ISSN 1472-2739 (on-line) 1472-2747 (printed) **1433**

Algebraic & Geometric Topology
Volume 5 (2005) 1433–1450
Published: 17 October 2005

Degree one maps between small 3-manifolds and Heegaard genus

Michel Boileau
Shicheng Wang

Abstract We prove a rigidity theorem for degree one maps between small 3-manifolds using Heegaard genus, and provide some applications and connections to Heegaard genus and Dehn surgery problems.

AMS Classification 57M50, 57N10

Keywords Degree one map, small 3-manifold, Heegaard genus

1 Introduction

All terminology not defined in this paper is standard, see [He] and [Ja].

Let M and N be two closed, connected, orientable 3-manifolds. Let H be a (not necessarily connected) compact 3-submanifold of N. We say that a degree one map $f : M \to N$ is a *homeomorphism outside* H if $f : (M, M - \text{int} f^{-1}(H), f^{-1}(H)) \to (N, N - \text{int} H, H)$ is a map between the triples such that the restriction $f| : M - \text{int} f^{-1}(H) \to N - \text{int} H$ is a homeomorphism. We say also that f is a *pinch* and N is obtained from M by *pinching* $W = f^{-1}(H)$ *onto* H.

Let H be a compact 3-manifold (not necessarily connected). We use $g(H)$ to denote the *Heegaard genus* of H, that is the minimal number of 1-handles used to build H.

We define $mg(H) = max\{g(H_i),\ H_i \textit{ runs over components of } H\}$. It is clear that $mg(H) \le g(H)$ and $mg(H) = g(H)$ if H is connected.

A path-connected subset X of a connected 3-manifold is said to carry $\pi_1 M$ if the inclusion homomorphism $\pi_1 X \to \pi_1 M$ is surjective.

In this paper, any incompressible surface in a 3-manifold is 2-sided and is not the 2-sphere. A closed 3-manifold M is *small* if it is orientable, irreducible and if it contains no incompressible surface.

It has been observed by Kneser, Haken and Waldhausen ([Ha], [Wa], see also [RW] for a quick transversality argument) that a degree one map $M \to N$ between two closed, orientable 3-manifolds is homotopic to a map which is a homeomorphism outside a handlebody corresponding to one side of a Heegaard splitting of N. This fact is known as "any degree one map between 3-manifolds is homotopic to a pinch".

A main result of this paper is the following rigidity theorem.

Theorem 1 *Let M and N be two closed, small 3-manifolds. If there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$, then either:*

(1) *There is a component U of H which carries $\pi_1 N$ and such that $g(U) \geq g(N)$, or*

(2) *M and N are homeomorphic.*

Remark 1 Given M and N two non-homeomorphic small 3-manifolds , Theorem 1 implies that N cannot be obtained from M by a sequence of pinchings onto submanifolds of genus smaller than $g(N)$. However Theorem 1 does not hold when M is not small. Below are easy examples:

- Let $f : P\#N \to N$ be a degree one map defined by pinching P to a 3-ball in N. Then f is a homeomorphism outside the 3-ball, which is genus zero and does not carry $\pi_1 N$.
- Let k be a knot in a closed, orientable 3-manifold N and let F be a once punctured closed surface. Let M be the 3-manifold obtained by gluing the boundaries of $F \times S^1$ and of $E(k)$ in such a way that $\partial F \times \{x\}$ is matched with the meridian of k, $x \in S^1$. Then a degree one map $f : M \to N$ pinching $F \times S^1$ to a tubular neigborhood $\mathcal{N}(k)$ of k, is a homeomorphism outside a handlebody of genus 1. If $\pi_1 N$ is not cyclic or tivial, then $g(\mathcal{N}(k)) < g(N)$ and $\mathcal{N}(k)$ does not carry $\pi_1 N$.

The pinched part of a degree one map between closed, orientable non-homeomorphic surfaces has incompressible boundary [Ed]. The following straigtforward corollary of Theorem 1 gives an analogous result for small 3-manifolds:

Corollary 1 *Let M and N be two closed, small, non-homeomorphic 3-manifolds. Let $f : M \to N$ be a degree one map and let $V \cup H = N$ be a minimal genus Heegaard splitting for N. Then the map f can be homotoped to be a homeomorphism outside H such that $f^{-1}(H)$ is ∂-irreducible.*

Remark 2 Corollary 1 remains true for any strongly irreducible heegaard splitting of N. Then the argument, using Casson-Gordon's result [CG], is essentially the same as [Le, Theorem 3.1], even if in [Le] it is only proved for the case $M = S^3$ and N a homotopy 3-sphere. The proof in [Le] is based on his main result [Le, Theorem 1.3], but one can also use a direct argument from degree one maps.

Theorem 1 follows directly from two rather technical Propositions (Proposition 1 and Proposition 2). Theorem 1 and its proof lead to some results about Heegaard genus of small 3-manifolds and Dehn surgery on null-homotopic knots.

Theorem 2 *Let M be a closed, small 3-manifold. Let $F \subset M$ be a closed, orientable surface (not necessary connected) which cuts M into finitely many compact, connected 3-manifolds $U_1, \dots, U_n$. Then there is a component U_i which carries $\pi_1 M$ and such that $g(U_i) \geq g(M)$.*

Remark 3 In general (see [La]) one has only the upper bound:

$$g(M) \leq \sum_{i=1}^{n} g(U_i)) - g(F).$$

Suppose that k is a null-homotopic knot in a closed orientable 3-manifold M. Its unknotting number $u(k)$ is defined as the minimal number of self-crossing changes needed to transform it into a trivial knot contained in a 3-ball in M.

Theorem 3 *Let k be a null-homotopic knot in a closed, small 3-manifold M. If $u(k) < g(M)$, then every closed 3-manifolds obtained by a non-trivial Dehn surgery along k is not small. In particular k is determined by its complements.*

This article is organized as follows.

In Section 2 we state and prove Proposition 1 which is the first step in the proof of Theorem 1. The second step, given by Proposition 2 is proved in Section 3; then Theorem 1 follows from these two propositions. Section 4 is devoted to the proof of Theorem 2, and Section 5 to the proof of Theorem 3.

Acknowledgements We would like to thank both the referee and Professor Scharlemann for their suggestions which enhance the paper. The second author is partially supported by MSTC and NSFC.

2 Making the preimage of H ∂-irreducible

The first step of the proof of Theorem 1 is given by the following proposition:

Proposition 1 *Let M and N be two closed, connected, orientable, irreducible 3-manifolds which have the same first Betti number, but are not homeomorphic.*

Suppose there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside a compact irreducible 3-submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$. Then there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset H_0$ such that:

- $\partial H \neq \emptyset$;
- $mg(H) \leq mg(H_0)$,
- *Any connected component of $f^{-1}(H)$ is either ∂-irreducible or a 3-ball, and there is at least one component of $f^{-1}(H)$ which is ∂-irreducible.*

Remark 4 Since M is not homeomorphic to N it is clear that at least one component of $f^{-1}(H)$ is not a 3-ball.

Proof In the whole proof, 3-manifolds M and N are supposed to meet all hypotheses given in the first paragraph of Proposition 1.

By the assumption there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$.

Let $\mathcal{H}_0$ be the set of all 3-submanifolds $H \subset H_0$ such that:

(1) There is a degree one map $f : M \to N$ which is a homeomorphism outside H;

(2) $\partial H \neq \emptyset$;

(3) $mg(H) \leq mg(H_0)$;

(4) H is irreducible.

For an element $H \in \mathcal{H}_0$, its complexity is defined as a pair

$$c(H) = (\sigma(\partial H), \pi_0(H))$$

with the lexicographic order, and where $\sigma(\partial H)$ is the sum of the squares of the genera of the components of ∂H, and $\pi_0(H)$ is the number of components of H.

Remark on $c(H)$ The second term of $c(H)$ is not used in this section, but will be used in the next two sections.

Clearly $\mathcal{H}_0$ is not the empty set, since by assumption $H_0 \in \mathcal{H}_0$.

A compressing disk for ∂H in H is a properly embedded 2-disk $(D, \partial D) \subset (H, \partial H)$ such that $\partial D = D \cap \partial H$ is an essential simple closed curve on ∂H (i.e. does not bound a disk on ∂H). In the following we shall denote by $H \backslash \mathcal{N}(D)$ the compact 3-manifold obtained from H by removing an open product neighborhood of D. The operation of removing such neighborhood is called *splitting* H *along* D.

Lemma 1 *Let* H *be a compact orientable 3-manifold and let* $(D, \partial D) \subset (H, \partial H)$ *be a compressing disk . Then* $mg(H_*) \leq mg(H)$, *where* $H_* = H \backslash \mathcal{N}(D)$ *is obtained by splitting* H *along* D. *Moreover* $c(H_*) < c(H)$.

Proof By Haken's lemma for boundary-compressing disk ([BO], [CG]), a minimal genus Heegaard surface for H can be isotoped to meet D along a single simple closed curve. It follows that $mg(H_*) \leq mg(H)$.

Since ∂D is an essential simple closed curve on ∂H, it is easy to see that $\sigma(\partial H_*) < \sigma(\partial H)$, therefore $c(H_*) < c(H)$. □

The proof of Proposition 1 follows from the following:

Lemma 2 *Let* $H \in \mathcal{H}_0$ *be an element which realizes the minimal complexity, then any component of* $f^{-1}(H)$ *which is not a 3-ball is* ∂*-irreducible.*

Proof Let $W_0 \subset W = f^{-1}(H)$ be a component which is not homeomorphic to a 3-ball. Such a component exits since M is not homeomorphic to N. To prove that W_0 is ∂-irreducible, we argue by contradiction.

If ∂W_0 is compressible in W, there is a compressing disc $(D, \partial D) \to (W, \partial W)$ whose boundary is an essential simple closed curve on ∂W.

Since $f : M \to N$ is a homeomorphism outside the submanifold $H \subset N$ the restriction $f| : (W, \partial W) \to (H, \partial H)$ maps ∂W homeomorphically onto ∂H. Therefore $f(\partial D)$ is an essential simple closed curve on ∂H which bounds the immersed disk $f(D)$ in H. By Dehn's Lemma, $f(\partial D)$ bounds an embedded disc D^* in H.

Lemma 3 *By a homotopy of* f, *supported on* $W = f^{-1}(H)$ *and constant on* ∂W, *we can achieve that:*

- *$f|: W \to H$ is a homeomorphism in a collar neighborhood of $\partial W \cup D$,*
- *$f|^{-1}(D^*) = D \cup S$, where S is a closed orientable surface.*

Proof We define a homotopy $F: W \times [0,1] \to H$ by the following steps:

(1) $F(x,0) = f(x)$ for every $x \in W$;

(2) $F(x,t) = F(x,0)$ for every $x \in \partial f^{-1}(H) = \partial W$ and for every $t \in [0,1]$;

(3) Then we extend $F(x,1): D \times \{1\} \to D^*$ by a homeomorphism.

We have defined F on $D \times \{0\} \cup \partial D \times [0,1] \cup D \times \{1\}$ which is homeomorphic to a 2-sphere S^2. Since H is irreducible, by the Sphere theorem $\pi_2(H) = \{0\}$. Hence:

(4) We can extend F to $D \times [0,1]$;

Now F has been defined on $W \times \{0\} \cup \partial W \times [0,1] \cup D \times [0,1]$, which is a deformation retract of $W \times [0,1]$, therefore:

(5) We can finally extend F on $W \times [0,1]$.

After this homotopy we may assume that $f(x) = F(x,1)$, for every $x \in W$. Then by construction this new f sends $\partial W \cup D$ homeomorphically to $\partial H \cup D^*$. By transversality, we may further assume that $f|: W \to H$ is a homeomorphism in a collar neighborhood of $\partial W \cup D$ and that $f|^{-1}(D^*) = D \cup S$, where S is a closed surface. □

The following lemma will be useful:

Lemma 4 *Suppose $f: M \to N$ is a degree one map between two closed orientable 3-manifolds with the same first Betti number $\beta_1(M) = \beta_1(N)$. Then $f_\star: H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is an isomorphism.*

Proof Since $f: M \to N$ is a degree one map, by [Br, Theorem I.2.5], there is a homomorphism $\mu: H_2(N;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ such that $f_\star \circ \mu: H_2(N;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is the identity, where $f_\star: H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is the homomorphism induced by f.

In particular $f_\star: H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is surjective. Then the injectivity follows from the fact that $H_2(M;\mathbb{Z})$ and $H_2(N;\mathbb{Z})$ are torsion free abelian groups with the same finite rank $\beta_2(M) = \beta_1(M) = \beta_1(N) = \beta_2(M)$. □

Since the degree one map $f : M \to N$ is a homeomorphism outside H, the Mayer-Vietoris sequence and Lemma 4 imply that $f_\star : H_2(W;\mathbb{Z}) \to H_2(H;\mathbb{Z})$ is an isomorphism.

Let S' be a connected component of S. Since $f(S') \subset D^*$, the homology class $[f(S')] = f_\star([S'])$ is zero in $H_2(H,\mathbb{Z})$. Hence the homology class $[S']$ is zero in $H_2(W,\mathbb{Z})$, because $f_\star : H_2(W,\mathbb{Z}) \to H_2(H,\mathbb{Z})$ is an isomorphism. It follows that S' is the boundary of a compact submanifold of W. Therefore S' divides W into two parts W_1 and W_2 such that $\partial W_2 = S'$ and W_1 contains $\partial W \cup D$.

We can define a map $g : W \to H$ such that:

(a) $g|_{W_1} = f|_{W_1}$ and $g(W_2) \subset D^*$.

Then by slightly pushing the image $g(W_2)$ to the correct side of D^*, we can improve the map $g : W \to H$ such that:

(b) $g|\partial W = f|\partial W$,

(c) $g^{-1}(D^*) = D \cup (S \setminus S')$ and $g : \mathcal{N}(D) \to \mathcal{N}(D^*)$ is a homeomorphism.

After finitely many such steps we get a map $h : W \to H$ such that:

(b) $h|\partial W = f|\partial W$,

(d) $h^{-1}(D^*) = D$ and $h : \mathcal{N}(D) \to \mathcal{N}(D^*)$ is a homeomorphism.

Let $H_* = H \backslash \mathcal{N}(D)$ obtained by splitting H along D. Then H_* is still an irreducible 3-submanifold of N with $\partial H_* \neq \emptyset$.

Now $f|_{M-\mathrm{int}W}$ and $h|_W$ together provide a degree one map $h : M \to N$. The transformation from f to h is supported in W, hence h is a homeomorphism outside the irreducible submanifold H_* of N.

Since H_* is obtained by splitting H along a compressing disk, we have $H_* \subset H_0$ and H_* belongs to $\mathcal{H}_0$. Moreover $mg(H_*) \leq mg(H)$ and $c(H_*) < c(H)$ by Lemma 1.

This contradiction finishes the proof of Lemma 2 and thus the proof of Proposition 1. □

3 Finding a closed incompressible surface in the domain

Since closed, orientable, small 3-manifolds are irreducible and have first Betti number equal to zero, Theorem 1 is a direct corollary of the following proposition:

Proposition 2 *Let M and N be two closed, connected, orientable, irreducible 3-manifolds whith the same first Betti number. Suppose that there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ such that for each connected component U of H_0, either $g(U) < g(N)$ or U does not carry $\pi_1 N$. Then either M contains an incompressible orientable surface or M is homeomorphic to N.*

Let (M, N) be a pair of closed orientable 3-manifolds such that there is a degree one map from M to N. We say that condition $(*)$ holds for the pair (M, N) if:

$$(*) \qquad \pi_1 N = \{1\} \quad \text{implies} \quad M = S^3.$$

For the proof we first assume that condition $(*)$ holds for the pair (M, N).

Proof of Proposition 2 under condition $(*)$

By the assumptions, there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$ and such that for each connected component U of H_0 either $g(U) < g(N)$ or U does not carry $\pi_1 N$. We assume moreover that M is not homeomorphic to N. Our goal is to show that M contains an incompressible surface.

Similar to Section 2, let $\mathcal{H}$ be the set of all 3-submanifolds $H \subset N$ such that:

(1) There is a degree one map $f : M \to N$ which is a homeomorphism outside H.

(2) ∂H is not empty.

(3) For each component U of H, either $g(U) < g(N)$ or U does not carry $\pi_1 N$.

(4) H is irreducible.

The set $\mathcal{H}$ is not empty by our assumptions.

The complexity $c(H) = (\sigma(\partial H), \pi_0(H))$ for the elements of $\mathcal{H}$ is defined like in Section 2.

Lemma 5 *Assume that there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$. If H contains 3-ball component B^3, then f can be homotoped to be a homeomorphism outside H_*, where $H_* = H - B^3$. Moreover if H is irreducible, then H_* is also irreducible.*

Proof By our assumption, there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$ and H contains a B^3 component. Since $f| : f^{-1}(\partial H) \to \partial H$ is a homeomorphism, then $f^{-1}(\partial B^3)$ is a 2-sphere $S^2_* \subset M$. Since M is irreducible, S^2_* bounds a 3-ball B^3_* in M. Then either

(a) $M - \mathrm{int} f^{-1}(B^3) = B^3_*$, or

(b) $f^{-1}(B^3) = B^3_*$.

In case (a), $N = f(B^3_*) \cup B^3$ is a union of two homotopy 3-balls with their boundaries identified homeomorphically, and clearly $\pi_1 N = \{1\}$. So $M = S^3$ by assumption $(*)$. Hence (b) holds in either case.

In case (b), by a homotopy of f supported in $f^{-1}(B^3)$, we can achieve that $f| : f^{-1}(B^3) \to B^3$ is a homeomorphism. Then f becomes a homeomorphism outside the irreducible 3-submanifold $H_* \subset N$, obtained from H by deleting the 3-ball B^3.

The last sentence in Lemma 5 is obviously true. □

Let $H \in \mathcal{H}$ be an element which realizes the minimal complexity. By Lemma 5 no component of H is a 3-ball, hence no component of ∂H is a 2-sphere since H is irreducible. Therefore no component of $f^{-1}(H)$ is a 3-ball and $\partial f^{-1}(H)$ is incompressible in $f^{-1}(H)$ by the proof of Lemma 2.

Since $f : M - \mathrm{int} f^{-1}(H) \to N - \mathrm{int} H$ is a homeomorphism, $\partial f^{-1}(H)$ is incompressible in $M - \mathrm{int} f^{-1}(H)$ if and only if ∂H is incompressible in $N - \mathrm{int} H$. For simplicity we will set $V = N - \mathrm{int} H$, then $N = V \cup H$.

Then the proof of Proposition 2 under condition $(*)$ follows from:

Lemma 6 *If ∂H is compressible in V, then there is $H_* \in \mathcal{H}$ such that $c(H_*) < c(H)$.*

Proof Suppose ∂H is compressible in V. Let $(D, \partial D) \subset (V, \partial V)$ be a compressing disc. By surgery along D, we get two submanifolds H_1 and V_1 as follows:

$$H_1 = H \cup \mathcal{N}(D), \qquad V_1 = V \backslash \mathcal{N}(D).$$

Since H_1 is obtained from H by adding a 2-handle, for each component U' of H_1 there is a component U of H_0 such that $g(U') \leq g(U)$ and $\pi_1 U'$ is a quotient of $\pi_1 U$, hence H_1 verifies the defining condition (3) of $\mathcal{H}$. Moreover f is still a homeomorphism outside H_1 because H_1 contains H as a subset.

Clearly $\partial H_1 \neq \emptyset$. Hence H_1 satisfies also the defining conditions (1) and (2) of $\mathcal{H}$. We notice that $c(H_1) < c(H)$ because $\sigma(\partial H_1) < \sigma(\partial H)$.

We will modify H_1 to become $H_* \in \mathcal{H}$ with $c(H_*) \leq c(H_1)$. The modification will be divided into two steps carried by Lemma 8 and Lemma 9 below. First the following standard lemma will be useful:

Lemma 7 *Suppose U is a connected 3-submanifold in N and let $B^3 \subset N$ be a 3-ball with $\partial B^3 = S^2$.*

(i) *Suppose $S^2 \subset \partial U$. If $\text{int}\, U \cap B^3 \neq \emptyset$, then $U \subset B^3$. Otherwise $U \cap B^3 = S^2$.*

(ii) *if $\partial U \subset B^3$, then either $U \subset B^3$, or $N - \text{int}\, U \subset B^3$.*

Proof For (i): Suppose first $\text{int}\, U \cap B^3 \neq \emptyset$. Let $x \in \text{int}\, U \cap B^3$. Since U is connected, then for any $y \in U$, there is a path $\alpha \subset U$ connecting x and y. Since S^2 is a component of ∂U, α does not cross S^2. Hence $\alpha \subset B^3$ and $y \in B^3$, therefore $U \subset B^3$.

Now suppose $\text{int}\, U \cap B^3 = \emptyset$. Let $x \in \partial U \cap B^3$. If $x \in \text{int}\, B^3$, then there is $y \in \text{int}\, U \cap B^3$. It contradicts the assumption. So $x \in \partial B^3 = S^2$.

For (ii): Suppose that U is not a subset of B^3, then there is a point $x \in U \cap (N - \text{int} B^3)$. Let $y \in N - \text{int}\, U$. If $y \in N - \text{int}\, B^3$, there is a path α in $N - \text{int}\, B^3$ connecting x and y, since $N - \text{int}\, B^3$ is connected. This path α does not meet ∂U, because $\partial U \subset B^3$. This would contradict that $x \in U$ and $y \in N - \text{int}\, U$. Hence we must have $y \in B^3$, and therefore $N - \text{int}\, U \subset B^3$. □

Lemma 8 *Suppose H_1 meets the defining conditions (1), (2) and (3) of the set $\mathcal{H}$. Then H_1 can be modified to be a 3-submanifold $H_* \subset N$ such that:*

(i) *∂H_* contains no 2-sphere;*

(ii) *$c(H_*) \leq c(H_1)$;*

(iii) *H_* still meets the the defining conditions (1) (2) (3) of $\mathcal{H}$.*

Proof We suppose that ∂H_1 contains a 2-sphere component S^2, otherwise Lemma 8 is proved. Then S^2 bounds a 3-ball B^3 in N since N is irreducible. We consider two cases:

Case (a) *S^2 does bound a 3-ball B^3 in H_1.*

In this case B^3 is a component of H_1. By Lemma 5, f can be homotoped to be a homeomorphism outside $H_2 = H_1 - B^3$.

Case (b) *S^2 does not bound a 3-ball B^3 in H_1.*

Let H_1' be the component of H_1 such that $S^2 \subset \partial H_1'$. By Lemma 7 (i), either:

(b′) $H_1' \subset B^3$, or

(b″) $H_1' \cap B^3 = S^2$.

In case (b′), let $H_2 = H_1 - B^3$. By Lemma 5 f can be homotoped to be a homeomorphism outside H_2. Note $H_2 \neq \emptyset$, otherwise M and N are homeomorphic, which contradicts our assumption.

In case (b″), let $H_2 = H_1 \cup B^3$, then ∂H_2 has at least one component less than ∂H_1. Since we are enlarging H_1, f is a homeomorphism outside H_2.

It is easy to check that in each case (a), (b′), (b″) the components of H_2 verify the defining condition (3) of $\mathcal{H}$ and $c(H_2) \leq c(H_1) < c(H)$. Moreover H_2 is not empty because M and N are not homeomorphic, and $\partial H_2 \neq \emptyset$ since $g(H_2) \leq g(H_1) < g(N)$. Hence each of the transformations (a), (b′) and (b″) preserves properties (ii) and (iii) in the conclusion of Lemma 8. Since each one strictly reduces the number of components of H_1 or of ∂H_1, after a finite number of such transformations we reach a 3-submanifold H_* of N such that H_* meets the properties (ii) and (iii) of Lemma 8, and ∂H_* contains no 2-sphere components. This proves Lemma 8. □

Lemma 9 *Suppose that H_1 meets conditions (i), (ii) and (iii) in the conclusion of Lemma 8. Then H_1 can be modified to be a 3-submanifold H_* of N such that:*

(a) *H_* is irreducible;*

(b) *$c(H_*) \leq c(H_1)$ is not increasing;*

(c) *H_* still meets the the defining conditions (1), (2), (3) of $\mathcal{H}$.*

In particular H_* belongs to $\mathcal{H}$.

Proof If there is an essential 2-sphere S^2 in H_1, it must separate N since N is irreducible. Let H_1' be the component of H_1 containing S^2. The 2-sphere S^2 induces a connected sum decomposition of H_1': it separates H_1' into two connected parts $K_\circ$ and $K_\circ'$, such that:

$$H_1' = K \#_{S^2} K' = K_\circ \cup_{S^2} K_\circ',$$

$K_\circ \subset H_1$ (resp. $K_\circ' \subset H_1$) is homeomorphic to a once punctured K (resp. a once punctured K').

By Haken's Lemma, we have:

$$g(H_1') = g(K) + g(K').$$

Neither $K_\circ$ nor $K_\circ'$ is a n-punctured 3-sphere, $n \geq 0$, because ∂H_1 contains no 2-sphere component, hence:

$$g(K) < g(H_1') \qquad \text{and} \qquad g(K') < g(H_1')$$

Since N is irreducible, S^2 bounds a 3-ball B^3 in N. We may assume that $\text{int}K_\circ \cap B^3 = \emptyset$ and $\text{int}K_\circ' \cap B^3 \neq \emptyset$. By Lemma 7 (i), we have $K_\circ \cap B^3 = S^2$ and $K_\circ' \subset B^3$.

Moreover $\partial H_1' \cap B^3 \neq \emptyset$, otherwise $K_\circ'$ is homeomorphic to B^3, in contradiction with the assumption that S^2 is a 2-sphere of connected sum.

Lemma 10 *$\partial H_1'$ is not a subset of B^3.*

Proof We argue by contradiction. If $\partial H_1'$ is a subset of B^3, we have $N - \text{int}H_1' \subset B^3$ by Lemma 7 (ii), since H_1' is not a subset of B^3. Then:

$$N = H_1' \cup (N - \text{int}H_1') = H_1' \cup B^3 = (K_\circ \#_{S^2} K_\circ') \cup B^3 = K_\circ \cup_{S^2} B^3 = K.$$

Hence K is homeomorphic to the whole N. If $g(H_1') < g(N)$, this contradicts the fact that $g(K) < g(H_1') < g(N)$. If H_1' does not carry $\pi_1 N$ this contadicts the fact that $K \subset H_1'$. □

By Lemma 10, $\partial H_1'$ (and therefore ∂H_1) has components disjoint from B^3. Therefore if we replace H_1 by $H_2 = H_1 \cup B^3$, then ∂H_2 is not empty and it has no component which is a 2-sphere. Moreover the application of Haken's Lemma above shows that $g(H_2) < g(H_1)$.

Since we are enlarging H_1, f is a homeomorphism outside H_2, and clearly H_2 still meets the the defining condition (3) of $\mathcal{H}$. Moreover $c(H_2) \leq c(H_1)$. Hence the transformation from H_1 to H_2 preserves properties (b) and (c) in the conclusion of Lemma 9. Since it strictly reduces $g(H_1)$, after a finite number of such transformations we will reach a 3-submanifolds $H_* \subset N$ such that H_* meets conditions (b) and (c) in the conclusion of Lemma 9, but does not contain any essential 2-sphere. This proves Lemma 9. □

Lemma 8 and Lemma 9 imply Lemma 6. Hence we have proved Proposition 2 under condition $(*)$. □

Proof of Proposition 2 Let M and N be two closed, small 3-manifolds which are not homeomorphic. Suppose there is degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$ such that: for each component U of H, either $g(U) < g(N)$ or U does not carry $\pi_1 N$.

Condition $(*)$ in the above proof of Proposition 2 is only used in the proof of Lemma 5, when H contains a 3-ball component B^3 and that $M - \mathrm{int} f^{-1}(B^3) = B^3_*$ and $f^{-1}(B^3) \neq B^3_*$. Indeed we can now prove that this case cannot occur.

If this case happens then $\pi_1 N = \{1\}$ and thus $mg(H) < g(N)$, since every component of H carries $\pi_1 N$. By replacing $f^{-1}(B^3)$ by a 3-ball $B^3_\#$, we obtain a degree one map $\bar{f} : S^3 = B^3_* \cup B^3_\# \to N$ defined by $\bar{f}|B_* = f|B_*$ and $\bar{f}| : B^3_\# \to B_3$ is a homeomorphism. Then $\bar{f} : S^3 \to N$ is a map which is a homeomorphism outside a submanifold $H' = H - B^3$. Clearly $mg(H') = mg(H) < g(N)$. Furthermore condition $(*)$ now holds.

Since Proposition 2 has been proved under condition $(*)$, we have that N must be homeomorphic to S^3, since S^3 does not contain any incompressible surface. It would follow that $mg(H) < 0$, which is impossible.

The proof of Proposition 2, and hence of Theorem 1 is now complete. □

4 Heegaard genus of small 3-manifolds

This section is devoted to the proof of Theorem 2.

Let M be a closed orientable irreducible 3-manifold. Let $F \subset M$ be a closed orientable surface (not necessary connected) which splits M into finitely many compact connected 3-manifolds $U_1, \ldots, U_n$.

Let $M \backslash \mathcal{N}(F)$ be the manifold M split along the surface F. We define the complexity of the pair (M, F) as

$$c(M,F) = \{\sigma(F), \pi_0(M \backslash \mathcal{N}(F))\},$$

where $\sigma(F)$ is the sum of the squares of the genera of the components of F and $\pi_0(M \backslash \mathcal{N}(F))$ is the number of components of $M \backslash \mathcal{N}(F)$.

Let $\mathcal{F}$ be the set of all closed surfaces F such that for each component U_i of $M \setminus F$, either $g(U_i) < g(M)$ or U_i does not carry $\pi_1 M$.

Remark 5 This condition implies that the surface $F \neq \emptyset$ for every $F \in \mathcal{F}$.

With the hypothesis of Theorem 2, the set $\mathcal{F}$ is not empty. Let $F \in \mathcal{F}$ be a surface realizing the minimal complexity. Then the following Lemma implies Theorem 2.

Lemma 11 *A surface $F \in \mathcal{F}$ realizing the minimal complexity contains no 2-sphere component and is incompressible.*

Proof The arguments are analogous to those used in the proof of Propositions 2. We argue by contradiction.

Suppose that F contains a 2-sphere component S^2. It bounds a 3-ball $B^3 \subset M$, since M is irreducible. Let U_1 and U_2 be the closures of the components of $M \setminus \mathcal{N}(F)$ which contain S^2. Then by Lemma 7 (i), either:

- $U_2 \subset B^3$ and $U_1 \cap B^3 = S^2$, or
- $U_1 \subset B^3$ and $U_2 \cap B^3 = S^2$.

Since those two cases are symmetric, we may assume that we are in the first case. We consider the surface F' corresponding to the decomposition $\{U'_1, \dots, U'_k\}$ of M with $U'_1 = U_1 \cup B^3$, after forgetting all $U_i \subset B^3$ and then re-indexing the remaining U_i's to be $U'_2, \dots, U'_k$. This operation does not increase the Heegaard genus of any one of the components of the new decomposition. Moreover if U_1 does not carry $\pi_1 M$, the same holds for U'_1. Hence F' still belongs to $\mathcal{F}$. However, this operation strictly decreases the number of components of F, hence $c(F') < c(F)$, in contradiction with our choice of F.

Suppose that the surface F is compressible. Then some essential simple closed curve γ on F bounds an embedded disk in M. Let D' be a such a compression disk with the minimum number of circles of intersection in $\text{int}D' \cap F$. Then a subdisk of D' bounded by an innermost circle of intersection is contained inside one of the U_i, say U_1.

Let $(D, \partial D) \subset (U_1, F \cap \partial U_1)$ be such an innermost disk. Let U_2 be adjacent to U_1 along F, such that $\partial D \subset \partial U_2$. By surgery along D, we get a new surface F' which gives a new decomposition $\{U'_1, \dots, U'_n\}$ of M as follows:

$$U'_1 = U_1 \backslash \mathcal{N}(D), \qquad U'_2 = U_2 \cup \mathcal{N}(D), \qquad U'_i = U_i, \text{ for } i \geq 3.$$

Then $g(U'_i) \leq g(U_i)$, for $i = 1, \dots, n$. Moreover if U_i does not carry $\pi_1 M$, the same holds for U'_i. Hence $F' \in \mathcal{F}$. However, $\sigma(F') < \sigma(F)$ since ∂D is an essential circle on F. Therefore $c(F') < c(F)$ and we reach a contradiction. □

5 Null-homotopic knot with small unknotting number

In this section we prove Theorem 3.

Suppose M is a closed, small 3-manifold and $k \subset M$ is a null-homotopic knot with $u(k) < g(M)$. Then clearly M is not the 3-sphere.

If k is a non-trivial knot in a 3-ball $B^3 \subset M$. Then the compact 3-manifold $B^3(k,\lambda)$ obtained by any non-trivial surgery of slope λ on k will not be a 3-ball by [GL]. Therefore $M(k,\lambda)$ contains an essential 2-sphere.

Hence below we assume that k is not contained in a 3-ball.

Since the knot $k \subset M$ is null-homotopic with unknotting number $u(k)$, k can be obtained from a trivial knot $k' \subset B^3 \subset M$ by $u(k)$ self-crossing changes. Let $D' \subset M$ be an embedded disk bounded by k'. If we let D' move following the self-crossing changes from k' to k, then each self-crossing change corresponds to an identification of pairs of arcs in D'. Hence one obtains a singular disk Δ in M with $\partial\Delta = k$ and with $u(k)$ clasp singularities. Since Δ has the homotopy type of a graph, its regular neighborhood $\mathcal{N}(\Delta)$ is a handlebody of genus $g(\mathcal{N}(\Delta)) = u(k) < g(M)$.

First we prove the following lemma which is a particular case of a more general result about Dehn surgeries on null-homotopic knots, obtained in [BBDM]. Since this paper is not yet available, we give here a simpler proof in this particular case.

Lemma 12 *With the hypothesis above, if the slope α is not the meridian slope of k, then $M(k,\alpha)$ is not homeomorphic to M.*

Proof Since M is irreducible and $k \subset M$ is not contained in a 3-ball, $M - \text{int}\mathcal{N}(k)$ is irreducible and ∂-irreducible. Hence $1 \leq u(k) < g(M)$ and M cannot be a lens space.

Let consider the set $\mathcal{W}$ of compact, connected, orientable, 3-submanifolds $W \subset M$ such that:

(1) $k \subset W$ is null-homotopic in W;

(2) there is no 2-sphere component in ∂W;

(3) $g(W) < g(M)$.

By hypothesis the set $\mathcal{W}$ is not empty since a regular neighborhood $\mathcal{N}(\Delta)$ of a singular unknotting disk for k is a handlebody of genus ≥ 1.

Claim 1 *For a 3-submanifold $W_0 \in \mathcal{W}$ with a minimal complexity $c(W_0) = \sigma(\partial W_0)$, the surface ∂W_0 is incompressible in the exterior $M - \mathrm{int}\mathcal{N}(k)$.*

Proof If ∂W_0 is compressible in $M - \mathrm{int}W_0$, let $(D, \partial D) \hookrightarrow (M - \mathrm{int}W_0, \partial W_0)$ be a compression disk for ∂W_0. The 3-manifold $W_1 = W_0 \cup \mathcal{N}(D)$, obtained by adding a 2-andle to W_0, is a compact, connected submanifold of M containing k.

Any 2-sphere in ∂W_1 bounds a 3-ball in $M - \mathrm{int}\mathcal{N}(k)$ since it is irreductible. Hence after gluing some 3-ball along the boundary, we may assume that W_1 contains no 2-sphere component. Moreover $k \subset W_1$ is null-homotopic in W_1 and $g(W_1) \leq g(W_0) < g(M)$. It follows that $W_1 \in \mathcal{W}$. Since $c(W_1) < c(W_0)$ we get a contradiction.

If ∂W_0 is compressible in $W_0 - \mathrm{int}\mathcal{N}(k)$, let $(D, \partial D) \hookrightarrow, (W_0 - \mathrm{int}\mathcal{N}(k), \partial W_0)$ be a compression disk for ∂W_0. Let W_2 be the component of the 3-manifold $W_0 \backslash \mathcal{N}(D)$ which contains k. As above, after possibly gluing some 3-ball along the boundary, we may assume that ∂W_2 contains no 2-sphere component. The knot $k \subset W_2$ is null-homotopic in W_2, since it is null-homotopic in W_0 and $\pi_1 W_2$ is a factor of the free product decomposition of W_0 induced by the $\partial-$ *compression* disk D. Moreover by Lemma 1 $g(W_2) \leq g(W_0) < g(M)$. It follows that $W_2 \in \mathcal{W}$ and $c(W_2) < c(W_0)$. As above this contradicts the minimality of $c(W_0)$. □

To finish the proof of Lemma 12 we distinguish two cases:

(a) *The surface ∂W_0 is compressible in $W_0(k, \alpha)$* Then one can apply Scharlemann's theorem [Sch, Thm 6.1]. The fact that $k \subset W_0$ is null-homotopic rules out cases a) and b) of Scharlemann's theorem. Moreover by [BW, Prop.3.2] there is a degree one map $g : W_0(k, \alpha) \to W_0$, and thus there is a simple closed curve on ∂W_0 which is a compression curve both in $W_0(k, \alpha)$ and in W_0. Therefore case d) of Scharlemann's theorem cannot occure. The remaining case c) of Scharlemann's theorem shows that $k \subset W_0$ is a non-trivial cable of a knot $k_0 \subset W_0$ and that the surgery slope α corresponds to the slope of the cabling annulus. But then the manifold $M(k, \alpha)$ is the connected sum of a non-trivial Lens space with a manifold obtained by Dehn surgery along k_0. If $M(k, \alpha)$ is homeomorphic to the small 3-manifold M, then M and $M(k, \alpha)$ both would be homeomorphic to a Lens space, which is impossible since $1 \leq u(k) < g(M)$.

(b) *The surface ∂W_0 is incompressible in $W_0(k,\alpha)$* Since ∂W_0 is incompressible in $M - \mathcal{N}(k)$, it is incompressible in $M(k,\alpha)$. Therefore $M(k,\alpha)$ and M cannot be homeomorphic since M is a small manifold. □

It follows from [BW, Prop.3.2] that there is a degree one map $f : M(k,\alpha) \to M$ which is a homeomorphism outside $\mathcal{N}(\Delta)$. Since $g(\mathcal{N}(\Delta)) = u(k) < g(M)$, Theorem 3 is a consequence of Theorem 1 and Lemma 12. □

References

[BBDM] **M Boileau**, **S Boyer**, **M Domergue**, **Y Mathieu**, *Killing slopes*, in preparation

[BW] **M Boileau**, **S Wang**, *Non-zero degree maps and surface bundles over S^1*, J. Differential Geom. 43 (1996) 789–806

[BO] **F Bonahon**, **J-P Otal**, *Scindements de Heegaard des espaces lenticulaires*, Ann. Sci. École Norm. Sup. (4) 16 (1983) 451–466 (1984)

[Br] **W Browder**, *Surgery on simply-connected manifolds*, Springer-Verlag, New York (1972)

[CG] **A J Casson**, **C M Gordon**, *Reducing Heegaard splittings*, Topology Appl. 27 (1987) 275–283

[Ed] **A L Edmonds**, *Deformation of maps to branched coverings in dimension two*, Ann. of Math. (2) 110 (1979) 113–125

[GL] **C McA Gordon**, **J Luecke**, *Reducible manifolds and Dehn surgery.* Topology 35 (1996) 385–409

[Ha] **W Haken**, *On homotopy 3-spheres*, Illinois J. Math. 10 (1966) 159–178

[He] **J Hempel**, 3*-Manifolds*, Princeton University Press, Princeton, N. J. (1976)

[Ja] **W H Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, American Mathematical Society, Providence, RI (1980)

[La] **M Lackenby**, *The Heegaard genus of amalgamated 3-manifolds*, Geom. Dedicata 109 (2004) 139–145

[Le] **F Lei**, *Complete systems of surfaces in* 3*-manifolds*, Math. Proc. Cambridge Philos. Soc. 122 (1997) 185–191

[RW] **Y W Rong**, **S C Wang**, *The preimages of submanifolds*, Math. Proc. Cambridge Philos. Soc. 112 (1992) 271–279

[Sch] **M Scharlemann**, *Producing reducible* 3*-manifolds by surgery on a knot*, Topology 29 (1990) 481–500

[Wa] **F Waldhausen**, *On mappings of handlebodies and of Heegaard splittings*, from: "Topology of Manifolds (Proc. Inst. Univ. of Georgia, Athens, Ga. 1969)", Markham, Chicago, Ill. (1970) 205–211

Laboratoire Émile Picard, CNRS UMR 5580, Université Paul Sabatier
118 Route de Narbonne, F-31062 TOULOUSE Cedex 4, France
and
LAMA Department of Mathematics, Peking University
Beijing 100871, China

Email: boileau@picard.ups-tlse.fr, wangsc@math.pku.edu.cn

Received: 20 July 2005