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# Splitting the concordance group of algebraically slice knots

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## Abstract

As a corollary of work of Ozsvath and Szabo [8], it is shown that the classical concordance group of algebraically slice knots has an in nite cyclic summand and in particular is not a divisible group.

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Let $\mathcal{A}$ denote the concordance group of algebraically slice knots, the kernel of Levine's homomorphism $\phi\colon \mathcal{C} \to \mathcal{G}$, where $\mathcal{C}$ is the classical knot concordance group and $\mathcal{G}$ is Levine's algebraic concordance group [6]. Little is known about the algebraic structure of $\mathcal{A}$: it is countable and abelian, Casson and Gordon [2] proved that $\mathcal{A}$ is nontrivial, Jiang [5] showed it contains a subgroup isomorphic to $\mathbf{Z}^{\infty}$, and the author [7] proved that it contains a subgroup isomorphic to $\mathbf{Z}_2^{\infty}$. We add the following theorem, a quick corollary of recent work of Ozsvath and Szabo [8].

**Theorem 1** *The group $\mathcal{A}$ contains a summand isomorphic to $\mathbf{Z}$ and in particular $\mathcal{A}$ is not divisible.*

**Proof** In [8] a homomorphism $\tau\colon \mathcal{C} \to \mathbf{Z}$ is constructed. We prove that $\tau$ is nontrivial on $\mathcal{A}$. The theorem follows since, because $\mathrm{Im}(\tau)$ is free, there is the induced splitting, $\mathcal{A} = \mathrm{Im}(\tau) \oplus \mathrm{Ker}(\tau)$. No element representing a generator of $\mathrm{Im}(\tau)$ is divisible.

According to [8], $|\tau(K)| \le g_4(K)$, where $g_4$ is the 4–ball genus of a knot, and there is the example of the $(4,5)$–torus knot $T$ for which $\tau(T) = 6$. We will show that there is a knot $T'$ algebraically concordant to $T$ with $g_4(T') < 6$. Hence, $T \# -T'$ is an algebraically slice knot with nontrivial $\tau$, as desired.

Recall that $T$ is a fibered knot with fiber $F$ of genus $(4-1)(5-1)/2 = 6$. Let $V$ be the $12 \times 12$ Seifert matrix for $T$ with respect to some basis for $H_1(F)$. The quadratic form $q(x) = xVx^t$ on $\mathbf{Z}^{12}$ is equal to the form given by $(V + V^t)/2$. Using [3] the signature of this symmetric bilinear form can be computed to be 8, so $q$ is indefinite, and thus by Meyer's theorem [4] there is a nontrivial primitive element $z$ with $q(z) = 0$. Since $z$ is primitive, it is a member of a symplectic basis for $H_1(F)$. Let $V'$ be the Seifert matrix for $T$ with respect to that basis. The canonical construction of a Seifert surface with Seifert matrix $V'$ ([9], or see [1]) yields a surface $F'$ such that $z$ is represented by a simple closed curve on $F'$ that is unknotted in $S^3$. Hence, $F'$ can be surgered in the 4–ball to show that its boundary $T'$ satisfies $g_4(T') < 6$. Since $T'$ and $T$ have the same Seifert form, they are algebraically concordant. □

**Addendum** An alternative proof of Theorem 1 follows from the construction of knots with trivial Alexander polynomial for which $\tau$ is nontrivial, to appear in a forthcoming paper.

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