$\begin{aligned} dV_{i\,b} &= (y-1)(xy+1)b_i - (x-1)(xy+1)c_{i+1} \\ dV_{i\,a} &= -(y-1)(xy+1)a_i + (x-1)(xy+1)c_i \\ dV_{i\,0} &= (xy+1)(c_i - c_{i+1}) \end{aligned}$$

Another fact which will be of importance in our calculations is that all the A_{ls}-coordinates of E_{ij} are zero except the A_{ij}-one.

Proposition 3.4 *The set $\{E_{ij} \mid 1 \le i < j \le n\}$ is a basis for* $\ker d_{\mathbb{Q}} = H_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$.

Proof It is easy to see that $dE_{ij} = 0$ for all $1 \le i < j \le n$. Moreover, since E_{ij} is the only element of $\{E_{ls} \mid 1 \le l < s \le n\}$ such that the A_{ij}-coordinate is nonzero, the set $\{E_{ij} \mid \le i < j \le n\}$ is linearly independent.

So, to prove Proposition 3.4, it suffices to show that $\dim(\ker d_{\mathbb{Q}}) \le n(n-1)/2$. To do so, we exhibit a linear subspace W of $C_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$ of codimension $n(n-1)/2$ and prove that $d_{\mathbb{Q}}|_W$ is injective.

Let $B = \{B_{ir} \mid 1 \leq i \leq n \text{ and } 1 \leq r \leq 3\}$, and let W be the linear subspace of $C_2(F_n;\Gamma) \otimes \mathbb{Q}(x,y)$ generated by B. The codimension of W is clearly $n(n-1)/2$. Let $\psi : C_1(F_n;\Gamma) \otimes \mathbb{Q}(x,y) \to W$ be the linear map defined by

$$\psi(a_i) = -(xy-y+1)B_{i\,1} - (y-1)B_{i\,2} + yB_{i\,3}$$
$$\psi(b_i) = -xyB_{i\,1} + x(y-1)B_{i\,2} + B_{i\,3}$$
$$\psi(c_i) = \begin{cases} -y(y-1)B_{i\,1} + (y-1)^2 B_{i\,2} - y(y-1)B_{i\,3} & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i = n+1 \end{cases}$$

Choose a linear ordering of B which satisfies $B_{ir} > B_{i+1\,s}$ for $1 \leq r,s \leq 3$. A straightforward calculation shows that the matrix of $(\psi \circ d_{\mathbb{Q}}|_W)$ with respect to the ordered basis B is a triangular matrix with nonzero entries on the diagonal, thus $(\psi \circ d_{\mathbb{Q}}|_W)$ is invertible and, therefore, $d_{\mathbb{Q}}|_W$ is injective. □

Remark 3.5 Let $U \in H_2(F_n;\Gamma) \otimes \mathbb{Q}(x,y)$. As pointed out before, E_{ij} is the only element of $\{E_{ls} \mid 1 \leq l < s \leq n\}$ such that the A_{ij}-coordinate is nonzero. So, if λ_{ij} is the E_{ij}-coordinate of U, then $\lambda_{ij}(y-1)(xy+1)$ is the A_{ij}-coordinate of U.

Proposition 3.6 *$H_2(F_n;\Gamma)$ is a free $\mathbb{Z}[H]$-module of rank $n(n-1)/2$.*

Proof Let

$$X_{ij} = \begin{cases} E_{ij} & \text{if } j = i+1 \\ (E_{1\,2} + E_{2\,3} - E_{1\,3})/(y-1) & \text{if } i = 1 \text{ and } j = 3 \\ (xyE_{i-1\,i} + (x-1)yE_{i\,i+1} \\ \quad -E_{i+1\,i+2} - xyE_{i-1\,i+1} \\ \quad +E_{i\,i+2})/(y-1)(xy+1) & \text{if } i \geq 2 \text{ and } j = i+2 \\ (E_{i+1\,j-1} - E_{i\,j-1} - E_{i+1\,j} \\ \quad +E_{ij})/(y-1)(xy+1) & \text{if } j > i+2 \end{cases}$$

and let $X = \{X_{ij} \mid 1 \leq i < j \leq n\}$. We shall prove that X is a $\mathbb{Z}[H]$-basis for $H_2(F_n;\Gamma)$.

Since X_{ij} is a linear combination (with coefficients in $\mathbb{Q}(x,y)$) of $\{E_{ls} \mid 1 \leq l < s \leq n\}$, one has $d_{\mathbb{Q}}X_{ij} = 0$. Moreover, one can easily verify

$$X_{ij} = \begin{cases} (xy+1)A_{1\,2} + (xy+1)A_{2\,3} - (xy+1)A_{1\,3} \\ \quad -(x-1)B_{2\,1} + (x^2-1)B_{2\,2} - (x-1)B_{2\,3} & \text{if } i = 1 \text{ and } j = 3 \\ xyA_{i-1\,i} + y(x-1)A_{i\,i+1} - A_{i+1\,i+2} \\ \quad -xyA_{i-1\,i+1} + A_{i\,i+2} \\ \quad +x(x-1)B_{i\,2} - (x-1)B_{i\,3} \\ \quad -(x-1)B_{i+1\,2} + (x-1)B_{i+1\,3} & \text{if } i \geq 2 \text{ and } j = i+2 \\ A_{ij} - A_{i\,j-1} - A_{i+1\,j} + A_{i+1\,j-1} & \text{if } j > i+2, \end{cases}$$

thus $X_{ij} \in C_2(F_n;\Gamma)$. So, $X_{ij} \in H_2(F_n;\Gamma)$.

Let $<$ be the linear ordering on $\{A_{ij} \mid 1 \le i < j \le n\}$ defined by $A_{ij} < A_{ls}$ if either $j-i < s-l$, or $j-i = s-l$ and $i < l$. The A_{ij}-coordinate of X_{ij} is nonzero and, for $A_{ls} > A_{ij}$, the A_{ls}-coordinate of X_{ij} is zero, thus X is linearly independent.

It remains to show that any element of $H_2(F_n;\Gamma)$ can be written as a linear combination of X with coefficients in $\mathbb{Z}[H]$. Suppose that there exists $U \in H_2(F_n;\Gamma)$ which cannot be written as a linear combination of X with coefficients in $\mathbb{Z}[H]$. Write

$$U = \sum \lambda_{ij} A_{ij} + \sum \mu_{ir} B_{ir}$$

where $\lambda_{ij}, \mu_{ir} \in \mathbb{Z}[H]$. By Remark 3.5, we have

$$U = \sum \frac{\lambda_{ij}}{(y-1)(xy+1)} E_{ij}.$$

Moreover, $U \neq 0$. Let A_{ij} be such that $\lambda_{ij} \neq 0$ and $\lambda_{ls} = 0$ for $A_{ls} > A_{ij}$. We choose U so that A_{ij} is minimal (with respect to the ordering defined above).

Suppose $j > i+2$. The A_{ij}-coordinate of X_{ij} is 1 and, for $A_{ls} > A_{ij}$, the A_{ls}-coordinate of X_{ij} is 0, thus $U - \lambda_{ij} X_{ij}$ would contradict the minimality of A_{ij}.

Suppose $i \ge 2$ and $j = i+2$. Again, the A_{ij}-coordinate of X_{ij} is 1 and, for $A_{ls} > A_{ij}$, the A_{ls}-coordinate of X_{ij} is 0, thus $U - \lambda_{ij} X_{ij}$ would contradict the minimality of A_{ij}.

Suppose $i = 1$ and $j = 3$. Recall the equality $U = \sum \lambda_{ls}/((y-1)(xy+1)) E_{ls}$. The B_{n1}-coordinate of U is $\mu_{n1} = (x-1)\lambda_{n-1\,n}/((y-1)(xy+1))$, thus $xy+1$ divides $\lambda_{n-1\,n}$. For $k = 4, \dots, n-1$, the B_{k1}-coordinate of U is $\mu_{k1} = (x-1)(\lambda_{k-1\,k} - xy\lambda_{k\,k+1})/((y-1)(xy+1))$. It succesively follows, for $k = n-1, n-2, \dots, 4$, that $xy+1$ divides $\lambda_{k-1\,k}$. The $B_{3,1}$-coordinate, $B_{2,1}$-coordinate, and $B_{1,1}$-coordinate of U are respectively:

$$\mu_{31} = (x-1)(\lambda_{23} + \lambda_{13} - xy\lambda_{34})/((y-1)(xy+1))$$
$$\mu_{21} = (x-1)(\lambda_{12} + y\lambda_{13} - xy\lambda_{13} - xy\lambda_{23})/((y-1)(xy+1))$$
$$\mu_{11} = -xy(x-1)(\lambda_{12} + \lambda_{13})/((y-1)(xy+1))$$

Thus

$$\lambda_{23} + \lambda_{13} \equiv 0 \pmod{xy+1}$$
$$\lambda_{12} + \lambda_{23} + (y+1)\lambda_{13} \equiv 0 \pmod{xy+1}$$
$$\lambda_{12} + \lambda_{13} \equiv 0 \pmod{xy+1}.$$

Hence $xy+1$ divides $\ _{13}$. Let $\ '_{13} \in \mathbb{Z}[H]$ be such that $\ _{13} = (xy+1)\ '_{13}$. The A_{13}-coordinate of X_{13} is $-(xy+1)$ and, for $A_{ls} > A_{13}$, the A_{ls}-coordinate of X_{13} is 0, thus $U + \ '_{13}X_{13}$ would contradict the minimality of $A_{ij} = A_{13}$.

Suppose $j = i+1$. The $B_{i+1\,1}$-coordinate of U is $\ _{i+1\,1} = (x-1)\ _{i\,i+1}/((y-1)(xy+1))$, thus $(y-1)(xy+1)$ divides $\ _{i\,i+1}$. Let $\ '_{i\,i+1} \in \mathbb{Z}[H]$ such that $\ _{i\,i+1} = (y-1)(xy+1)\ '_{i\,i+1}$. The $A_{i\,i+1}$-coordinate of $X_{i\,i+1}$ is $(y-1)(xy+1)$ and, for $A_{ls} > A_{i\,i+1}$, the A_{ls}-coordinate of $X_{i\,i+1}$ is 0, thus $U - \ '_{i\,i+1}X_{i\,i+1}$ would contradict the minimality of $A_{ij} = A_{i\,i+1}$. □

4 Computing the action

We shall see in the next section how to interpret the "forks" of Krammer and Bigelow in our terminology, and, from this interpretation, how to use Bigelow's calculations [1, Sec.4] to recover the action of B_n on $H_2(F_n;\Gamma)$. In this section, we shall apply our techniques for calculating the action of B_n on $H_2(F_n;\Gamma)$. Since most of the results of the section are well-known, some technical details will be left to the reader.

Let $k \in \{1,\dots,n-1\}$. Choose some small $\varepsilon > 0$ (say $\varepsilon < 1/4$) and an embedding $V\colon S^1 \ [0,1] \to \mathbb{C}$ which satisfies:

$$\mathrm{im}V = \{z \in \mathbb{C}\,; 1/2 - \varepsilon \ \ |z-k-1/2| \ \ 1/2+\varepsilon\};$$

$$V(\ ;1/2) = k + 1/2 + \ /2, \text{ for all } \ \in S^1.$$

Consider the *Dehn twist* $T_k^0\colon \mathbb{C} \to \mathbb{C}$ defined by

$$(T_k^0 \ \ V)(\ ;t) = V(e^{-2i\ \ t}\ ;t)$$

for all $(\ ;t) \in S^1 \ [0,1]$, and T_k^0 is the identity outside the image of V. Note that T_k^0 interchanges k and $k+1$ and fixes the other points of $\{1,\dots,n\}$. Consider now the diagonal homeomorphism $(T_k^0 \ T_k^0)\colon \mathbb{C}^2 \to \mathbb{C}^2$. One has $(T_k^0 \ T_k^0)(F_n^0) = F_n^0$, and $(T_k^0 \ T_k^0)$ commutes with the action of $\ _2$ (namely, $(T_k^0 \ T_k^0) \ \ g = g \ \ (T_k^0 \ T_k^0)$ for all $g \in \ _2$), thus $(T_k^0 \ T_k^0)$ induces a homeomorphism $T_k\colon F_n^0/\ _2 = F_n \to F_n$. Recall that $\ _1,\dots,\ _{n-1}$ denote the standard generators of the braid group B_n. Then T_k represents $\ _k$, namely $\ (T_k) = \ _k\colon \ _1(F_n) \to \ _1(F_n)$.

We assume that $P_b = (n+1,0)$ is the basepoint of F_n, we denote by the unique vertex of $Sal(F_n)$, and choose a homotopy equivalence $\ \colon Sal(F_n) \to F_n$ which sends to P_b.

Let $\ \ _h$ denote two loops in F_n based at P_b that are homotopic.

Lemma 4.1 *Let* $k \in \{1, \dots, n-1\}$. *Then*

$$T_k(\ (a_i)) \ _h \begin{cases} (a_{i-1}) & \text{if } k = i-1 \\ (a_i a_{i+1} a_i^{-1}) & \text{if } k = i \\ (a_i) & \text{otherwise} \end{cases}$$

$$T_k(\ (b_i)) \ _h \begin{cases} (b_i b_{i-1} b_i^{-1}) & \text{if } k = i-1 \\ (b_{i+1}) & \text{if } k = i \\ (b_i) & \text{otherwise} \end{cases}$$

$$T_k(\ (c_i)) \ _h \begin{cases} (a_{i-1} b_i c_i b_i^{-1} a_{i-1}^{-1}) & \text{if } k = i-1 \\ (c_i) & \text{otherwise.} \end{cases}$$

Proof The homotopy relations for (a_i) follow from the fact that (a_i) can be drawn in the plane $\mathbb{C} \ \{0\}$ as shown in Figure 4, $\mathbb{C} \ \{0\}$ is invariant by T_k, and T_k acts on $\mathbb{C} \ \{0\}$ as the Dehn twist T_k^0. The other homotopy relations can be proved in the same way. □

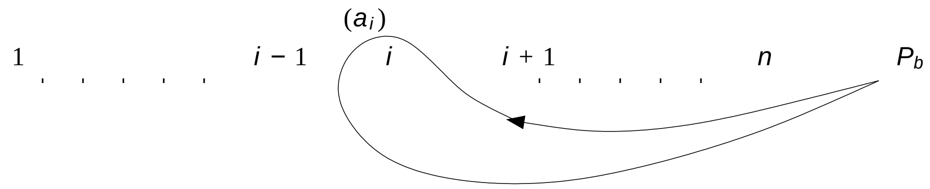

Figure 4: The curve (a_i) in $\mathbb{C} \ \{0\}$

A straightforward consequence of Lemma 4.1 is:

Corollary 4.2 *The action of* B_n *on* $H_1(F_n)$ *is given by:*

$$_k([a_i]) = \begin{cases} [a_{i-1}] & \text{if } k = i-1 \\ [a_{i+1}] & \text{if } k = i \\ [a_i] & \text{otherwise} \end{cases}$$

$$_k([c_i]) = [c_i]$$

We now consider the homomorphism $\ : \ _1(F_n) \to H$ defined in Section 3.

Corollary 4.3

(i) ker is invariant by the action of B_n. *In particular, the action of* B_n *on* $_1(F_n)$ *induces an action of* B_n *on* H.

(ii) *The action of* B_n *on* H *is trivial.*

So, as pointed out in Section 1, this implies:

Corollary 4.4 *The locally trivial fibration $p\colon N_n \to M_n$ induces a representation $\rho\colon B_n = \pi_1(M_n) \to \mathrm{Aut}_{\mathbb{Z}[H]}(H(F_n;\Gamma))$.*

We turn now to compute the action of B_n on $H_2(F_n;\Gamma)$.

Let $k \in \{1,\dots,n-1\}$. Define the map $S_k\colon Sal_1(F_n) \to Sal(F_n)$ by:

$$S_k(\epsilon) = \epsilon$$

$$S_k(a_i) = \begin{cases} a_{i-1} & \text{if } k = i-1 \\ a_i a_{i+1} a_i^{-1} & \text{if } k = i \\ a_i & \text{otherwise} \end{cases}$$

$$S_k(b_i) = \begin{cases} b_i b_{i-1} b_i^{-1} & \text{if } k = i-1 \\ b_{i+1} & \text{if } k = i \\ b_i & \text{otherwise} \end{cases}$$

$$S_k(c_i) = \begin{cases} a_{i-1} b_i c_i b_i^{-1} a_{i-1}^{-1} & \text{if } k = i-1 \\ c_i & \text{otherwise} \end{cases}$$

By Lemma 4.1, S_k induces a homomorphism

$$(S_k)_*\colon \pi_1(Sal(F_n)) \to \pi_1(Sal(F_n))$$

which is equal to σ_k. Moreover, by [4], $Sal(F_n)$ is aspherical, thus S_k extends to a map $S_k\colon Sal(F_n) \to Sal(F_n)$ which is unique up to homotopy.

Let K and K' be two CW-complexes. Call a map $f\colon K \to K'$ a *combinatorial map* if:

- the image of any cell C of K is a cell of K';
- if $\dim C = \dim f(C)$, then $f|_C\colon C \to f(C)$ is a homeomorphism.

We can, and will, suppose that every cell D of $Sal(F_n)$ is endowed with a cellular decomposition such that $S_k|_D\colon D \to Sal(F_n)$ is a combinatorial map. Under this assumption, the map S_k determines a $\mathbb{Z}[H]$-homomorphism $(S_k)_*\colon C_2(F_n;\Gamma) \to C_2(F_n;\Gamma)$ as follows.

Let $D \in \mathcal{C}_2$ be a 2-cell of $Sal(F_n)$. Recall that D is endowed with the cellular decomposition such that $S_k|_D\colon D \to Sal(F_n)$ is a combinatorial map. Let $\mathcal{C}_2^0(D)$ denote the set of 2-cells R in D such that $S_k(R)$ is a 2-cell of $Sal(F_n)$.

Let $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$ be the standard disc. In the definition of the differential $d\colon C_2(F_n;\Gamma) \to C_1(F_n;\Gamma)$, for a given 2-cell $D \in \mathcal{C}_2$, the expression $\partial D = \alpha_1^{\varepsilon_1} \cdots \alpha_l^{\varepsilon_l}$ means that D is endowed with a cellular map $\phi_D\colon \mathbb{D}^2 \to D$ such that the restriction of ϕ_D to the interior of $\mathbb{D}^2$ is a homeomorphism onto the interior of D, and $\partial\sigma_D = \alpha_1^{\varepsilon_1}\dots\alpha_l^{\varepsilon_l}$, where $\partial\sigma_D\colon [0,1] \to D$ is defined by $\partial\sigma_D(t) = \sigma_D(e^{2i\pi t})$. Now, every 2-cell $R \in C_2^0(D)$ is also endowed with a cellular map $\sigma_R\colon \mathbb{D}^2 \to R$ defined by $S_k \circ \sigma_R = \sigma_{S_k(R)}$. For $R \in C_2^0(D)$, we set $Q_R = \sigma_R(1)$. This point should be understood as the starting point of the reading of the boundary of R.

For every $R \in C_2^0(D)$ we set $\varepsilon(R) = 1$ if $S_k\colon R \to S_k(R)$ preserves the orientation and $\varepsilon(R) = -1$ otherwise, and we choose a path γ_R from Q_D to Q_R in the 1-skeleton of D. Then

$$(S_k)_*(D) = \sum_{R \in C_2^0(D)} \varepsilon(R)(\rho \circ S_k)(\gamma_R)\, S_k(R)$$

The sub-module $\ker d = H_2(F_n;\Gamma)$ is invariant by $(S_k)_*$ and the restriction of $(S_k)_*$ to $H_2(F_n;\Gamma)$ is equal to the action of τ_k on $H_2(F_n;\Gamma)$.

Lemma 4.5 *One can choose S_k such that:*

$$(S_k)_*(A_{ij}) = \begin{cases} (1-x)A_{ij} + xA_{i-1\,j} & \text{if } k = i-1 \\ A_{i+1\,j} & \text{if } k = i < j-1 \\ U_i & \text{if } k = i = j-1 \\ A_{i\,j-1} & \text{if } i < j-1 = k \\ (1-x)A_{ij} + xA_{i\,j+1} & \text{if } k = j \\ A_{ij} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i1}) = \begin{cases} xB_{i3} & \text{if } k = i-1 \\ B_{i1} + xB_{i+1\,1} - x^2 B_{i+1\,3} & \text{if } k = i \\ B_{i1} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i2}) = \begin{cases} U_{i-1} + B_{i3} - xB_{i-1\,1} + xB_{i-1\,2} & \text{if } k = i-1 \\ B_{i1} + xB_{i+1\,2} - xB_{i+1\,3} + yU_i & \text{if } k = i \\ B_{i2} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i3}) = \begin{cases} B_{i3} + xB_{i-1\,3} - x^2 B_{i-1\,1} - x(y-1)U_{i-1} & \text{if } k = i-1 \\ (y-1)U_i + xB_{i1} & \text{if } k = i \\ B_{i3} & \text{otherwise} \end{cases}$$

where

$$U_i = (x-1)(B_{i1} - B_{i2} - B_{i+1\,2} + B_{i+1\,3}) - yA_{i\,i+1}.$$

Proof The method for constructing the extension of $S_k\colon Sal_1(F_n) \to Sal(F_n)$ is as follows. For every $D \in C_2$, we compute $S_k(\partial D)$, and, from this result, we construct a cellular decomposition of D and a combinatorial map $S_{k,D}\colon D \to Sal(F_n)$ which extends the restriction of S_k to ∂D. This can be done case by case without any difficulty. The maps $S_{k,D}$, $D \in C_2$, determine the required extension $S_k\colon Sal(F_n) \to Sal(F_n)$. With this construction, it is easy to compute $(S_k)_*(D)$ from the definition given above. ☐

Now, from Lemma 4.5 , one can easily compute the action of B_n on $H_2(F_n;\Gamma)$ and obtain the following formulae.

Proposition 4.6 *Let $k \in \{1,\dots,n-1\}$ and $1 \le i < j \le n$. Then*

$$\sigma_k(E_{ij}) = \begin{cases} xE_{i-1\,j} + (1-x)E_{ij} & \text{if } k = i-1 \\ E_{i+1\,j} - xy(x-1)E_{k\,k+1} & \text{if } k = i < j-1 \\ -x^2yE_{k\,k+1} & \text{if } k = i = j-1 \\ E_{ij} - y(x-1)^2E_{k\,k+1} & \text{if } i < k < j-1 \\ E_{i\,j-1} - xy(x-1)E_{k\,k+1} & \text{if } i < j-1 = k \\ xE_{i\,j+1} + (1-x)E_{ij} & \text{if } k = j \\ E_{ij} & \text{otherwise.} \end{cases}$$

Now we can prove our main theorem.

Proof of Theorem 1.2 Let V be the $\mathbb{Z}[H]$-submodule generated by $\{E_{ij} \mid 1 \le i < j \le n\}$.

Proof of (i) By Proposition 3.4, V is a free $\mathbb{Z}[H]$-module with basis $\{E_{ij} \mid 1 \le i < j \le n\}$, and, by Proposition 4.6, V is a sub-representation of $H_2(F_n;\Gamma)$, and is an LKB representation.

Proof of (ii) The element X_{13} of the proof of Proposition 3.6 lies in $H_2(F_n;\Gamma)$ but does not lie in V.

Proof of (iii) Let V' be a sub-representation of $H_2(F_n;\Gamma)$ such that V' is an LKB representation. By definition, V' is a free $\mathbb{Z}[H]$-module, and there exist a basis $\{E'_{ij} \mid 1 \le i < j \le n\}$ for V' and an automorphism $\varphi\colon \mathbb{Z}[H] \to \mathbb{Z}[H]$ such that

$$\sigma_k(E'_{ij}) = \begin{cases} \varphi(x)E'_{i-1\,j} + \varphi(1-x)E'_{ij} & \text{if } k = i-1 \\ E'_{i+1\,j} - \varphi(xy(x-1))E'_{k\,k+1} & \text{if } k = i < j-1 \\ -\varphi(x^2y)E'_{k\,k+1} & \text{if } k = i = j-1 \\ E'_{ij} - \varphi(y(x-1)^2)E'_{k\,k+1} & \text{if } i < k < j-1 \\ E'_{i\,j-1} - \varphi(xy(x-1))E'_{k\,k+1} & \text{if } i < j-1 = k \\ \varphi(x)E'_{i\,j+1} + \varphi(1-x)E'_{ij} & \text{if } k = j \\ E'_{ij} & \text{otherwise.} \end{cases}$$

Note that V' is generated as a $\mathbb{Z}[H]$-module by the B_n-orbit of E'_{12}, thus, in order to prove that $V' \subset V$, it suffices to show that $E'_{12} \in V$.

For $j = 3, \dots, n$, let

$$F_j = (xy - 1)E_{1j} - (xy - 1)E_{2j} + y(1 - x)E_{12}$$
$$G_j = x(x^2y + 1)E_{1j} + (x^2y + 1)E_{2j} + x^2y(1 - x)E_{12}.$$

It is easy to see that:

$$
\begin{array}{ll}
{}_1(E_{12}) = -x^2yE_{12} & \\
{}_1(F_j) = -xF_j & \text{for } 3 \le j \le n \\
{}_1(G_j) = G_j & \text{for } 3 \le j \le n \\
{}_1(E_{ij}) = E_{ij} & \text{for } 3 \le i < j \le n
\end{array}
$$

The set $\{E_{12}\} \cup \{F_j, G_j \mid 3 \le j \le n\} \cup \{E_{ij} \mid 3 \le i < j \le n\}$ is a basis for $H_2(F_n; \Gamma) \otimes \mathbb{Q}(x, y)$, thus the eigenvalues of ${}_1$ are $-x^2y$ of multiplicity 1, $-x$ of multiplicity $(n-2)$, and 1 of multiplicity $(n-1)(n-2)/2$. The same argument shows that $-(x^2y)$ is an eigenvalue of ${}_1$ of multiplicity 1 and E'_{12} is an eigenvector associated to this eigenvalue. Since $n \ge 4$, it follows that $(-x^2y) = -x^2y$ and E'_{12} is a multiple of E_{12}.

Write $E'_{12} = E_{12}$, where $\in \mathbb{Q}(x, y)$. The A_{12}-coordinate of E'_{12}, and B_{13}-coordinate of E'_{12} are $(y-1)(xy+1)$ and $(x-1)$, respectively, and both coordinates lie in $\mathbb{Z}[H]$, thus $\in \mathbb{Z}[H]$ and $E'_{12} = E_{12} \in V$.

Proof of (iv) This follows directly from Proposition 3.4. □

5 Computing the action with the forks

Recall that $\tilde{F}_n \to F_n$ denotes the regular covering associated to $: \pi_1(F_n) \to H$. Choose some $\varepsilon > 0$ (say $\varepsilon < 1/4$), and, for $p = 1, \dots, n$, write $v(p) = \{z \in \mathbb{C} \mid |z - p| < \varepsilon\}$. Let T be a fork with vertices P_b, p, q, z. Let $U(T)$ be the set of pairs $\{x, y\}$ in F_n such that either x or y lies in $v(p) \cup v(q)$, and let $\tilde{U}(T)$ be the pre-image of $U(T)$ in $\tilde{F}_n$. Bigelow [1] associated to any fork T a disc $S^{(1)}(T)$ embedded in $\tilde{F}_n$ whose boundary lies in $\tilde{U}(T)$, and proved that $(x-1)^2(xy+1)\,\partial S^{(1)}(T)$ is a boundary in $\tilde{U}(T)$. Thus, there exists an immersed surface $S^{(2)}(T)$ whose boundary is equal to $(x-1)^2(xy+1)\,\partial S^{(1)}(T)$. Note that the surface $S^{(2)}(T)$ is unique since $H_2(\tilde{U}(T); \mathbb{Z}) = 0$. So, the element $S(T) = S^{(2)}(T) - (x-1)^2(xy+1)\,S^{(1)}(T)$ is a well-defined 2-cycle which represents a non-trivial element of $H_2(\tilde{F}_n; \mathbb{Z}) = H_2(F_n; \Gamma)$. Moreover, the mapping $T \mapsto S(T)$ is equivariant by the action of B_n.

In [5], Krammer defined a family $T = \{T_{pq} \mid 1 \le p < q \le n\}$ of forks, called *standard forks*, proved that T is a basis for the LKB representation V, defined as a quotient of the free $\mathbb{Z}[H]$-module generated by the isotopy classes of forks, and explicitly computed the action of B_n on T. Let $\widetilde{Sal}(F_n) \to Sal(F_n)$ be the regular covering associated to $\pi : \pi_1(Sal(F_n)) \to H$. Let T_{pq} be a standard fork with vertices P_b, p, q, z, and assume $p+1<q$. Let

$$Sal(T_{pq}) = (\cup_{r=1}^{3} B_{pr}) \cup (\cup_{r=1}^{3} B_{qr}) \cup A_{pq} \cup (\cup_{k=p+1}^{q-1} A_{kq}) \cup (\cup_{k=p+1}^{q-1} A_{pk}),$$

and let $\widetilde{Sal}(T_{pq})$ be the pre-image of $Sal(T_{pq})$ in $\widetilde{Sal}(F_n)$. Then the pair $(\widetilde{Sal}(F_n), \widetilde{Sal}(T_{pq}))$ is homotopy equivalent to $(\tilde{F}_n, \tilde{U}(T_{pq}))$. Let

$$X_{pq}^{(1)} = \cup_{k=p+1}^{q-1} x^{k-1} B_{k1}.$$

The set $X_{pq}^{(1)}$ is a disc embedded in $\widetilde{Sal}(F_n)$ whose boundary lies in $\widetilde{Sal}(T_{pq})$, and one can choose the homotopy equivalence

$$(\widetilde{Sal}(F_n), \widetilde{Sal}(T_{pq})) \to (\tilde{F}_n, \tilde{U}(T_{pq}))$$

such that $X_{pq}^{(1)}$ is sent to $S^{(1)}(T_{pq})$. Let

$$X_{pq}^{(2)} = x^p(x-1)V_{pb} + x^{q-1}(x-1)V_{qa} + x^{q-1}(y-1)(xy+1)A_{pq} + \\ - \sum_{k=p+1}^{q-1} x^{k-1}(x-1)(y-1)(xy+1)A_{pk}.$$

$X_{pq}^{(2)}$ is a 2-chain in $C_2(\widetilde{Sal}(T_{pq}); \mathbb{Z})$ and one has

$$dX_{pq}^{(2)} = (x-1)^2(xy+1)\, dX_{pq}^{(1)} = \\ = (x-1)^2(xy+1)\left(- \sum_{k=p+1}^{q-1} x^{k-1}(y-1)a_k - x^p c_{p+1} + x^{q-1} c_q \right).$$

(Here, $X_{pq}^{(1)} = \sum_{k=p+1}^{q-1} x^{k-1} B_{k1}$ is viewed as a 2-chain). In particular, one has the equality $S(T_{pq}) = X_{pq}^{(2)} - (x-1)^2(xy+1)X_{pq}^{(1)}$ in $H_2(F_n; \Gamma) = H_2(Sal(F_n); \Gamma)$.

A similar argument shows that $S(T_{pq}) = x^p E_{pq}$ if $q = p+1$.

Let

$$X_{pq} = \begin{cases} x^p E_{pq} & \text{if } q = p+1 \\ X_{pq}^{(2)} - (x-1)^2(xy+1)X_{pq}^{(1)} & \text{if } q > p+1. \end{cases}$$

Note that, by Remark 3.5, one has

$$X_{pq} = x^{q-1}E_{pq} - \sum_{k=p+1}^{q-1} x^{k-1}(x-1)E_{pk}.$$

The set $\{X_{pq} \mid 1 \le p < q \le n\}$ is a $\mathbb{Z}[H]$-basis for the LKB representation V of the statement of 1.2 and, by the above remarks, the action of B_n on the X_{pq}'s is given by the formulae of [1, Thm. 4.1].

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