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Global structure of the mod two symmetric algebra, $H^*(BO;\mathbb{F}_2)$, over the Steenrod Algebra

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Abstract The algebra S of symmetric invariants over the field with two elements is an unstable algebra over the Steenrod algebra A, and is isomorphic to the mod two cohomology of BO, the classifying space for vector bundles. We provide a minimal presentation for S in the category of unstable A-algebras, i.e., minimal generators and minimal relations.

From this we produce minimal presentations for various unstable A-algebras associated with the cohomology of related spaces, such as the $BO(2^m-1)$ that classify finite dimensional vector bundles, and the connected covers of BO. The presentations then show that certain of these unstable A-algebras coalesce to produce the Dickson algebras of general linear group invariants, and we speculate about possible related topological realizability.

Our methods also produce a related simple minimal A-module presentation of the cohomology of infinite dimensional real projective space, with filtered quotients the unstable modules $F(2^p-1)/A\overline{A}_{p-2}$, as described in an independent appendix.

AMS Classification 55R45; 13A50, 16W22, 16W50, 55R40, 55S05, 55S10

Keywords Symmetric algebra, Steenrod algebra, unstable algebra, classifying space, Dickson algebra, BO, real projective space.

1 Introduction

We continue our study [9] of invariant algebras as unstable algebras over the Steenrod algebra A by proving a structure theorem for the algebra S of symmetric invariants over the field $\mathbb{F}_2$. The algebra S is isomorphic to the mod two cohomology of BO, the classifying space for vector bundles [8], and we identify the two. We also make several applications to the cohomology of related spaces, which then reveal a relationship between S and the Dickson algebras [13].

Our goal is to provide a minimal presentation for $S^* = H^*(BO;\mathbb{F}_2)$ in the category of unstable A-algebras [11], beginning with a minimally presented generating A-module and then introducing a minimal set of A-algebra relations. This reveals how a minimal set of A-module building blocks for S^* fit together in its A-algebra structure. In brief, our main result (Theorem 3.5) is that $S^* = H^*(BO;\mathbb{F}_2)$ is minimally presented in the category of unstable A-algebras as the free unstable A-algebra on the two-power Stiefel-Whitney classes w_{2^k} modulo relations expressing the fact that, for each $i \le k-2$, $Sq^{2^i} w_{2^k}$ differs from $Sq^{2^{k-1}} Sq^{2^i} w_{2^{k-1}}$ by a decomposable. (By contrast, and at first seemingly paradoxically, we shall also see (Theorem 2.3) that while S^* is generated as an A-algebra by $\{w_{2^k} : k \ge 0\}$, with relations linking the resulting algebra generators, in fact the A-submodule of S^* generated by $\{w_m : m \ge 0\}$ is a free unstable A-module on all the Stiefel-Whitney classes.)

We apply this structure theorem to characterize similarly the cohomology images $B^*(n)$ for the connected covers of BO (Theorem 4.2) [3], which include the full cohomology algebras of BSO, $BSpin$, and $BO\langle 8\rangle$. We likewise characterize the quotients $H^*(BO(q);\mathbb{F}_2)$ for the classifying spaces of finite dimensional vector bundles [8], and in particular (Theorem 4.3) we analyze $H^*(BO(2^{n+1}-1);\mathbb{F}_2)$.

Finally, we shall produce an A-algebra epimorphism from $S^* = H^*(BO;\mathbb{F}_2)$ to each of the mod two Dickson algebras (Theorem 4.4), which we characterized in [9] as unstable A-algebras. In fact we shall show that the $(n+1)$-st Dickson algebra has the role of capturing precisely the quotient of $S^* = H^*(BO;\mathbb{F}_2)$ common to the cohomology of the n-th distinct connected cover $BO\langle\phi(n)\rangle$ and to $BO(2^{n+1}-1)$. We speculate about how this phenomenon may relate to spaces beyond the range in which Dickson algebras are directly realizable topologically.

Our minimal A-algebra presentations for all the above objects will devolve naturally from our main presentation of S^*, and in that sense these A-algebras are all "parallel" to the main presentation.

In Appendix I, which is independent of the rest of the paper, we present a related result, in which the unstable A-modules $F(2^p-1)/A\overline{A}_{p-2}$ appear as the filtered quotients of a simple minimal A-presentation for $H^*(RP^\infty;\mathbb{F}_2)$.

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The first author dedicates this paper to his parents, Daphne M. and Eric T. Pengelley, in memoriam.

2 Motivation, first steps, and a plan

The unstable A-algebra of symmetric invariants $S = H^*(BO;\mathbb{F}_2)$ is a polynomial algebra $\mathbb{F}_2[w_m : m \geq 0; w_0 = 1]$, with each elementary symmetric function (Stiefel-Whitney class) w_m having degree m [8]. The action of the Steenrod algebra is completely determined from the Wu formulas [3, 12, 14]

$$Sq^j w_m = \sum_{l=0}^{j} \binom{m-j+l-1}{l} w_{j-l} w_{m+l}$$

and the Cartan formula on products [11].

To ease into our categorical point of view, and to illustrate our approach and methods, let us begin by seeing that abstract Stiefel-Whitney classes, taken all together as free unstable A-algebra generators, along with imposed “Wu formulas”, actually “present” S. This is something one might easily take for granted, but should actually prove, since in principle there might be “other” relations lurking in S beyond those inherent in the Wu formulas. To avoid confusion from notational abuse, we build from abstract classes t_m which will correspond to the actual Stiefel-Whitney classes under an isomorphism.

Proposition 2.1 (Wu formulas present S) *The unstable A-algebra $S = H^*(BO;\mathbb{F}_2)$ is isomorphic to the quotient of the abstract free unstable A-algebra on classes t_m in each degree $m \geq 1$, modulo the left A-ideal generated by abstract “Wu formulas” formed by writing t's in place of w's in the Wu formulas above.*

Proof Iterating the abstract Wu formulas via the Cartan formula shows that the abstract classes $\{t_m : m \geq 1\}$ actually generate the abstract A-algebra quotient considered merely as an algebra, i.e., its (algebra) indecomposable quotient has rank at most one in each degree. On the other hand, by its construction the abstract A-algebra quotient must map onto S by sending each t_m to w_m, since the respective Wu formulas correspond. Thus the two must be isomorphic, since S is free as a commutative algebra. □

Notice, however, that this presentation of S is far from minimal in the category of unstable A-algebras, since it used vastly more generators than needed. What we seek instead is to achieve three features for a minimal presentation:

Step 1 Find a minimal A-submodule of S that will generate S as an A-algebra.

Step 2 Find a minimal presentation of this A-submodule, i.e., with minimal generators and minimal relations.

Step 3 Form the free unstable A-algebra U on this module, and find minimal relations on U so that its A-algebra quotient produces S.

To begin, let us find a minimal set of A-algebra generators for S. Consider the (algebra) indecomposable quotient QS, i.e., the vector space with basis $\{w_m : m \geq 1\}$ and induced A-action

$$Sq^j w_m = \binom{m-1}{j} w_{m+j}.$$

Since $\binom{m-1}{j}$ is always zero mod two when $m+j$ is a two-power, and never zero when m is a two-power and j is less than m, we see that the A-module indecomposables of QS have basis exactly $\{w_{2^k} : k \geq 0\}$.

Since our philosophy is to begin the presentation at the A-module level, with minimal A-algebra generators and minimal module relations, we thus start with

Definition 2.2 Let M be the free unstable A-module on abstract classes $\{t_{2^k} : k \geq 0\}$, where subscripts indicate the topological degree of each class.

We wish to map M to S via $t_{2^k} \mapsto w_{2^k}$, and need first to ask whether M injects. In other words, is the A-submodule of $S = H^*(BO;\mathbb{F}_2)$ generated by $\{w_{2^k} : k \geq 0\}$ free? Or are there, to the contrary, A-relations amongst the two-power Stiefel-Whitney classes, which will compel us to introduce module relations on M in order to complete steps 1 and 2 above? The Wu formulas appear to suggest that no such relations exist. In fact we can prove something even stronger.

Theorem 2.3 (Stiefel-Whitney classes inject freely) *The A-submodule of $S = H^*(BO;\mathbb{F}_2)$ generated by $\{w_m : m \geq 0\}$ is free unstable on these classes.*

The proof is in Section 5.

Remark 2.4 The proof also shows that in

$$H^*(BO(q);\mathbb{F}_2) = H^*(BO;\mathbb{F}_2)/(w_m : m > q),$$

the A-submodule generated by $\{w_m : 0 \leq m \leq q\}$ is free unstable on these classes.

Remark 2.5 The fact that the free unstable A-module F_m on a single class in degree m injects into $H^*(BO(m);\mathbb{F}_2)$ on the class w_m is clear from the already known result [4, page 55] that F_m is isomorphic to the invariants $\left(F_1^{\otimes m}\right)^{\Sigma_m}$; which clearly inject naturally into $(H^*(RP^\infty;\mathbb{F}_2)^{\otimes m})^{\Sigma_m} = H^*(BO(m);\mathbb{F}_2)$ on w_m. Theorem 2.3 generalizes this by handling all F_m simultaneously, showing that they do not interfere when simultaneously perched on the Stiefel-Whitney classes in the symmetric algebra $S = H^*(BO;\mathbb{F}_2)$.

Corollary 2.6 *The A-submodule of $S = H^*(BO;\mathbb{F}_2)$ generated by $\{w_{2^k} : k \geq 0\}$ is free unstable, so M injects naturally into S.*

This completes steps 1 and 2 of our goal, and we can begin step 3.

Definition 2.7 Let U be the free unstable A-algebra on M, in other words, U is the free unstable A-algebra on abstract classes $\{t_{2^k} : k \geq 0\}$.

Clearly U maps via $t_{2^k} \to w_{2^k}$ onto the desired A-algebra S, but the map has an enormous kernel, since QS is the vector space $\mathbb{F}_2\{w_m : m \geq 1\}$, while QU is much larger. Our goal in step 3 is to describe a minimal set of A-algebra relations producing S from U, i.e., minimal generators for the kernel as an A-ideal.

Let us explore a prototype example in degree five, which is the first place a difference occurs. There QS has only w_5, whereas $Sq^1 t_4$ and $Sq^2 Sq^1 t_2$ are distinct indecomposables in QU (recall that $QU = \Sigma\Omega M$, and that a basis for M consists of the unstable admissible monomials on the A-generators t_{2^k} [11]). A few calculations with the Wu formulas show that in S we have

$$\begin{aligned} Sq^1 w_4 &= w_5 + w_1 w_4 \text{ and} \\ Sq^2 Sq^1 w_2 &= w_5 + w_1 w_4 + w_2 w_3 + w_1 w_2^2 + w_1^2 w_3 + w_1^3 w_2. \end{aligned}$$

Thus to imitate S abstractly via U, we must impose an algebra relation on U decreeing that

$$Sq^1 t_4 = Sq^2 Sq^1 t_2 + \text{some decomposable},$$

per the calculations above. One challenge in doing even this, though, is that it is not clear how to describe that needed decomposable difference in U, since there we have no name as yet for the element corresponding to w_3. To remedy this, and to describe general formulas for relationships like the one we have just discovered, we wish to use the Wu formulas to focus our understanding as much as possible on both two-power Steenrod squares and two-power Stiefel-Whitney classes. Thus one of our formulas in the next section will express each Stiefel-Whitney class purely in this way (Lemma 3.2).

While the plethora of algebra relations, such as the one above, needed to obtain S from U may appear intractable to specify, recall that our chosen task is actually somewhat different. Since we are working in the category of A-algebras, we seek relations in U whose A-algebra consequences, not just their algebra consequences, will produce S. We shall show that this requires only a much smaller and more tractable set of relations, for which our illustration in degree five serves as perfect prototype. Specifically, the relationship between $Sq^{2^i}w_{2^k}$ and $Sq^{2^{k-1}}Sq^{2^i}w_{2^{k-1}}$ for every $i \leq k-2$ will be the key place to focus attention. We shall impose one abstract relation on U for each such pair (k, i), and prove that these are precisely the minimal relations producing $S = H^*(BO; \mathbb{F}_2)$ in the category of A-algebras.

Our general plan is as follows. Form our abstract presentation candidate as just outlined; call it G. The construction of G will immediately provide a natural A-algebra epimorphism to S. The hard part now is showing that our (k, i)-indexed family of A-algebra relations leaves no remaining kernel, i.e., that we have put in enough relations to generate the kernel as an A-ideal. To achieve this we show that the epimorphism $G \to S$ induces a monomorphism $QG \to QS$, on the indecomposable quotients, by computing a basis for QG. For this we appeal to our earlier understanding [9], via the Kudo-Araki-May algebra K [10] (see Appendix II), of bases for the unstable cyclic A-modules arising in the analogous structure theorem for the Dickson algebras. With $QG \to QS$ an isomorphism, $G \to S$ must be an isomorphism also, since S is a free commutative algebra. The minimality of the (k, i)-family of relations is then not hard to see by appropriate filtering.

3 Main theorem

We first identify the key A-algebra relations in $S = H^*(BO; \mathbb{F}_2)$.

Analysis of the binomial coefficients in the Wu formulas shows that if $r \geq 1$, then

$$Sq^{2^{j-1}}w_{r2^j} = w_{2^{j-1}}w_{r2^j} + w_{2^{j-1}+r2^j}. \tag{3.1}$$

This formula will serve two purposes. It will guide us below in how to specify any Stiefel-Whitney class from just the two-power ones, which is needed for creating our abstract presentation. But before this it will lead us to the key relations needed from S.

To find these, recall from the previous section that we seek a relation involving a decomposable difference between $Sq^{2^i} w_{2^k}$ and $Sq^{2^{k-1}} Sq^{2^i} w_{2^{k-1}}$ for every $i \leq k-2$. We begin with a special case of equation (3.1): For $i \leq k-2$, we have

$$Sq^{2^i} w_{2^{k-1}} = w_{2^i} w_{2^{k-1}} + w_{2^{k-1}+2^i}.$$

Applying $Sq^{2^{k-1}}$, we get

$$Sq^{2^{k-1}} Sq^{2^i} w_{2^{k-1}} = Sq^{2^{k-1}}(w_{2^i} w_{2^{k-1}}) + Sq^{2^{k-1}}\left(w_{2^{k-1}+2^i}\right).$$

Using a Wu formula on the last term, analyzing the binomial coefficients, and using (3.1) again, the reader may check that we obtain the following relations.

Proposition 3.1 (Key relations in S) *For $i \leq k-2$,*

$$\begin{aligned} Sq^{2^{k-1}} Sq^{2^i} w_{2^{k-1}} = {} & Sq^{2^i} w_{2^k} + \\ & Sq^{2^{k-1}}(w_{2^i} w_{2^{k-1}}) + \sum_{l=0}^{2^{k-i-1}-2} w_{2^{k-1}-2^i l} w_{2^{k-1}+2^i+2^i l}, \end{aligned} \tag{3.2}$$

These show explicitly how the elements $Sq^{2^i} w_{2^k}$ and $Sq^{2^{k-1}} Sq^{2^i} w_{2^{k-1}}$ differ by a decomposable, and will guide us to the corresponding abstract relations needed in G. However, the relations we have found here involve non-two-power Stiefel-Whitney classes, which still have as yet no analogs in U. We remedy this problem now by extending equation (3.1).

Mixing notations, we write (3.1) as

$$w_{2^{j-1}+r2^j} = (Sq^{2^{j-1}} + w_{2^{j-1}}) w_{r2^j}$$

(i.e., $(Sq^m + w_m)x$ means $Sq^m x + w_m \cdot x$). The following lemma is then immediate.

Lemma 3.2 (Expressing Stiefel-Whitney classes) *Every Stiefel-Whitney class can be expressed in terms of two-power classes and two-power squares as follows: If we write any $m = 2^{n_1} + \cdots + 2^{n_s}$, where $n_1 > \cdots > n_s$, we have*

$$w_m = \left(Sq^{2^{n_s}} + w_{2^{n_s}}\right) \cdots \left(Sq^{2^{n_2}} + w_{2^{n_2}}\right) w_{2^{n_1}}. \tag{3.3}$$

We are now ready to define formally the abstract presentation G.

Definition 3.3 In U, extend the set of generators $\{t_{2^k}; k \geq 0\}$, to define elements t_m for all $m \geq 1$, by first writing $m = 2^{n_1} + \cdots + 2^{n_s}$; where $n_1 > \cdots > n_s$. Then by analogy with equation (3.3) set

$$t_m = (Sq^{2^{n_s}} + t_{2^{n_s}}) \cdots (Sq^{2^{n_2}} + t_{2^{n_2}})\, t_{2^{n_1}}.$$

Definition 3.4 (Abstract key relations) Imitating equation (3.2), let G be the the A-algebra quotient of U by the left A-ideal generated by the elements

$$\rho(k,i) = Sq^{2^i} t_{2^k} + Sq^{2^{k-1}} Sq^{2^i} t_{2^{k-1}} + Sq^{2^{k-1}}(t_{2^{k-1}} t_{2^i}) + \sum_{l=0}^{2^{k-i-1}-2} t_{2^{k-1}-2^i l}\, t_{2^{k-1}+2^i+2^i l} \tag{3.4}$$

for $i \leq k-2$.

Theorem 3.5 (Structure of S) *The symmetric algebra $S = H^*(BO;\mathbb{F}_2)$ is isomorphic to G as an algebra over the Steenrod algebra. Moreover, the relations (3.4) generating the A-ideal are minimal, i.e., nonredundant.*

The proof is in Section 5.

4 Applications and speculation

We apply the main structure theorem to the cohomology images from the connected covers of BO, and to the cohomology of the spaces $BO(q)$ for classifying finite dimensional vector bundles. Finally we shall see how these descriptions naturally converge into the Dickson invariant algebras.

First we consider cohomology images from the connected covers.

Definition 4.1 Following [3], let $B(n)$ be the cohomology image of the map induced by the projection

$$BO\langle\phi(n)\rangle \to BO,$$

where $BO\langle\phi(n)\rangle$ is the n-th distinct connected cover of BO. That is, $BO\langle\phi(n)\rangle$ is $(\phi(n)-1)$-connected, where $n = 4s + t$, $0 \leq t \leq 3$, and $\phi(n) = 8s + 2^t$.

In particular, for $n = 0, 1, 2, 3$ the projections are surjective in cohomology, so the unstable A-algebras $B(n)$ are isomorphic to the cohomologies of BO, BSO, $BSpin$, and $BO\langle 8\rangle$ [3]. In general, $B(n)$ is $(2^n - 1)$-connected, and is the quotient of $B(0) = H^*BO = S$ by the A-ideal generated by $\{w_{2^k} : k < n\}$ [3].

Theorem 4.2 (Structure of connected cover images) *An abstract presentation of $B^*(n)$ is obtained from that of $B^*(0) = H^*BO = S$ (Theorem 3.5) as the quotient by the A-ideal generated by $\{t_{2^k} : k < n\}$. This produces a minimal presentation as follows.*

Let K_n denote the direct sum of the A-module $M(n,0)$ on t_{2^n} with the free unstable A-module on the t_{2^k}, $k \ge n+1$. Here $M(n,0)$ is as defined in [9], namely the free unstable A-module on one generator t_{2^n} modulo the left A-submodule generated by $Sq^{2^i} t_{2^n}$, $i \le n-2$.

Then $B^*(n)$ is isomorphic to the quotient of the free unstable A-algebra on K_n by the left A-ideal generated by the elements (k,i), $k \ge n+1$, $i \le k-2$, subject to the requirement that all appearances in (k,i) of t_m, $0 < m < 2^n$, are replaced by zero.

The proof is in Section 5.

For our second application, we note that the presentation for H^*BO in our main theorem will immediately produce presentations for the cohomologies of the classifying spaces $H^*BO(q)$, since each is just the algebra quotient (actually also A-algebra quotient) of H^*BO by the ideal generated by $\{w_m : m > q\}$ [8], and w_m corresponds to t_m, which we defined in the presentation of H^*BO. The resulting presentation becomes both tractable and useful for $H^*BO(2^{n+1}-1)$.

Theorem 4.3 (Structure of $H^*BO(2^{n+1}-1)$) *An abstract presentation of $H^*BO(2^{n+1}-1)$ is obtained from that of $B^*(0) = H^*BO = S$ (Theorem 3.5) as the quotient by the A-ideal generated by $\{t_{2^k} : k \ge n+1\}$. This produces a minimal presentation as follows.*

$H^*BO(2^{n+1}-1)$ is presented by the free unstable A-algebra on abstract classes $\{t_{2^k} : 0 \le k \le n\}$, modulo the left A-ideal generated by the elements (k,i) for $k \ge n+1$, $i \le k-2$, (using Definition 3.3 of t_m for $m < 2^{n+1}$), subject to the requirement that when $k = n+1$, the term $Sq^{2^i} t_{2^{n+1}}$ is replaced by zero for each i (all other terms involve only t's in degrees less than 2^{n+1}).

The proof is in Section 5.

Finally, combining the relations on $S = H^*(BO;\mathbb{F}_2)$ from the two theorems above will produce the common A-algebra quotient of $B^*(n)$ and $H^*BO(2^{n+1}-1)$. Since the first of these is (2^n-1)-connected, while the second is decomposable beyond degree $2^{n+1}-1$, we will obtain an A-algebra with algebra generators in the range 2^n through $2^{n+1}-1$. Surprisingly, this much smaller quotient of $S = H^*BO$ turns out to be already familiar. We will show now that as an A-algebra it is isomorphic to the n-th Dickson algebra W_{n+1} (see Figure 1). In this sense one can say that the Dickson algebra captures precisely the cohomology common to $BO\langle\phi(n)\rangle$ and $BO(2^{n+1}-1)$ from H^*BO, i.e., it is the A-algebra pushout.

$$\begin{tikzcd} W_{n+1} & H^*BO(2^{n+1}-1) \arrow[l] \\ B(n) \arrow[u] & H^*BO \arrow[l] \arrow[u] \end{tikzcd}$$

Figure 1

Theorem 4.4 (Convergence to Dickson algebras) *The quotient of the symmetric algebra S by the left A-ideal generated by $\{w_{2^k} : k \neq n\}$ is isomorphic to the $n+1$-st mod 2 Dickson algebra, W_{n+1}. Specifically, using the notation of the presentation of Theorem 3.5, as an A-algebra it is minimally presented by the free unstable A-algebra on the module $M(n;0)$ (defined in Theorem 4.2), subject to the single A-algebra relation*

$$Sq^{2^n} Sq^{2^{n-1}} t_{2^n} = t_{2^n} Sq^{2^{n-1}} t_{2^n}.$$

We proved in [9] that this precisely characterizes the Dickson algebra W_{n+1}.

The proof is in Section 5.

Let us speculate on how Figure 1 might fit in with something topologically realizable. It is known that W_{n+1} is realizable precisely for $n \leq 3$ [6], and that $B(n) \to H^*BO\langle\phi(n)\rangle$ is an isomorphism also precisely in this range [3]. Thus for $n \leq 3$ it is reasonable to expect that Figure 1 be realizable. For general n it is perhaps reasonable to hope for the existence of a space X_n and a homotopy commutative square (Figure 2) whose cohomology is compatible with Figure 1 in the sense of combining to produce the commutative diagram of Figure 3. Additionally we would like X_n to have the property that the outer square in Figure 3 is also a pushout of unstable A-algebras. In other words, X_n does its best to realize a Dickson algebra, even when this is no longer possible.

5 Proofs

Proof of Theorem 2.3 Let F_m be the free unstable A-module (equivalently K module) on a generator t_m in degree m. We shall show that the A-module map $f\colon \bigoplus_{m \geq 0} F_m \to H^*BO$ determined by $f(t_m) = w_m$ is injective.

$$\begin{array}{ccc} X_n & \to & BO(2^{n+1}-1) \\ \downarrow & & \downarrow \\ BO\langle\phi(n)\rangle & \to & BO \end{array}$$

Figure 2

$$\begin{array}{ccccc} H_*X_n & \leftarrow & W_{n+1} & \leftarrow & H_*BO(2^{n+1}-1) \\ \uparrow & & \uparrow & & \uparrow \\ H_*BO\langle\phi(n)\rangle & \leftarrow & B_*(n) & \leftarrow & H_*BO \end{array}$$

Figure 3

From [10], basis elements for the domain of f consist of $D_J t_m$ where $J = (j_1, \ldots, j_s)$ and $0 \le j_1 \le \cdots \le j_s < m$. (Appendix II recalls the features of the elements D_J in the Kudo-Araki-May algebra K essential to what follows.)

On the other side of f, basis monomials of the range H_*BO can be written as $\cdots w_{n_2} w_{n_1}$ with nondecreasing indices, i.e., labeled by finitely nonzero tuples $(\ldots, n_2, n_1)$ with $0 \le \cdots \le n_2 \le n_1$. We order the latter reverse lexicographically.

Now for each basis element $D_J t_m$, we consider its image $f(D_J t_m) = D_J w_m$, and we claim that this element of H_*BO has a "leading" monomial term, i.e., that

$$D_J w_m = \underbrace{w_{m-j_s}^{2^{s-1}} w_{m-j_{s-1}}^{2^{s-2}} \cdots w_{m-j_2}^{2} w_{m-j_1} w_m}_{z} + \text{ higher order terms.}$$

This will complete the proof, since distinct $D_J w_m$ clearly produce distinct leading monomials, with remaining terms always of higher order; so the $D_J w_m$ are all linearly independent, and thus f is injective.

We will use the following notation: As a subscript, "$> k$" (resp. "$< k$") denotes any index greater (resp. less) than k, each occurrence of an unsubscripted w denotes any element of H_*BO, and expressions involving any of these mean any sum of expressions of such form.

We prove our claim by induction on s, based on the Wu formula

$$D_j w_m = Sq^{m-j} w_m = w_{m-j} w_m + \text{ higher order terms of form } w w_{>m}.$$

Clearly the claim holds for lengths 0 and 1. For the inductive step, consider $D_{\hat{J}}$ of length $s+1$, and note that application of any nontrivially-acting D_J always increases the order of a monomial in H^*BO. Now calculate, using the K-Cartan formula [10] as needed, and recalling that the leading term z was defined above:

$$
\begin{aligned}
D_{\hat{J}} w_m & \\
&= D_{j_1} D_{j_2} \cdots D_{j_{s+1}} w_m = D_{j_1} \left(D_{j_2} \cdots D_{j_{s+1}} w_m \right) \\
&= D_{j_1} \left(\underbrace{w_{m-j_{s+1}}^{2^{s-1}} \cdots w_{m-j_2}}_{x} w_m + \text{ higher order terms than } x w_m \right) \\
&= x^2 w_{m-j_1} w_m + x^2 w w_{>m} + w D_{<j_1} w_m \\
&\quad + D_{j_1} \left(w w_{>m} + \underbrace{\text{higher order terms than } x w_m}_{v} \right) \\
&= z + w w_{>(m-j_1)} w_m + \text{ higher order terms than } z \\
&\quad + \left(w w_{>m} + v^2 w_{m-j_1} w_m + w w_{>(m-j_1)} w_m \right) \\
&= z + \text{ higher order terms than } z,
\end{aligned}
$$

since the terms of v^2 have higher order than x^2. □

Proof of Theorem 3.5 There is a map of A-algebras $U \to S$ obtained by taking t_{2^k} to w_{2^k}; and from Lemma 3.2 and Definition 3.3 this map takes each t_m to w_m. Since the relations (3.4) that define G map to those also satisfied in S (3.2), there is an induced A-algebra epimorphism $G \to S$. We shall show that this map is monic by showing that the induced map on the indecomposable quotients is monic, essentially a counting argument.

To start with, note that the indecomposables are

$$QU = \left\langle Sq^I t_{2^k} : k \geq 0,\ I \text{ admissible, of excess } < 2^k \right\rangle .$$

Then QG is QU modulo the A-relations (degenerate versions of $(k;i)=0$)

$$Sq^{2^i} t_{2^k} = Sq^{2^{k-1}} Sq^{2^i} t_{2^{k-1}},\ i \leq k-2.$$

There is an A-module filtration

$$F_p QU = \left\langle Sq^I t_{2^k} : 0 \leq k \leq p,\ I \text{ admissible, of excess } < 2^k \right\rangle ;$$

which induces an A-module filtration F_pQG. Then

$$F_pQG/F_{p-1}QG = \left\langle Sq^I t_{2^p} : I \text{ admissible, of excess } < 2^p \right\rangle / A\left\{ Sq^{2^i} t_{2^p} : i \le p-2 \right\}.$$

This is the suspension of the module $M(p,1)$ analyzed in [9, Theorem 2.11][1], and the basis described there suspends to

$$\{D_I t_{2^p} : I = (2^{a_1}, \ldots, 2^{a_l}), \text{ where } 0 \le a_1 \le \cdots \le a_l < p\}.$$

(As in the proof of Theorem 2.3, we refer the reader to Appendix II for essentials concerning the elements D_I in the Kudo-Araki-May algebra K.)

We shall finish the proof of isomorphism by showing that the above basis elements for $\bigoplus_{p \ge 0} F_pQG/F_{p-1}QG$ are in distinct degrees; in fact we claim there is exactly one in each positive degree (The appendix discusses the modules $M(p,1)$ in relation to the literature, and points out an alternative path for substantiating our claim.). Let m be a positive integer. Then m may be written uniquely in the form

$$m = 2^r - \sum_{j=1}^{s} 2^{b_j},$$

where $s \ge 0$ and $0 \le b_1 < \cdots < b_s < r-1$. The reader may check by induction on s that the unique basis element in degree m is $D_I t_{2^p}$, where $p = r - s$ and $I = (2^{a_1}, \ldots, 2^{a_s})$, with $a_j = b_j - j + 1$. With both QG and QS having rank one in each degree, $QG \to QS$ is an isomorphism. Then since S is a free commutative algebra, the epimorphism $G \to S$ must be an isomorphism also.

That the relations are minimal (nonredundant) is clear from the fact that in $F_pQU/F_{p-1}QU$, which is the suspension of the free unstable module on a class in degree $2^p - 1$, the induced relations are simply $Sq^{2^i} t_{2^p} = 0$, for $i \le p-2$, and these are all nonredundant. □

Proof of Theorem 4.2 We have already mentioned that according to [3], $B(n)$ is isomorphic to the quotient of S by the A-ideal generated by $\{w_{2^k} : k \le n-1\}$. Hence the images under the projection $S \to B(n)$ of all w_m, $1 \le m \le 2^n - 1$, are certainly zero from Lemma 3.2. From [3] we also have that $B(n)$ is a polynomial algebra generated by certain remaining w_m (see below). We denote the images of the w_m in $B(n)$ by the same symbols w_m.

[1] $M(p,1)$ is defined in [9] as the quotient of the free unstable A-module on a class in degree $2^p - 1$ modulo the action of Sq^{2^i} for $i \le p-2$; in other words, in usual notation, $M(p,1) = F(2^p - 1)/A\overline{A}_{p-2}$.

Let H_n denote the quotient of the free unstable A-algebra on K_n by the left A-ideal generated by the elements $(k;i)$, for $k \geq n+1$, subject to the requirement that all appearances of $t_m;\ 0 < m < 2^n$, are replaced by zero, as in the statement of the theorem.

We begin by defining a map from K_n to $B(n)$ by, as in the preceding proof, assigning t_{2^k} to w_{2^k} for $k \leq n$. Since the defining relations for K_n are clearly satisfied in $B(n)$ (from equation (3.2)), this assignment extends to the desired map. And since the defining relations for the algebra H_n are also clearly satisfied in $B(n)$, this extends to an A-algebra map $H_n \to B(n)$. This map is epimorphic (since $B(n)$ is generated by certain w_m with $i \geq 2^n$), so as in the preceding proof, we need only show the the induced map on indecomposables is monomorphic.

According to [3][2], the polynomial generators of $B(n)$ are the w_m for which $\alpha(m-1)$; the number of ones in the binary representation of $m-1$; is at least n. We filter QH_n as in the proof of the previous theorem,

$$F_pQH_n = \left\{Sq^I t_{2^k} \in QH_n : k \leq p\right\},$$

and as in the previous proof the filtered quotient $F_pQH_n/F_{p-1}QH_n$ is the suspension of the module $M(p;1)$ for $p \geq n$, and 0 for $p < n$. It is straightforward to check that the alpha numbers of one less than the degrees of the elements

$$\{D_I t_{2^p} : I = (2^{a_1}; \ldots ; 2^{a_l}); \text{ where } 0 \leq a_1 \leq \cdots \leq a_l < p\}$$

are exactly $p \geq n$, so these are all in degrees where $B(n)$ has generators. Since we showed in the previous proof that these elements are also in distinct degrees, this similarly completes the proof. Minimality follows as in the previous proof. □

Proof of Theorem 4.3 It is clear that the presentation of S collapses in the manner stated. Minimality follows for most of the relations as in the previous proofs. We comment only that to confirm that the collapsed top relations

$$0 = (n+1;i) \equiv Sq^{2^n} Sq^{2^i} t_{2^n} + \text{decomposables for } i \leq n-1$$

are also all nonredundant, one can observe that there is a natural map of the new presentation without these final relations to the presentation for S, and compute that on indecomposables, each $Sq^{2^n}Sq^{2^i}t_{2^n}$ maps to $w_{2^{n+1}+2^i}$. Now from the Wu formulas, $QH\,BO$ is filtered over A by $F_pQH\,BO = \{w_m : \alpha(m-1) \leq p\}$, and $w_{2^{n+1}+2^i}$ is in filtration exactly $i+1$. Thus

[2] Kochman describes degrees of generators in terms of $\alpha(m) + \nu(m)$ (ν is the 2-divisibility), but we equivalently use $\alpha(m-1) = \alpha(m) + \nu(m) - 1$.

$\{w_{2^{n+1}+2^i} : i \le n-1\}$ must be a minimal generating set for the A-submodule it generates in QH^*BO. The same then must be true of $\{\varphi(n+1,i) : i \le n-1\}$ in the indecomposables of the new presentation without these final relations; so they too are minimal. □

Proof of Theorem 4.4 In [9] we proved that the $(n+1)$-st Dickson algebra W_{n+1} is isomorphic to the quotient of the free unstable A-algebra on the module $M(n,0)$ on generator x_{2^n} by the single A-algebra relation

$$Sq^{2^n} Sq^{2^{n-1}} x_{2^n} = x_{2^n} Sq^{2^{n-1}} x_{2^n},$$

and that $M(n,0)$ injects into W_{n+1} ([9], proof of Theorem 2.11). In other words, this is a minimal presentation in our sense.

Now let us turn to the quotient of the symmetric algebra that combines the relations from the previous two theorems, i.e., the quotient by the left A-ideal generated by $\{t_{2^k}, k \ne n\}$. Let us denote this quotient by J_n. In J_n, the relations $\varphi(k,i)$ are all trivial except when k is $n+1$ or n. When $k = n$, they reduce to $Sq^{2^i} t_{2^n} = 0$, $i \le n-2$, the defining relations for $M(n,0)$. When $k = n+1$, we have the relations

$$0 = \varphi(n+1,i) \equiv Sq^{2^n} Sq^{2^i} t_{2^n} + Sq^{2^i} t_{2^{n+1}} + Sq^{2^n}(t_{2^n} t_{2^i}) + \sum_{l=0}^{2^{n-i}-2} t_{2^n - 2^i l} t_{2^n + 2^i + 2^i l}$$

for $i \le n-1$. These reduce to

$$Sq^{2^n} Sq^{2^i} t_{2^n} = t_{2^n} t_{2^n+2^i}.$$

Now since

$$t_{2^n} t_{2^n+2^i} = t_{2^n}\left(Sq^{2^i} t_{2^n} + t_{2^i} t_{2^n}\right) = t_{2^n} Sq^{2^i} t_{2^n},$$

the relations can be rewritten as

$$Sq^{2^n} Sq^{2^i} t_{2^n} = t_{2^n} Sq^{2^i} t_{2^n}.$$

Since $Sq^{2^i} t_{2^n} = 0$ for $i < n-1$, these are trivial for $i < n-1$, and yield

$$Sq^{2^n} Sq^{2^{n-1}} t_{2^n} = t_{2^n} Sq^{2^{n-1}} t_{2^n}$$

for $i = n-1$. This precisely matches the single relation (stated above) characterizing the Dickson algebra, so we obtain an isomorphism of A-algebras from J_n to W_{n+1} by taking $t_{2^n} \in J_n$ to the generator $x_{2^n} \in W_{n+1}$. □

6 Appendix I: The unstable modules $F(2^p-1)/A\overline{A}_{p-2}$ and a minimal A-presentation for $H^*(RP^\infty)$

For each $p \geq 0$, the module $M(p,1)$ is defined in [9] as the quotient of the free unstable A-module on a class x_{2^p-1} in degree 2^p-1 modulo the action of Sq^{2^i} for $i \leq p-2$; in other words, in usual notation,

$$M(p,1) = F(2^p-1)/A\overline{A}_{p-2}.$$

These modules are tractable, important, and interesting, and we shall show they are the filtered quotients of a simple minimal A-presentation for H^*RP^∞.

In the proof of our primary Theorem 3.5 above, we appealed to our development in [9, Theorem 2.11] of bases for these modules. The proof used the bases to “count” that the direct sum of the modules (we were actually dealing with their suspensions in that theorem) has rank exactly one in each nonnegative degree. In fact we know the rank separately for each module:

Theorem 6.1 (Rank of $M(p,1)$) *The module $M(p,1)$ has precisely a single nonzero element in each degree with alpha number p, i.e., with p ones in its binary expansion, and nothing else.*

Proof The basis for $M(p,1)$ provided in [9, Theorem 2.11] is

$$\{D_I x_{2^p-1} : \text{the multi-index } I \text{ consists of nonnegative, nondecreasing entries of form } 2^k-1,\ k < p\}.$$

The reader may check that the degrees of these elements are precisely those with alpha number p (see Appendix II for a recollection of essentials regarding the elements D_I in the Kudo-Araki-May algebra K). ◻

This suggests a connection to the cohomology of RP^∞. Recall that

$$H^*RP^\infty = \mathbb{F}_2[y] \text{ with } Sq^j y^l = \binom{l}{j} y^{l+j}, \tag{6.1}$$

from which one sees that H^*RP^∞ is A-filtered by the number of ones in the binary expansion of degrees. Indeed it is now not hard to prove

Theorem 6.2 ($M(p,1)$ and H^*RP^∞) *The A-module $M(p,1)$ is isomorphic to the p-th filtered quotient of H^*RP^∞.*

Proof The module $M(p;1)$ clearly maps nontrivially to the p-th filtered quotient of H^*RP^∞, since the quotient begins with y^{2^p-1}, and $Sq^{2^i}y^{2^p-1}$ lies in lower filtration for $i \leq p-2$. The map is onto because one sees from (6.1) that the p-th filtered quotient of H^*RP^∞ is generated over A from degree 2^p-1. Now the previous theorem shows that the ranks agree, so the two are isomorphic. □

Remark 6.3 This result also follows from [2], where it essentially appears in a stabilized form. Indeed, in [2] the A-modules

$$\Sigma^{2^p-1}A/A\left\{Sq^{2^j} : j \neq p-1\right\}$$

are studied with stable purposes in mind. Each of these modules obviously maps onto the corresponding $M(p;1)$, and thus the two would clearly be isomorphic if it were known that the domain module is unstable, which does not seem obvious. In fact, though, it is proven in [2] that these modules are isomorphic to the same filtered quotients of H^*RP^∞. Thus they are indeed unstable and isomorphic to the modules $M(p;1)$. The theorem follows.

Remark 6.4 The modules $M(p;1)$ are also used in [5], where Remark 2.6 claims that in an unpublished manuscript [7], William Massey calculated that $M(p;1)$ is A-isomorphic to the p-th filtered quotient of H^*RP^∞, i.e., the theorem above. However, this does not actually seem to appear explicitly in [7]. Finally, we note that the filtered quotients of H^*RP^∞ arise again in [1, after Prop. 3.1] in a fashion closely related both to [5] and [7].

We are now equipped to show

Theorem 6.5 (Minimal A-presentation of $H^*(RP^\infty)$) *There is a minimal unstable A-module presentation of $H^*(RP^\infty;\mathbb{F}_2)$, as the quotient of the free unstable module on abstract classes s_{2^k-1} in degrees 2^k-1 by the relations*

$$Sq^{2^i}s_{2^k-1} = Sq^{2^{k-1}}Sq^{2^i}s_{2^{k-1}-1}; \quad i \leq k-2.$$

Proof There is an A-module map from the abstract quotient to H^*RP^∞, carrying each A-generator nontrivially, since the given relations are easily calculated also to hold amongst the nonzero classes in H^*RP^∞. Moreover this is epic, since H^*RP^∞ is generated over A from degrees one less than a two-power. To see that the two are isomorphic, we need merely show that these relations are enough, i.e., that the abstract quotient has only rank one in each degree. This we do by considering the A-filtration of the abstract quotient in which the p-th filtration is the A-submodule generated by $\{s_1, \ldots, s_{2^p-1}\}$. The p-th filtered quotient is clearly $M(p, 1)$. That the union of these has rank one in each nonnegative degree follows from either of the two previous theorems.

Minimality of the presentation is clear. The nonzero classes in H^*RP^∞ in degrees one less than a power of two cannot be reached from below, so the generating set is minimal, and unique. The nonredundancy of all the relations is clear from the filtered quotients and the fact that two-power squares are minimal generators of A.

An alternative proof would be to obtain this presentation simply by collapsing the relations (3.4) in the A-algebra presentation of H^*BO in Theorem 3.5 to the indecomposable quotient, since $\Sigma H^*RP^\infty = QH^*BO$ as A-modules (Wu formulas). □

7 Appendix II: The Kudo-Araki-May algebra K

We recall here just the bare essentials about K needed to understand the proofs in this paper. We refer the reader to [10] for much more extensive information about K.

The mod two Kudo-Araki-May algebra K is the $\mathbb{F}_2$-bialgebra (with identity) generated by elements $\{D_i : i \geq 0\}$ subject to homogeneous (Adem) relations [10, Def. 2.1], with coproduct ψ determined by the formula

$$\psi(D_i) = \sum_{t=0}^{i} D_t \otimes D_{i-t}.$$

It is bigraded by length and topological degrees ($|D_i| = i$), which behave skew-additively under multiplication [10, Def. 2.1].

The $\mathbb{F}_2$-cohomology of any space is an unstable algebra over the Steenrod algebra, and there is a correspondence between unstable A-algebras and unstable K-algebras, completely determined by iterating the conversion formulae: On any element x_l of degree l, and for all $j \geq 0$, one has

$$D_j x_l = Sq^{l-j} x_l, \text{ equivalently, } Sq^j x_l = D_{l-j} x_l.$$

Since the degree of the element is involved in the conversion, and this changes as operations are composed, the algebra structures of A and K are very different, and the skew additivity of the bigrading in K reflects this.

The requirements for an unstable K-algebra, corresponding to the nature and requirements of an unstable A-algebra, are: On any element x_l of degree l,

$$D_l x_l = x_l,\ D_j x_l = 0 \text{ for } j > l, \text{ and } D_0 x_l = x_l^2.$$

Finally, and used in our proofs, the K-algebra structure obeys the (Cartan) formula according to the coproduct Δ in K:

$$D_i(xy) = \sum_{t=0} D_t(x) D_{i-t}(y).$$

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