ISSN 1364-0380 (on line) 1465-3060 (printed) **355**

Geometry & Topology
Volume 6 (2002) 355–360
Published: 28 June 2002

Burnside obstructions to the Montesinos–Nakanishi 3–move conjecture

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Abstract

Yasutaka Nakanishi asked in 1981 whether a 3–move is an unknotting operation. In Kirby's problem list, this question is called *The Montesinos–Nakanishi 3–move conjecture*. We define the nth Burnside group of a link and use the 3rd Burnside group to answer Nakanishi's question; ie, we show that some links cannot be reduced to trivial links by 3–moves.

AMS Classification numbers Primary: 57M27

Secondary: 20D99

Keywords: Knot, link, tangle, 3–move, rational move, braid, Fox coloring, Burnside group, Borromean rings, Montesinos–Nakanishi conjecture, branched cover, core group, lower central series, associated graded Lie ring, skein module

Proposed: Robion Kirby Received: 5 May 2002
Seconded: Walter Neumann, Vaughan Jones Revised: 19 June 2002

One of the oldest elementary formulated problems in classical Knot Theory is the 3–move conjecture of Nakanishi. A 3–move on a link is a local change that involves replacing parallel lines by 3 half-twists (Figure 1).

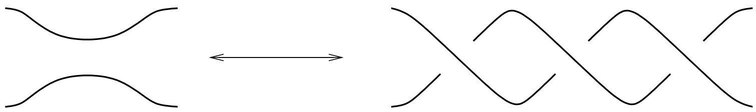

Figure 1

Conjecture 1 (Montesinos–Nakanishi, Kirby's problem list; Problem 1.59(1), [4]) *Any link can be reduced to a trivial link by a sequence of 3–moves.*

The conjecture has been proved to be valid for several classes of links by Chen, Nakanishi, Przytycki and Tsukamoto (eg, closed 4–braids and 4–bridge links).

Nakanishi, in 1994, and Chen, in 1999, have presented examples of links which they were not able to reduce: L_{2BR}, the 2–parallel of the Borromean rings, and $\hat{\gamma}$, the closure of the square of the center of the fifth braid group, ie, $\gamma = (\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$.

Remark 2 In [6] it was noted that 3–moves preserve the first homology of the double branched cover of a link L with Z_3 coefficients ($H_1(M_L^{(2)}; Z_3)$). Suppose that $\hat{\gamma}$ (respectively L_{2BR}) can be reduced by 3–moves to the trivial link T_n. Since $H_1(M_{\hat{\gamma}}^{(2)}; Z_3) = Z_3^4$, $H_1(M_{L_{2BR}}^{(2)}; Z_3) = Z_3^5$ and $H_1(M_{T_n}^{(2)}; Z_3) = Z_3^{n-1}$ where T_n is a trivial link of n components, it follows that $n = 5$ (respectively n=6).

We show below that neither $\hat{\gamma}$ nor L_{2BR} can be reduced by 3–moves to trivial links.

The tool we use is a non-abelian version of Fox n–colorings, which we shall call the nth Burnside group of a link, $B_L(n)$.

Definition 3 The nth Burnside group of a link is the quotient of the fundamental group of the double branched cover of S^3 with the link as the branch set divided by all relations of the form $a^n = 1$. Succinctly: $B_L(n) = \pi_1(M_L^{(2)})/(a^n)$.

Proposition 4 $B_L(3)$ *is preserved by 3–moves.*

Proof In the proof we use the core group interpretation of $\pi_1(M_L^{(2)})$. Let D be a diagram of a link L. We define (after [3, 2]) the associated core group $\Pi_D^{(2)}$ of D as follows: generators of $\Pi_D^{(2)}$ correspond to arcs of the diagram. Any crossing v_s yields the relation $r_s = y_i y_j^{-1} y_i y_k^{-1}$ where y_i corresponds to the overcrossing and y_j, y_k correspond to the undercrossings at v_s (see Figure 2). In this presentation of $\Pi_L^{(2)}$ one relation can be dropped since it is a consequence of others. Wada proved that $\Pi_D^{(2)} = \pi_1(M_L^{(2)}) * Z$, [10] (see [7] for an elementary proof using only Wirtinger presentation). Furthermore, if we put $y_i = 1$ for any fixed generator, then $\Pi_D^{(2)}$ reduces to $\pi_1(M_L^{(2)})$. The last part of our proof is illustrated in Figure 2. □

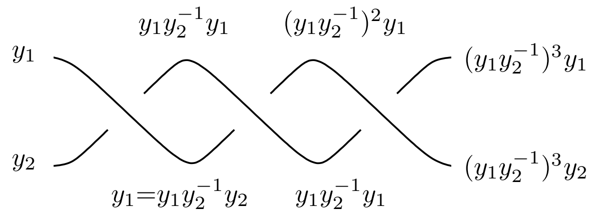

Figure 2

Lemma 5 *$B_{\hat{\gamma}}(3) = \{x_1, x_2, x_3, x_4 \mid a^3$ for any word a, $P_1, P_2, P_3, P_4\}$, where*

$$P_i = x_1 x_2^{-1} x_3 x_4^{-1} x_1^{-1} x_2 x_3^{-1} x_4 x_i x_4 x_3^{-1} x_2 x_1^{-1} x_4^{-1} x_3 x_2^{-1} x_1 x_i^{-1}.$$

Proof Consider the 5–braid $\gamma = (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^{10}$ (Figure 3). If we label initial arcs of the braid by x_1, x_2, x_3, x_4 and x_5, and use core relations (progressing from left to right) we obtain labels Q_1, Q_2, Q_3, Q_4 and Q_5 on the final arcs of the braid where

$$Q_i = x_1 x_2^{-1} x_3 x_4^{-1} x_5 x_1^{-1} x_2 x_3^{-1} x_4 x_5^{-1} x_i x_5^{-1} x_4 x_3^{-1} x_2 x_1^{-1} x_5 x_4^{-1} x_3 x_2^{-1} x_1.$$

For a group $\Pi_{\hat{\gamma}}^{(2)}$, of the closed braid $\hat{\gamma}$, we have relations $Q_i = x_i$. To obtain $\pi_1(M_{\hat{\gamma}}^{(2)})$ we can put $x_5 = 1$, and delete one relation, say $Q_5 x_5^{-1}$. These lead to the presentation of $B_{\hat{\gamma}}(3)$ described in the lemma. □

Theorem 6 *The links $\hat{\gamma}$ and L_{2BR} are not 3–move reducible to trivial links.*

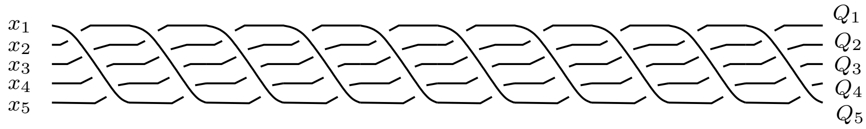

Figure 3

Proof Let $B(n,3)$ denote the classical free n generator Burnside group of exponent 3. As shown by Burnside [1], $B(n,3)$ is a finite group. Its order, $|B(n,3)|$, is equal to $3^{n+\binom{n}{2}+\binom{n}{3}}$. For a trivial link: $B_{T_k}(3) = B(k-1,3)$. In order to prove that $\hat{\gamma}$ and L_{2BR} are not 3–move reducible to trivial links, it suffices to show that $B_{\hat{\gamma}}(3) \neq B(4,3)$ and $B_{L_{2BR}}(3) \neq B(5,3)$ (see Remark 2). We have demonstrated these to be true both by manual computation, and by using the programs GAP, Magnus and Magma. More details in the case of $\hat{\gamma}$ are provided below.

For the manual calculations, one first observes that for any i, P_i is in the third term of the lower central series of $B(4,3)$. In particular, for $u = x_1x_2^{-1}x_3x_4^{-1}$ and $\bar{u} = x_1^{-1}x_2x_3^{-1}x_4$, one has $u\bar{u} \in [B(4,3), B(4,3)]$ and $P_i = [u\bar{u}, x_i\bar{u}]$. It is known ([9]), that $B(4,3)$ is of class 3 (the lower central series has 3 terms), and that the third term is isomorphic to Z_3^4 with basis: $e_1 = [[x_2,x_3],x_4]$, $e_2 = [[x_1,x_3],x_4]$, $e_3 = [[x_1,x_2],x_4]$ and $e_4 = [[x_1,x_2],x_3]$. It now takes an elementary linear algebra calculation (see Lemma 7 below) to show that P_1,P_2,P_3,P_4 form another basis of the third term of the lower central series of $B(4,3)$. Thus $|B_{\hat{\gamma}}(3)| = 3^{10}$. □

Lemma 7 *P_1,P_2,P_3, and P_4 form a basis of the third term of the lower central series of $B(4,3)$.*

Proof In the associated graded Lie ring $L(4,3)$ of $B(4,3)$ ([9]), the third term (denoted L_3) is isomorphic to Z_3^4 with basis e_1,e_2,e_3,e_4. In $L(4,3)$, which is a linear space over Z_3, one uses an additive notation and the bracket in the group becomes a (non-associative) product ([9]). In this notation $e_1 = x_2x_3x_4$, $e_2 = x_1x_3x_4$, $e_3 = x_1x_2x_4$ and $e_4 = x_1x_2x_3$. In the calculation expressing P_i in the basis we use the following identities in L_3 ([9]; page 89).

$$xyzt = 0, xyz = yzx = zxy = -xzy = -zyx = -yxz, xyy = 0.$$

Now we have: $P_i = (u\bar{u})(x_i\bar{u})(u\bar{u})^{-1}(x_i\bar{u})^{-1} = [(u\bar{u})^{-1},(x_i\bar{u})^{-1}] = [u\bar{u}, x_i\bar{u}]$ as the last term of the lower central series is in the center of $B(4,3)$. Furthermore, we have $u\bar{u} = x_1x_2^{-1}x_3x_4^{-1}x_1^{-1}x_2x_3^{-1}x_4 = [x_2^{-1}x_3x_4^{-1}, x_1^{-1}][x_3x_4^{-1}, x_2][x_4^{-1}, x_3^{-1}]$.

Writing P_i additively in L_3 one obtains:

$$P_i = ((-x_2 + x_3 - x_4)(-x_1) + (x_3 - x_4)x_2 + x_4x_3)(x_i - x_1 + x_2 - x_3 + x_4).$$

After simplifications one gets:

$$P_1 = -e_1, P_2 = e_1 + e_2, P_3 = e_1 - e_2 - e_3, \text{ and } P_4 = e_1 - e_2 + e_3 + e_4.$$

The matrix expressing P_i's in terms of e_i's is the upper triangular matrix with the determinant equal to 1. Therefore the lemma follows. □

A similar calculation establishes that $|B_{L_{2BR}}(3)| < |B(5,3)|$. $B(5,3)$ is of class 3 and has 3^{25} elements. Considering L_{2BR} as a closed 6–braid we note that $B_{L_{2BR}}(3)$ is obtained from $B(5,3)$ by adding 5 relations $R_1, ..., R_5$. Relations $\{R_i\}$ are in the last term of the lower central series of $B(5,3)$ (and of the associated graded algebra $L(5,3)$). Relations form a 4–dimensional subspace in $L_3 = Z_3^{10}$. Thus $|B_{L_{2BR}}(3)| = 3^{21}$.

For a computer verification showing that $B_{\hat{\gamma}}(3) \neq B(4,3)$ consider any presentation of $B(4,3)$ (eg, Magma solution by Mike Newman [5]) and add the relations P_i to obtain a presentation of $B_{\hat{\gamma}}(3)$. Using any of the algebra programs mentioned above, one verifies that $|B_{\hat{\gamma}}(3)| = 3^{10}$ while $|B(4,3)| = 3^{14}$.

The solution of the Nakanishi–Montesinos 3–move conjecture, presented above, is the first instance of application of Burnside groups of links. It was motivated by the analysis of cubic skein modules of 3–manifolds. The next step is the application of Burnside groups to rational moves on links. This, in turn, should have deep implications to the theory of skein modules [7].

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