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Global rigidity of solvable group actions on S^1

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Abstract

In this paper we find all solvable subgroups of $\mathrm{Diff}^\omega(S^1)$ and classify their actions. We also investigate the C^r local rigidity of actions of the solvable Baumslag–Solitar groups on the circle.

The investigation leads to two novel phenomena in the study of infinite group actions on compact manifolds. We exhibit a finitely generated group Γ and a manifold M such that:

- Γ has exactly countably infinitely many effective real-analytic actions on M, up to conjugacy in $\mathrm{Diff}^\omega(M)$;
- every effective, real analytic action of Γ on M is C^r locally rigid, for some $r \geq 3$, and for every such r, there are infinitely many nonconjugate, effective real-analytic actions of Γ on M that are C^r locally rigid, but not C^{r-1} locally rigid.

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Introduction

This paper describes two novel phenomena in the study of infinite group actions on compact manifolds. We exhibit a finitely generated group Γ and a manifold M such that:

- Γ has exactly countably infinitely many effective real-analytic actions on M, up to conjugacy in $\mathrm{Diff}^{\omega}(M)$;
- every effective, real analytic action of Γ on M is C^r locally rigid, for some $r \geq 3$, and for every such r, there are infinitely many nonconjugate, effective real-analytic actions of Γ on M that are C^r locally rigid, but not C^{r-1} locally rigid.

In the cases we know of where an infinite group Γ has exactly countably many smooth effective actions on a manifold M, that countable number is finite, and indeed usually 0. While many actions have been shown to to be C^r locally rigid, in the cases where a precise cutoff in rigidity has been established, it occurs between $r = 1$ and $r = 2$. For a survey of some of the recent results on smooth group actions, see the paper of Labourie [9].

Our manifold M is the circle S^1 and our group Γ is the solvable Baumslag–Solitar group:

$$\mathrm{BS}(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle,$$

where $n \geq 2$.

As a natural by-product of our techniques, we obtain a classification of all solvable subgroups of $\mathrm{Diff}^{\omega}(S^1)$. We show that every such subgroup G is either conjugate in $\mathrm{Diff}^{\omega}(S^1)$ to a subgroup of a ramified affine group $\mathrm{Aff}^{s}(\mathbb{R})$, or, for some $m \in \mathbb{Z}$, the group $G^m := \{g^m : g \in G\}$ is abelian. The ramified affine groups are defined and their properties discussed in Section 2.2. Each ramified affine group is abstractly isomorphic to a direct product $\mathrm{Aff}_+(\mathbb{R}) \times H$, where $\mathrm{Aff}_+(\mathbb{R})$ is the group of orientation-preserving affine transformations of $\mathbb{R}$, and H is a subgroup of a finite dihedral group.

1 Statement of results

1.1 Notation and preliminary definitions

In places, we shall use two different analytic coordinatizations of the circle S^1. To denote an element of the additive group, $\mathbb{R}/\mathbb{Z}$, we will use u, and for an element of the real projective line $\mathbb{R}P^1$, we will use x. These coordinate systems are identified by: $u \in \mathbb{R}/\mathbb{Z} \mapsto x = \tan(\pi u) \in \mathbb{R}P^1$. When we are not specifying a coordinate system, we will use p or q to denote an element of S^1. We fix an orientation on S^1 and use "$<$" to denote the counterclockwise cyclic ordering on S^1.

If G is a group, then we denote by $R^r(G)$ the set of all representations $\rho_0 \colon G \to \mathrm{Diff}^r(S^1)$, and we denote by $R^r_+(G)$ the set of all orientation-preserving representations in $R^r(G)$. Two representations $\rho_1, \rho_2 \in R^r(G)$ are *conjugate (in* $\mathrm{Diff}^r(S^1)$*)* if there exists $h \in \mathrm{Diff}^r(S^1)$ such that, for every $\gamma \in G$, $h\rho_1(\gamma)h^{-1} = \rho_2(\gamma)$.

We use the standard C^k topology on representations of a finitely-generated group into $\mathrm{Diff}^r(S^1)$, $r \in \{1, \ldots, \infty, \omega\}$ and $k \leq r$. If Γ is a finitely-generated group, then the C^k–open sets in $R^r(\Gamma)$ take generators in a fixed generating set for Γ into C^k–open sets. A representation $\rho_0 \in R^r(\Gamma)$ is *(C^r) locally rigid* if there exists a C^1 neighborhood U of ρ_0 in $R^r(\Gamma)$ such that every $\rho \in U$ is conjugate in $\mathrm{Diff}^r(S^1)$ to ρ_0. Finally, we say that Γ is *globally rigid in* $\mathrm{Diff}^r(S^1)$ if there exists a countable set of locally rigid representations in $R^r(\Gamma)$ such that every faithful representation in $R^r(\Gamma)$ is conjugate to an element of this set.

To construct the subgroups and representations in this paper, we use a procedure we call real ramified lifting.

Definition A real analytic surjection $\pi \colon S^1 \to S^1$ is called a *ramified covering map over* $p \in S^1$ if the restriction of π to $\pi^{-1}(S^1 \setminus \{p\})$ is a regular analytic covering map onto $S^1 \setminus \{p\}$ of degree $d \geq 1$. The *degree* of of π is defined to be this integer d.

Examples and properties of ramified covering maps and ramified lifts are described in Section 2.

Let $\pi \colon S^1 \to S^1$ be a ramified covering map over $p \in S^1$, and let $f \colon S^1 \to S^1$ be a real analytic diffeomorphism that fixes p. We say that $\hat{f} \in \mathrm{Diff}^\omega(S^1)$ is a π*–ramified lift* of f if the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xrightarrow{\hat{f}} & S^1 \\ \big\downarrow \pi & & \big\downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

More generally, let $\rho \colon \Gamma \to \mathrm{Diff}^\omega(S^1)$ be a representation with global fixed point p. A representation $\hat{\rho} \colon \Gamma \to \mathrm{Diff}^\omega(S^1)$ is a π*–ramified lift of* ρ if $\hat{\rho}(\gamma)$ is a ψ–ramified lift $\hat{\rho}(\gamma)$, for every $\gamma \in \Gamma$. We will show in Proposition 2.3 that a representation can have more than one ψ–ramified lift.

For G a subgroup of $\mathrm{Diff}^\omega(S^1)$ with a global fixed point p, we define $\hat{G}_\psi$, the *ψ–ramified lift of G* to be the collection of all ψ–ramified lifts of elements of G. By Proposition 2.3 and Propostion 2.8, $\hat{G}_\psi$ is a subgroup of $\mathrm{Diff}^\omega(S^1)$, abstractly isomorphic to an H–extension of G_+, where $G_+ = \mathrm{Diff}^\omega_+(S^1) \cap G$, and H is a subgroup of a dihedral group determined by ψ.

1.2 Rigidity of solvable Baumslag–Solitar groups

In real projective coordinates on $\mathbb{RP}^1$, the *standard representation* ρ_n of $\mathrm{BS}(1,n)$ into $\mathrm{Diff}^\omega_+(S^1)$ takes the generators a and b to the affine maps

$$x \mapsto nx; \qquad \text{and} \qquad x \mapsto x+1.$$

This representation has a global fixed point $\infty \in \mathbb{RP}^1$. Our first result states that $\mathrm{BS}(1,n)$ is globally rigid in $\mathrm{Diff}^\omega(S^1)$:

Theorem 1.1 *For each $n \geq 2$, there are exactly countably infinitely many faithful representations of $\mathrm{BS}(1,n)$ into $\mathrm{Diff}^\omega(S^1)$, up to conjugacy in $\mathrm{Diff}^\omega(S^1)$. Each conjugacy class contains a ψ–ramified lift of ρ_n, where $\psi\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ is a rational map that is ramified over ∞. Furthermore, if $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^\omega(S^1)$ is not faithful, then there exists a $k \geq 1$ such that $\rho(b)^k = id$.*

We give an explicit description of these conjugacy classes in Section 2.1.

The conclusion of Theorem 1.1 does not hold when C^ω is replaced by a lower differentiability class such as C^∞, even when analytic conjugacy is replaced by topological conjugacy in the statement. Nonetheless, as r increases, there is a sort of "quantum rigidity" phenomenon. Let $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^2(S^1)$ be a representation, and let $f = \rho(a)$. We make a preliminary observation:

Lemma 1.2 *If the rotation number of f is irrational, then $g^k = id$, for some $k \leq n+1$, where $g = \rho(b)$.*

(See the beginning of Section 5 for a proof). Hence, if $\rho \in R^2(\mathrm{BS}(1,n))$ is faithful, then f must have periodic points. For $\rho \in R^2(\mathrm{BS}(1,n))$ a faithful representation, we define the *inner spectral radius* $\sigma(\rho)$ by:

$$\sigma(\rho) = \sup\{|(f^k)'(p)|^{\frac{1}{k}} \mid p \in \mathrm{Fix}(f^k) \text{ and } |(f^k)'(p)| \leq 1\}.$$

For the standard representation, $\sigma(\rho_n) = \frac{1}{n}$, and if $\hat{\rho}_n$ is a ramified lift of ρ_n, then $\sigma(\hat{\rho}_n) = \left(\frac{1}{n}\right)^{\frac{1}{s}}$, for some $s \in \mathbb{N}_{\geq 1}$.

Theorem 1.3 *Let* $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^r(S^1)$ *be a faithful representation, where* $r \in [2,\infty]$. *If either* $r < \infty$ *and* $\nu(\rho) \leq \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$, *or* $r = \infty$ *and* $\nu(\rho) < 1$, *then* ρ *is conjugated by an element of* $\mathrm{Diff}^r(S^1)$ *into a unique conjugacy class in* $R^\omega(\mathrm{BS}(1,n))$.

If ρ *takes values in* $\mathrm{Diff}^r_+(S^1)$, *then* ρ *is conjugated by an element of* $\mathrm{Diff}^r_+(S^1)$ *into a unique conjugacy class in* $R^\omega_+(\mathrm{BS}(1,n))$.

Theorem 1.3 has the following corollary:

Corollary 1.4 *Every representation* $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^\omega(S^1)$ *is* C^∞ *locally rigid. Further, if* $\nu(\rho) < \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$, *then* ρ *is* C^r *locally rigid.*

This corollary implies that the standard representation is C^r locally rigid, for all $r \geq 3$, and every representation in $R^\omega(\mathrm{BS}(1,n))$ is locally rigid in some finite differentiability classes. This local rigidity breaks down, however, if the differentiability class is lowered.

Proposition 1.5 *For every representation* $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^\omega(S^1)$, *if* $\nu(\rho) = \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$, *for some* $r \geq 2$ *then there exists a family of representations* $\rho_t \in R^r(\mathrm{BS}(1,n))$, $t \in (-1,1)$, *with the following properties:*

(1) $\rho_0 = \rho$,

(2) $t \mapsto \rho_t$ *is continuous in the* C^{r-1} *topology on* $R^r(\mathrm{BS}(1,n))$,

(3) *for every* $t_1, t_2 \in (-1,1)$, *if* ρ_{t_1} *is conjugate to* ρ_{t_2} *in* $\mathrm{Diff}^1(S^1)$ *then* $t_1 = t_2$.

It follows from our characterization of the conjugacy classes in $R^\omega(\mathrm{BS}(1,n))$ in the next section that, for each value of $r \in [1,\infty)$, there are infinitely many nonconjugate representations $\rho \in R^\omega(\mathrm{BS}(1,n))$ satisfying $\nu(\rho) = \left(\frac{1}{n}\right)^{\frac{1}{r}}$. Hence, for each $r \geq 3$ there are infinitely many distinct (nonconjugate) representations in $R^\omega(\mathrm{BS}(1,n))$ that are C^r locally rigid, but not C^{r-1} locally rigid.

A. Navas has given a complete classification of C^2 solvable group actions, up to finite index subgroups and topological semiconjugacy. One corollary of his result is that every faithful C^2 representation ρ of $\mathrm{BS}(1,n)$ into $\mathrm{Diff}^2(S^1)$ is is virtually topologically semiconjugate to the standard representation:

Theorem 1.6 [11] *Let* $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^r(S^1)$ *be a representation, where* $r \geq 2$. *Then either* ρ *is unfaithful, in which case* $\rho(b)^m = id$, *for some* m, *or there exists an integer $m \geq 1$, a finite collection of closed, connected sets $I_1, \ldots, I_k$, and a surjective continuous map $\varphi\colon S^1 \to \mathbb{R}P^1$ with the following properties:*

(1) $\rho(b)^m$ *is the identity on each set* I_k*;*

(2) φ *sends each set* I_k *to* ∞*;*

(3) *the restriction of* φ *to* $S^1 \setminus \bigcup_{i=1}^{k} I_i$ *is a* C^r *covering map of* $\mathbb{R}$*;*

(4) *For every* $\gamma \in \mathrm{BS}(1,n)$*, the following diagram commutes:*

$$\begin{array}{ccc} S^1 & \xrightarrow{\rho(\gamma^m)} & S^1 \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S^1 & \xrightarrow{\rho_n(\gamma^m)} & S^1 \end{array}$$

where $\rho_n\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^{\omega}(S^1)$ *is the standard representation.*

The map φ in Theorem 1.6 is a sort of "broken C^r ramified cover." The regularity of φ at the preimages of the point ∞ can be poor, and the map can be infinite-to-one on the sets $I_1, \ldots, I_k$, but a map with these features is nothing more than a deformation of a ramified covering map. Combining Theorem 1.6 with Theorem 1.3 and the proof Proposition 1.5, we obtain:

Corollary 1.7 *Let* $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^r(S^1)$*, be any representation, with* $r \geq 2$*. Then either:*

(1) ρ *is not faithful, and there exists an* $m \geq 1$ *such that* $\rho(b)^m = id$*;*

(2) ρ *admits* C^{r-1} *deformations as in Proposition 1.5; or*

(3) ρ *is* C^r *conjugated into a unique conjugacy class in* $R^{\omega}(\mathrm{BS}(1,n))$*.*

Since the statement of Theorem 1.6 does not appear explicitly in Navas's paper, and we don't use this result elsewhere in the paper, we sketch the proof at the end of Section 5.

Finally, note that the trivial representation $\rho_0(a) = \rho_0(b) = id$ is not rigid in any topology; it can be approximated by the representation $\rho(b) = id, \rho(a) = f$, where f is an any diffeomorphism close to the identity. Another nice consequence of Navas's theorem is that this is the *only* way to C^2 deform the trivial representation.

Corollary 1.8 *There is a* C^2 *neighborhood* $U \subset R^2(\mathrm{BS}(1,n))$ *of the trivial representation such that, for all* $\rho \in U$*,* $\rho(b) = id$*.*

Proof Let ρ be a C^2 representation. Since $\rho(b)$ is conjugate by $\rho(a)$ to $\rho(b)^n$, it will have rotation number of the form $\frac{k}{n-1}$ if $\rho(a)$ is orientation-preserving, and of the form $\frac{k}{n+1}$ if $\rho(a)$ is orientation-reversing. Therefore, if ρ is sufficiently C^0–close to ρ_0 and if $\rho(b)^m = id$, for some $m \geq 1$, then $m = 1$. So we may assume that there exists a map φ as in Theorem 1.6 and that $m = 1$. On a component of $S^1 \setminus \bigcup I_i$, φ is a diffeomorphism conjugating the action of ρ to the restriction of the standard representation ρ_n to $\mathbb{R}$ (in general φ fails to extend to a diffeomorphism at either endpoint of $\mathbb{R}$). But in the standard action, the element $\rho_n(a)$ has a fixed point in $\mathbb{R}$ of derivative n. If ρ is sufficiently C^1 close to ρ_0, this can't happen. □

We remark that, in contrast to the results in this paper, there are uncountably many topologically distinct faithful representations of $\mathrm{BS}(1,n)$ into $\mathrm{Diff}^\omega(\mathbb{R})$ (see [3], Proposition 5.1). The proof of our results uses the existence of a global fixed point on S^1 for a finite index subgroup of $\mathrm{BS}(1,n)$; such a point need not exist when $\mathrm{BS}(1,n)$ acts on $\mathbb{R}$. Farb and Franks [3] studied actions of Baumslag–Solitar groups on the line and circle. Among their results, they prove that if $m > 1$, the (nonsolvable) Baumslag–Solitar group:

$$\mathrm{BS}(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle,$$

has no faithful C^2 actions on S^1 if m does not divide n. They ask whether the actions of $B(1,n)$ on the circle can be classified. This question inspired the present paper.

1.3 Classification of solvable subgroups of $\mathrm{Diff}^\omega(S^1)$

Several works address the properties of solvable subgroups of $\mathrm{Diff}^r(S^1)$; we mention a few here. Building on work of Kopell [8], Plante and Thurston [12] showed that any nilpotent subgroup of $\mathrm{Diff}^2(S^1)$ is in fact abelian. Ghys [6] proved that any solvable subgroup of $\mathrm{Diff}^\omega(S^1)$ is metabelian, ie, two-step solvable. In the same work, he remarks that there are solvable subgroups of $\mathrm{Diff}^\infty(S^1)$ that are not metabelian. The subgroups he constructs contain *infinitely flat elements* — nontrivial diffeomorphisms $g \in \mathrm{Diff}^\infty(S^1)$ with the property that for some $p \in S^1$, $g(p) = p$, $g'(p) = 1$, and $g^{(k)}(p) = 0$ for all $k \geq 2$.

Navas [11] constructed further examples of solvable subgroups of $\mathrm{Diff}^\infty(S^1)$ with arbitrary degree of solvability, again using infinitely flat elements. As mentioned above, Navas's work also contains a topological classification of solvable subgroups of $\mathrm{Diff}^2(S^1)$. As part of a study of ergodicity of actions of discrete groups on S^1, Rebelo and Silva [13] also study the solvable subgroups of $\mathrm{Diff}^\omega(S^1)$.

Our main result in this part of the paper, Theorem 1.9, implies that any solvable subgroup of $\mathrm{Diff}^\infty(S^1)$ that does not contain infinitely flat elements is either virtually abelian or conjugate to a subgroup of a ramified lift of the affine group:

$$\mathrm{Aff}(\mathbb{R}) = \{x \mapsto cx + d : \mathbb{R}P^1 \to \mathbb{R}P^1 \mid c, d \in \mathbb{R}, c \neq 0\}.$$

Theorem 1.9 *Let $G < \mathrm{Diff}^r(S^1)$ be a solvable group, where $r \in \{\infty, \omega\}$. Then either:*

1. *for some $m \in \mathbb{Z}$, the group $G^m := \{g^m : g \in G\}$ is abelian,*
2. *G contains infinitely flat elements (which can't happen if $r = \omega$), or*
3. *G is conjugate in $\mathrm{Diff}^r(S^1)$ to a subgroup of a ψ–ramified lift of $\mathrm{Aff}(\mathbb{R})$, where $\psi : \mathbb{R}P^1 \to \mathbb{R}P^1$ is a ramified cover over ∞.*

Further, if $G < \mathrm{Diff}^r_+(S^1)$ and (3) holds, then the conjugacy can be taken in $\mathrm{Diff}^r_+(S^1)$.

In Section 2, we characterize the ramified lifts of $\mathrm{Aff}(\mathbb{R})$. To summarize the results there, we have:

Theorem 1.10 *There exists a collection*

$$RAFF := \{\widehat{\mathrm{Aff}}^{\mathbf{s}}(\mathbb{R}) < \mathrm{Diff}^\omega(S^1) \mid \mathbf{s} \in \overline{S}\},$$

where $\overline{S}$ is a countably infinite index set, with the following properties:

1. *if $\mathbf{s}_1, \mathbf{s}_2 \in \overline{S}$ and $\widehat{\mathrm{Aff}}^{\mathbf{s}_1}(\mathbb{R})$ is conjugate to $\widehat{\mathrm{Aff}}^{\mathbf{s}_2}(\mathbb{R})$ in $\mathrm{Diff}^1(S^1)$, then $\mathbf{s}_1 = \mathbf{s}_2$;*
2. *for each $\mathbf{s} \in \overline{S}$, there exists a subgroup H of a dihedral group such that $\widehat{\mathrm{Aff}}^{\mathbf{s}}(\mathbb{R}) \simeq \mathrm{Aff}_+(\mathbb{R}) \times H$,*
3. *for each finite dihedral or cyclic group H, there exist infinitely many $\mathbf{s} \in \overline{S}$ so that $\widehat{\mathrm{Aff}}^{\mathbf{s}}(\mathbb{R}) \simeq \mathrm{Aff}_+(\mathbb{R}) \times H$,*
4. *each element of $RAFF$ is the ψ–ramified lift of $\mathrm{Aff}(\mathbb{R})$, for some rational ramified cover $\psi : \mathbb{R}P^1 \to \mathbb{R}P^1$ over ∞, and every ψ–ramified lift of $\mathrm{Aff}(\mathbb{R})$ is conjugate in $\mathrm{Diff}^\omega(S^1)$ to an element of $RAFF$.*

There also exists a collection

$$RAFF_+ := \{\widehat{\mathrm{Aff}}^{\mathbf{s}}_+(\mathbb{R}) < \mathrm{Diff}^\omega_+(S^1) \mid \mathbf{s} \in \overline{S}_+\},$$

with the same properties, except that in (1) and (4), the conjugacy is orientation-preserving, and in (2) and (3), H is cyclic.

Hence we have found all solvable groups that act effectively on the circle as real-analytic diffeomorphisms.

2 Introduction to ramified lifts

Let G be a group and let $\rho\colon G \to \mathrm{Diff}^\omega(S^1)$ be a representation with a global fixed point p. Restricting each element of this representation to a suitably small neighborhood of p, we obtain a representation $\tilde\rho\colon G \to G^\omega$, where G^ω is the group of analytic germs of diffeomorphisms. It is known [1, 10, 2] that if if G is solvable, then for some $k \geq 1$, $\tilde\rho$ is conjugate in G^ω to a representation taking values in the *ramified affine group* $\mathrm{Aff}^k(\mathbb{R})$:

$$\mathrm{Aff}^k(\mathbb{R}) = \left\{ \frac{x}{(ax^k + b)^{\frac{1}{k}}} \mid a, b \in \mathbb{R}, b > 0 \right\}$$

(see [6] for a proof in the context of circle diffeomorphisms). The name "ramified affine group" is explained by the fact that the elements of $\mathrm{Aff}^k(\mathbb{R})$ are lifts of the elements of the affine group under the branched (or ramified) cover $z \mapsto z^k$. These lifts are well-defined as holomophic germs, but do not extend to diffeomorphisms of $\mathbb{C}P^1$.

The key observation of this paper is that the elements of $\mathrm{Aff}(\mathbb{R})$ do admit global ramified lifts as diffeomorphisms of $\mathbb{R}P^1$. The reason is that, in contrast to a ramified cover $\pi\colon \mathbb{C}P^1 \to \mathbb{C}P^1$, which must be ramified over 2 points, a ramified cover $\pi\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ is ramified over one point, which can be chosen to coincide with the global fixed point of $\mathrm{Aff}(\mathbb{R})$.

Examples of real ramified covers The map $\pi_1\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $\pi_1(u) = \sin^2(\pi u)$ is a ramified covering map over 0, with critical points of order 2 at $\pi_1^{-1}(0) = \{0, \frac{1}{2}\}$.

The rational map $\pi_2\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ given by:

$$\pi_2(x) = \frac{(x+1)^2(x-1)^2}{x(x^2+1)}$$

is also a ramified covering map over 0, with critical points of order 2 at ± 1. It is clear that ± 1 are critical points of π_2, and one verifies directly that the other critical points of π_2 in $\mathbb{C}P^1$ occur off of $\mathbb{R}P^1$, at $\pm i\sqrt{3 \pm \sqrt{8}}$.

We will define an equivalence relation on ramified covering maps in which π_1 and π_2 are equivalent, and show that, under this notion of equivalence, all possible ramified covering maps occur as rational maps.

If $\phi\colon S^1 \to S^1$ is a ramified covering map over p then for each $q \in \phi^{-1}(p)$, there exists an integer $s(q) \geq 1$ such that the leading (nonconstant) term in the Taylor expansion of ϕ at q is of order $s(q)$. A regular covering map is a ramified covering map; in this case, d is the topological degree of the map, and $s(q) = 1$, for each $q \in \phi^{-1}(p)$. As the examples show, a ramified covering map need not be a regular covering map (even topologically), as it is possible to have $s(q) > 1$.

Let ϕ be a ramified covering map over p, and let $q_1, \ldots, q_d$ be the elements of $\phi^{-1}(p)$, ordered so that $p \leq q_1 < q_2 < \cdots < q_d < p$. For each $i \in \{1, \ldots, d\}$ we define $o_i \in \{\pm 1\}$ by:

$$o_i = \begin{cases} 1 & \text{if } \phi|_{(q_i, q_{i+1})} \text{ is orientation-preserving,} \\ -1 & \text{if } \phi|_{(q_i, q_{i+1})} \text{ is orientation-reversing.} \end{cases}$$

We call the vector $\mathrm{s}(\phi) = (s(q_1), s(q_2), \ldots, s(q_d), o_1, \ldots o_d) \in \mathbb{N}^d \times \{\pm 1\}^d$ the *signature* of ϕ. Geometrically, we think of a signature as a regular d–gon in $\mathbb{R}^2$ with vertices labelled by $s_1, \ldots, s_d$ and edges labelled by $o_1, \ldots, o_d$. Every signature vector $\mathrm{s} = (s_1, \ldots, s_d, o_1, \ldots, o_d)$ has the following two properties:

(1) The number of vertices with an even label is even: $\#\{1 \leq i \leq d \mid s_i \in 2\mathbb{N}\} \in 2\mathbb{N}.$

(2) If a vertex has an odd label, then both edges connected to that vertex have the same label, and if a vertex has an even label, then the edges have opposite labels: $(-1)^{s_i+1} = o_{i-1} o_i,$ where addition is mod d.

We will call any vector $\mathrm{s} \in \mathbb{N}^d \times \{\pm 1\}^d$ with these properties a *signature vector*. Note that a signature vector of length $2d$ is determined by its first $d+1$ entries. Let S_d be the set of all signature vectors with length $2d$, and let S be the set of all signature vectors.

Proposition 2.1 *Given any* $\mathrm{s} \in S$ *and* $p \in S^1$, *there is a ramified covering map* $\phi\colon S^1 \to S^1$ *over* p *with signature* s.

Proof Let $\mathrm{s} = (s_1, \ldots, s_d, o_1, \ldots, o_d)$ be a signature, and let $p \in \mathbb{R}/\mathbb{Z}$. Choose points $u_1 < \cdots < u_d$ evenly spaced in $\mathbb{R}/\mathbb{Z}$, and let $F\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/Z$ be the piecewise affine map that sends the u_i to p, and which sends each component of $\mathbb{R}/\mathbb{Z} \setminus \{u_1, \ldots, u_d\}$ onto $\mathbb{R}/Z \setminus \{p\}$, with orientation determined by o_i.

Put a new analytic structure on $\mathbb{R}/\mathbb{Z}$ as follows. In the intervals $I_j = (u_j, u_{j+1})$ use the standard analytic charts, but in each interval $J_j = (u_j - \epsilon, u_j + \epsilon)$ compose the standard chart (that identifies u_j with 0) with the homeomorphism

$\phi_j\colon \mathbb{R} \to \mathbb{R}$ defined by:

$$\phi_j(x) = \begin{cases} x^{1/s_j} & \text{if } x > 0 \\ -|x|^{1/s_j} & \text{if } x \leq 0. \end{cases}$$

Since the overlaps are analytic, this defines a real anaytic atlas on $\mathbb{R}/\mathbb{Z}$.

Note that the map $F\colon (\mathbb{R}/\mathbb{Z}, \text{new structure}) \to (\mathbb{R}/\mathbb{Z}, \text{standard structure})$ is analytic: in charts around u_j and $p = F(u_j)$, the map F takes the form $x \mapsto x^{s_j}$. Since there is a unique real analytic structure on the circle, there is an analytic homeomorphism of the circle $h\colon (\mathbb{R}/\mathbb{Z}, \text{standard structure}) \to (\mathbb{R}/\mathbb{Z}, \text{new structure})$. Let $\phi = F \circ h$. Then ϕ is a a ramified covering map over 0 with signature s. □

In fact, ramified covers exist in the purely algebraic category; every signature can be realized by a rational map. We have:

Proposition 2.2 *Given any* s $\in \mathcal{S}$ *and* $p \in \mathbb{R}P^1$, *there is a rational map* $\phi\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ *that is a ramified cover over* p *with signature* s.

Proof Since the proof of Proposition 2.2 is somewhat lengthy, we omit the details. The construction proceeds as follows. Let s $= (s_1, \ldots, s_d; o_1, \ldots, o_d)$ be a signature, and assume that $p = 0 \in \mathbb{R}P^1$ and $o_1 = 1$. Choose a sequence of real numbers $a_0 < a_1 < \cdots < a_{2d-2}$, let $P(x) = (x - a_0)^{s_1}(x - a_2)^{s_2} \cdots (x - a_{2d-2})^{s_d}$ and let $Q(x) = (x - a_1)(x - a_3) \cdots (x - a_{2d-3})$. The desired rational function ϕ will be a modification of P/Q.

Let $h(x)$ be a polynomial of even degree with no zeros, with critical points of even degree at a_i, $0 \leq i \leq 2d - 2$, and with no other critical points. One first shows that, for N sufficiently large, the rational function:

$$\phi_0 = \frac{Ph^N}{Q}$$

has zeroes of order $s_1, \ldots, s_d$ at $a_0, a_2, \ldots, a_{2d-2}$, simple poles at $a_1, a_3, \ldots, a_{2d-3}$, a pole of odd order at ∞, and no other zeroes, poles or critical points. Hence ϕ_0 is a ramified covering map over 0 with signature s, except at ∞, where it may fail to be a diffeomorphism.

Choose such an N, and let $2m + 1$ be the order of the pole ∞ for ϕ_0. One then shows that for ε sufficiently small, the rational function:

$$\phi(x) = \frac{\phi_0(x)}{1 + \varepsilon x^{2m}}$$

has the same properties as ϕ_0, except that ∞ is now a simple pole; it is the desired ramified cover. □

If is a ramified covering map, then the cyclic and dihedral groups:

$$C_d = \langle b : b^d = id \rangle; \quad \text{and} \quad D_d = \langle a, b : b^d = id, a^2 = id, aba^{-1} = b^{-1} \rangle,$$

respectively, act on $^{-1}(p)$ and on the set $E(\ )$ of oriented components of $S^1 \setminus {}^{-1}(p)$ in a natural way. By an orientation-preserving homeomorphism, we identify the circle with a regular oriented d–gon, sending the elements of $^{-1}(p)$ to the vertices and the elements of $E(\ )$ to the edges. The groups $C_d \quad D_d$ act by symmetries of the d–gon, inducing actions on $^{-1}(p)$ and $E(\ )$ that are clearly independent of choice of homeomorphism. For $q \in {}^{-1}(p)$, $e \in E(\ )$, and $\ \in D_d$, we write (q) and (e) for their images under this action.

These symmetry groups also act on the signature vectors in S_d in the natural way, permuting both vertex labels and edge labels. For $\ \in D_d$, we will write (s) for the image of s $\in S_d$ under this action. In this notation, the action is generated by:

$$b(s_1, \ldots, s_d, o_1, \ldots, o_d) = (s_2, s_3, \ldots, s_d, s_1, o_2, o_3, \ldots, o_d, o_1),$$

and

$$a(s_1, \ldots, s_d, o_1, \ldots, o_d) = (s_1, s_d, s_{d-1} \ldots, s_3, s_2, -o_d, -o_{d-1}, \ldots, -o_2, -o_1).$$

Denote by $\mathrm{Stab}_{C_d}(\mathrm{s})$ and $\mathrm{Stab}_{D_d}(\mathrm{s})$ the stabilizer of s in C_d and D_d, respectively, under this action:

$$\mathrm{Stab}_H(\mathrm{s}) = \{\ \in H \mid \ (\mathrm{s}) = \mathrm{s}\},$$

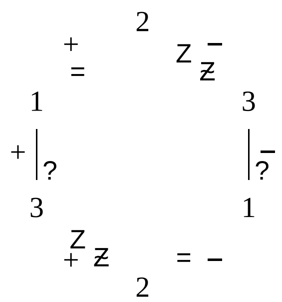
for $H = C_d$ or D_d.

Examples The signature vector of $_1(u) = \sin^2(\ u)$ is $\mathrm{s}_1 = (2, 2, 1, -1)$. The stabilizer of s_1 in D_d is $\mathrm{Stab}_{D_d}(\mathrm{s}_1) = \langle a \rangle$, and the stabilizer of s_1 in C_d is trivial. The signature vector of $_2(x) = ((x-1)^2(x+1)^2)/(x(x^2+1))$ is $\mathrm{s}_2 = (2, 2, -1, 1)$. Note that s_2 lies in the C_d–orbit of s_1, and so $\mathrm{Stab}_{C_d}(\mathrm{s}_2)$ and $\mathrm{Stab}_{D_d}(\mathrm{s}_2)$ must be conjugate to $\mathrm{Stab}_{C_d}(\mathrm{s}_1)$ and $\mathrm{Stab}_{D_d}(\mathrm{s}_1)$, respectively, by an element of C_d. In this simple case, the stabilizers are equal.

For another example, consider the signature vector

$$(2, 3, 1, 2, 3, 1, -1, -1, -1, 1, 1, 1),$$

which geometrically is represented by the following labelled graph:



This labelling has no symmetries, despite the fact that the edge labels have a flip symmetry and the vertex labels have a rotational symmetry. By contrast, the signature $(2,1,4,2,1,4,-1,-1,1,-1,-1,1)$ has a 180 degree rotational symmetry corresponding to the element $b^3 \in C_6$, and so both stabilizer subgroups are $\langle b^3 \rangle$.

2.1 Characterization of ramified lifts of the standard representation ρ_n of $\mathrm{BS}(1,n)$

The next proposition gives the key tool for lifting representations under ramified covering maps.

Proposition 2.3 *Let G be a group, and let $\rho\colon G \to \mathrm{Diff}^{\omega}_{+}(S^1)$ be a representation with global fixed point p. Let $\phi\colon S^1 \to S^1$ be a ramified covering map over p with signature $\mathrm{s} \in S_d$, for some $d \geq 1$.*

Then for every homomorphism $h\colon G \to \mathrm{Stab}_{D_d}(\mathrm{s})$, there is a unique representation

$$\hat{\rho} = \hat{\rho}(\phi, h)\colon G \to \mathrm{Diff}^{\omega}(S^1)$$

such that, for all $\gamma \in G$,

(1) *$\hat{\rho}$ is a ϕ–ramified lift of ρ;*

(2) *$\hat{\rho}(\gamma)(q) = h(\gamma)(q)$, for each $q \in \phi^{-1}(p)$;*

(3) *$\hat{\rho}(\gamma)(e) = h(\gamma)(e)$, for each oriented component $e \in E(\phi)$;*

Furthermore, if h takes values in $\mathrm{Stab}_{C_d}(\mathrm{s})$, then $\hat{\rho}$ takes values in $\mathrm{Diff}^{\omega}_{+}(S^1)$.

Proposition 2.3 is a special case of Proposition 4.4, which is proved in Section 4. Note that the representation ρ in Proposition 2.3 must be orientation preserving, although the lift $\hat{\rho}$ might not be, depending on where the image of h lies. There is also a criterion for lifting representations into $\mathrm{Diff}^{\omega}(S^1)$ that are not necessarily orientation-preserving. We discuss this issue in the next subsection.

Lemma 2.4 *Suppose that ψ_1 and ψ_2 are two ramified covering maps over p such that $s(\psi_2)$ lies in the D_d–orbit of $s(\psi_1)$; that is, suppose there exists $\beta \in D_d$ such that $s(\psi_2) = \beta(s(\psi_1))$. Then given any representation $\rho\colon G \to \mathrm{Diff}^{\omega}_{+}(S^1)$ with global fixed point p and homomorphism $h\colon G \to \mathrm{Stab}_{D_d}(s_1)$, the representations $\hat\rho(\psi_1, h)$ and $\hat\rho(\psi_2, \beta h \beta^{-1})$ are conjugate in $\mathrm{Diff}^{\omega}(S^1)$, where*

$$(\beta h \beta^{-1})(\gamma) := \beta h(\gamma) \beta^{-1}.$$

Furthermore, if $\beta \in C_d$ and h takes values in $\mathrm{Stab}_{C_d}(s)$, then $\hat\rho(\psi_1, h)$ and $\hat\rho(\psi_2, \beta h \beta^{-1})$ are conjugate in $\mathrm{Diff}^{\omega}_{+}(S^1)$.

Lemma 2.4 follows from Lemma 4.5, which is proved in Section 4. We now characterize the countably many conjugacy classes in $R^{\omega}(\mathrm{BS}(1,n))$. Note that the elements of S_d are totally ordered by the lexicographical order on $\mathbb{R}^n$. Hence we can write S_d as a disjoint union of C_d–orbits:

$$S_d = \bigsqcup_{\alpha \in A_+} C_d(s_\alpha),$$

where for each $\alpha \in A_+$, s_α is the smallest element in its C_d–orbit. Similarly, there is an index set $A \subset A_+$ such that:

$$S_d = \bigsqcup_{\alpha \in A} D_d(s_\alpha).$$

Let $\overline{S}_d = \{s_\alpha \mid \alpha \in A\}$, and let $\overline{S}_d^{+} = \{s_\alpha \mid \alpha \in A_+\}$. Finally, let $\overline{S} = \bigcup_d \overline{S}_d$ and let $\overline{S}^{+} = \bigcup_d \overline{S}_d^{+}$.

Definition Let $\rho_n\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^{\omega}_{+}(S^1)$ denote the standard projective action, with global fixed point at $\infty \in \mathbb{R}P^1$. Then we define:

$$V = \{\hat\rho_n(\psi_s, h) \mid s \in \overline{S}_d,\ h \in \mathrm{Hom}(\mathrm{BS}(1,n), \mathrm{Stab}_{D_d}(s))/\sim,\ d \in \mathbb{N}, d \geq 1\},$$

and let

$$V_+ = \{\hat\rho_n(\psi_s, h) \mid s \in \overline{S}_d^{+},\ h \in \mathrm{Hom}(\mathrm{BS}(1,n), \mathrm{Stab}_{C_d}(s)),\ d \in \mathbb{N}, d \geq 1\},$$

where, for $s \in S_d$, $\psi_s\colon S^1 \to S^1$ is the rational ramified cover over ∞ with signature s given by Proposition 2.2, and $\sim$ denotes conjugacy in $\mathrm{Stab}_{D_d}(s)$.

Proposition 2.5 *Each element of V and V_+ represents a distinct conjugacy class of faithful representations.*

That is, if $\hat\rho_n(\psi_{s_1}, h_1), \hat\rho_n(\psi_{s_2}, h_2) \in V$ (resp. $\in V_+$) are conjugate in $\mathrm{Diff}^{\infty}(S^1)$ (resp. in $\mathrm{Diff}^{\infty}_{+}(S^1)$), then $s_1 = s_2$ and $h_1 = h_2$.

Proposition 2.5 is proved at the end of Section 4. Our main result, Theorem 1.1, states that the elements of V and V_+ are the only faithful representations of $\mathrm{BS}(1,n)$, up to conjugacy in $\mathrm{Diff}^{\omega}(S^1)$ and $\mathrm{Diff}^{\omega}_{+}(S^1)$, respectively.

2.2 Proof of Theorem 1.10

To characterize the ramified lifts of $A_\Gamma(\mathbb{R})$, we need to define ramified lifts of orientation-reversing diffeomorphisms. In the end, our description is complicated by the following fact: in contrast to lifts by regular covering maps, ramified lifts of orientation-preserving diffeomorphisms can be orientation-reversing, and vice versa.

To deal with this issue, we introduce another action of the dihedral group $D_d = \langle a, b \mid a^2 = 1, b^d = 1, aba^{-1} = b^{-1} \rangle$ on S_d that ignores the edge labels completely. To distinguish from the action of D_d on S_d already defined, we will write $\gamma^{\#}\colon S_d \to S_d$ for the action of an element $\gamma \in D_d$. In this notation, the action is generated by:

$$b^{\#}(s_1, \ldots, s_d, o_1, \ldots, o_d) = (s_2, s_3, \ldots, s_d, s_1, o_1, o_2, \ldots, o_d),$$

and

$$a^{\#}(s_1, \ldots, s_d, o_1, \ldots, o_d) = (s_1, s_d, s_{d-1} \ldots, s_3, s_2, o_1, o_2, \ldots, o_d).$$

For $\mathbf{s} \in S_d$, we denote by $\mathrm{Stab}^{\#}_{D_d}(\mathbf{s})$ and $\mathrm{Stab}^{\#}_{C_d}(\mathbf{s})$ the stabilizers of $\mathbf{s}$ in D_d and C_d, respectively under this action.

Lemma 2.6 *For each* $\mathbf{s} \in S_d$*, there exists a homomorphism*

$$\rho_{\mathbf{s}}\colon \mathrm{Stab}^{\#}_{D_d}(\mathbf{s}) \to \mathbb{Z}/2\mathbb{Z},$$

such that $\mathrm{Stab}_{D_d}(\mathbf{s}) = \ker(\rho_{\mathbf{s}})$.

Proof Let $\mathbf{s} \in S_d$ be given. Clearly $\mathrm{Stab}_{D_d}(\mathbf{s})$ is a subgroup of $\mathrm{Stab}^{\#}_{D_d}(\mathbf{s})$. Let $I\colon S_d \to S_d$ be the involution:

$$I(s_1, \ldots, s_d, o_1, \ldots, o_d) = (s_1, \ldots, s_d, -o_1, \ldots, -o_d).$$

We show that for every $\gamma \in \mathrm{Stab}^{\#}_{D_d}(\mathbf{s})$, either $\gamma(\mathbf{s}) = \mathbf{s}$ (so that $\gamma \in \mathrm{Stab}_{D_d}(\mathbf{s})$) or $\gamma(\mathbf{s}) = I(\mathbf{s})$. This follows from the property (2) in the definition of signature vector, which implies that every element of S_d is determined by its first $d+1$ entries. Hence we may define $\rho_{\mathbf{s}}(\gamma)$ to be 0 if $\gamma(\mathbf{s}) = \mathbf{s}$ and 1 otherwise. Since I is an involution, $\rho_{\mathbf{s}}$ is a homomorphism. □

Example Consider the signature

$$\mathbf{s} = (2, 1, 2, 1, 2, 1, 2, 1, 1, 1, -1, -1, 1, 1, -1, -1).$$

For this example we have $\mathrm{Stab}_{C_8}(\mathbf{s}) = \langle b^4\rangle$, $\mathrm{Stab}_{D_8}(\mathbf{s}) = \langle a, b^4\rangle$, $\mathrm{Stab}^{\#}_{C_8}(\mathbf{s}) = \langle b^2\rangle$, and $\mathrm{Stab}^{\#}_{D_8}(\mathbf{s}) = \langle a, b^2\rangle$. In this example the homomorphism $\psi_{\mathbf{s}}$ is surjective, with nontrivial kernel. For $\mathbf{s} = (2, 1, 4, 1, 2, 1, 4, 1, 1, 1, -1, -1, 1, 1, -1, -1)$, on the other hand, the image of $\psi_{\mathbf{s}}$ is trivial, and $\mathrm{Stab}_{D_8}(\mathbf{s}) = \mathrm{Stab}^{\#}_{D_8}(\mathbf{s}) = \langle a, b^4\rangle$, $\mathrm{Stab}_{C_8}(\mathbf{s}) = \mathrm{Stab}^{\#}_{C_8}(\mathbf{s}) = \langle b^4\rangle$.

For a third example, recall that the stabilizer $\mathrm{Stab}_{D_6}(\mathbf{s})$ of the signature vector $\mathbf{s} = (2, 3, 1, 2, 3, 1, -1, -1, -1, 1, 1, 1)$ is trivial. Because of the rotational symmetry of the vertex labels, however, $\mathrm{Stab}^{\#}_{C_6}(\mathbf{s}) = \mathrm{Stab}^{\#}_{D_6}(\mathbf{s}) = \langle b^3\rangle \simeq \mathbb{Z}/2\mathbb{Z}$. In this example, $\psi_{\mathbf{s}}$ is an isomorphism.

Let $G < \mathrm{Diff}^{\omega}(S^1)$ be a subgroup with global fixed point $p \in S^1$: $f(p) = p$, for all $f \in G$. We now show how to assign, to each $\mathbf{s} \in \mathcal{S}$, a subgroup $\hat{G}^{\mathbf{s}}$ consisting of ramified lifts of elements of G. We first write $G = G_+ \sqcup G_-$, where $G_+ = G \cap \mathrm{Diff}^{\omega}_+(S^1)$ is the kernel of the homomorphism $O\colon G \to \mathbb{Z}/2\mathbb{Z}$ given by:

$$O(f) = \begin{cases} 0 & \text{if } f \text{ is orientation-preserving,} \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $\sigma\colon S^1 \to S^1$ is a ramified covering map over p. Then, for every $f \in G_+$, Proposition 2.3 implies that for every $\alpha \in \mathrm{Stab}_{D_d}(\mathbf{s}(\sigma))$, there is a unique lift $\hat{f}(\sigma, \alpha) \in \mathrm{Diff}^{\omega}(S^1)$ satisfying:

(1) $\hat{f}(\sigma, \alpha)$ is a σ–ramified lift of f,

(2) $\hat{f}(\sigma, \alpha)(q) = \alpha(q)$, for all $q \in \sigma^{-1}(p)$, and

(3) $\hat{f}(\sigma, \alpha)(e) = \alpha(e)$, for all $e \in E(p)$

(Further, this lift is orientation-preserving if $\alpha \in \mathrm{Stab}_{C_d}(\mathbf{s})$.) Suppose, on the other hand, that $f \in G_-$. In Section 4, we prove Lemma 4.2, which implies that if $\alpha \in \mathrm{Stab}^{\#}_{D_d}(\mathbf{s})$ satisfies:

$$\alpha(\mathbf{s}) = I(\mathbf{s}), \tag{1}$$

then there exists a unique lift $\hat{f}(\sigma, \alpha) \in \mathrm{Diff}^{\omega}(S^1)$ satisfying (1)–(3) (and, further, $\hat{f}(\sigma, \alpha) \in \mathrm{Diff}^{\omega}_+(S^1)$ if $\alpha \in \mathrm{Stab}^{\#}_{C_d}(\mathbf{s})$.) We can rephrase condition (1) as:

$$\psi_{\mathbf{s}}(\alpha) = 1.$$

To summarize this discussion, we have proved the following:

Lemma 2.7 *If* $f \in G$, *and* $\alpha \in \mathrm{Stab}^{\#}_{D_d}(\mathrm{s})$, *then there exists a lift* $\hat f(\psi_{\mathrm{s}};\alpha)$ *satisfying* (1)–(3) *if and only if:*

$$O(f) = \chi_{\mathrm{s}}(\alpha).$$

For $\mathrm{s} \in S$, let ψ_{s} be the ramified covering map over p with signature s given by Proposition 2.1. If $G < \mathrm{Diff}^{\omega}(S^1)$ has global fixed point p, we define $\hat G^{\mathrm{s}} \subset \mathrm{Diff}^{\omega}(S^1)$ to be the fibered product of G and $\mathrm{Stab}^{\#}_{D_d}(\mathrm{s})$ with respect to O and χ_{s}:

$$\hat G^{\mathrm{s}} := \{\hat f(\psi_{\mathrm{s}};\alpha) \mid (f,\alpha) \in G \times \mathrm{Stab}^{\#}_{D_d}(\mathrm{s});\ O(f) = \chi_{\mathrm{s}}(\alpha)\}.$$

Similarly, we define:

$$\hat G^{\mathrm{s}}_{+} := \{\hat f(\psi_{\mathrm{s}};\alpha) \mid (f,\alpha) \in G \times \mathrm{Stab}^{\#}_{C_d}(\mathrm{s});\ O(f) = \chi_{\mathrm{s}}(\alpha)\}.$$

Lemma 2.7 tells us that $\hat G^{\mathrm{s}}$ coincides with $\hat G^{\psi_{\mathrm{s}}}$, the set of all ψ_{s}–ramified lifts of G, and, similarly, that $\hat G^{\mathrm{s}}_{+} = \hat G^{\psi_{\mathrm{s}}} \cap \mathrm{Diff}_{+}(S^1)$. It follows from Lemma 4.3 that $\hat G^{\mathrm{s}}$ and $\hat G^{\mathrm{s}}_{+}$ are subgroups of $\mathrm{Diff}^{\omega}(S^1)$ and $\mathrm{Diff}^{\omega}_{+}(S^1)$, respectively, with:

$$\hat f_1(\psi_{\mathrm{s}};\alpha_1) \circ \hat f_2(\psi_{\mathrm{s}};\alpha_2) = \widehat{f_1 \circ f_2}(\psi_{\mathrm{s}};\alpha_1\alpha_2).$$

Further, we have:

Proposition 2.8 *Assume that* G_- *is nonempty. Then* $\hat G^{\mathrm{s}}$ *and* $\hat G^{\mathrm{s}}_{+}$ *are both finite extensions of* G_+*; there exist exact sequences:*

$$1 \to G_+ \to \hat G^{\mathrm{s}} \to \mathrm{Stab}^{\#}_{D_d}(\mathrm{s}) \to 1, \tag{2}$$

and

$$1 \to G_+ \to \hat G^{\mathrm{s}}_{+} \to \mathrm{Stab}^{\#}_{C_d}(\mathrm{s}) \to 1. \tag{3}$$

Furthermore, if the sequence

$$1 \to G_+ \to G \xrightarrow{O} O(G) \to 1$$

splits, where $O\colon G \to \mathbb{Z}/2Z$ *is the orientation homomorphism, then the sequences (2) and (3) are split, and so is the sequence*

$$1 \to \hat G^{\mathrm{s}}_{+} \to \hat G^{\mathrm{s}} \to O(\hat G^{\mathrm{s}}) \to 1. \tag{4}$$

Proof The maps in the first sequence (2) are given by:

$$\iota\colon G_+ \to \hat G^{\mathrm{s}} \qquad f \mapsto \hat f(\psi_{\mathrm{s}}; id)$$

$$\pi\colon \hat G^{\mathrm{s}} \to \mathrm{Stab}^{\#}_{D_d}(\mathrm{s}) \qquad \hat f(\psi_{\mathrm{s}};\alpha) \mapsto \alpha.$$

It is easy to see that is injective and is surjective. Moreover, $\hat{f}(\ _{\mathbf{s}};\ )$ is in the kernel of if and only if $= id$, if and only if f is orientation-preserving, if and only if $\hat{f}(\ _{\mathbf{s}}; id)$ is in the image of . Hence the first sequence is exact. Similarly, the second sequence (3) is exact.

Now suppose that $1 \to G_+ \to G \to O(G) \to 1$ is split exact. If $O(G)$ is trivial then $G_+ = G$, and there is nothing to prove. If $O(G) = \mathbb{Z}/2\mathbb{Z}$, then G contains an involution $g \in G_-$ with $g^2 = id$, namely the image of 1 under the homomorphism $O(G) \to G$. We use g to define a homomorphism $j\colon \mathrm{Stab}^{\#}_{D_d}(\mathrm{s}) \to \hat{G}^{\mathbf{s}}$ as follows:

$$j(\ ) = \begin{cases} \hat{id}(\ _{\mathbf{s}};\ ) & \text{if } \ _{\mathbf{s}}(\ ) = 0 \\ \hat{g}(\ _{\mathbf{s}};\ ) & \text{if } \ _{\mathbf{s}}(\ ) = 1. \end{cases}$$

Hence the sequence (2) is split. The restriction of j to $\mathrm{Stab}^{\#}_{C_d}(\mathrm{s})$ splits the sequence (3).

If $O(\hat{G}^{\mathbf{s}})$ is trivial, then the last sequence (4) is trivially split. If $O(\hat{G}^{\mathbf{s}}) = \mathbb{Z}/2\mathbb{Z}$, then there exists a $\ \in \mathrm{Stab}^{\#}_{D_d}(\mathrm{s})$ such that $\ _{\mathbf{s}}(\ ) = 1$. We then define $k\colon O(\hat{G}^{\mathbf{s}}) \to G^{\mathbf{s}}$ by $k(0) = id$, $k(1) = \hat{g}(\ _{\mathbf{s}};\ )$, which implies that (4) is split. □

Setting $G = \mathrm{A}\ (\mathbb{R})$, which has the global fixed point $\infty \in \mathbb{R}P^1$, we thereby define $\hat{\mathrm{A}}^{\mathbf{s}}(\mathbb{R})$ and $\hat{\mathrm{A}}^{\mathbf{s}}_+(\mathbb{R})$, for $\mathrm{s} \in S$. Let $\overline{S}$ and $\overline{S}_+$ be the sets of signatures defined at the end of the previous subsection. The elements of $\overline{S}$ and $\overline{S}_+$ are representatives of distinct orbits in S under the dihedral and cyclic actions, respectively. We now define:

$$RAFF := \{\hat{\mathrm{A}}^{\mathbf{s}}(\mathbb{R}) \mid \mathrm{s} \in \overline{S}\}; \quad \text{and} \quad RAFF_+ := \{\hat{\mathrm{A}}^{\mathbf{s}}_+(\mathbb{R}) \mid \mathrm{s} \in \overline{S}_+\}.$$

Since $\mathrm{A}\ (\mathbb{R})$ contains the involution $x \mapsto -x$, the sequence

$$1 \to \mathrm{A}\ _+(\mathbb{R}) \to \mathrm{A}\ (\mathbb{R}) \overset{}{\to} \mathbb{Z}/2\mathbb{Z} \to 1$$

splits. Proposition 2.8 implies that

$$\hat{\mathrm{A}}^{\mathbf{s}}(\mathbb{R}) \cong \mathrm{A}\ _+(\mathbb{R}) \quad \mathrm{Stab}^{\#}_{D_d}(\mathrm{s}); \qquad \hat{\mathrm{A}}^{\mathbf{s}}_+(\mathbb{R}) \cong \mathrm{A}\ _+(\mathbb{R}) \quad \mathrm{Stab}^{\#}_{C_d}(\mathrm{s});$$

and either $\hat{\mathrm{A}}^{\mathbf{s}}(\mathbb{R}) = \hat{\mathrm{A}}^{\mathbf{s}}_+(\mathbb{R})$ or $\hat{\mathrm{A}}^{\mathbf{s}}(\mathbb{R}) \cong \hat{\mathrm{A}}^{\mathbf{s}}_+(\mathbb{R}) \quad \mathbb{Z}/2\mathbb{Z}$, depending on whether $\ _{\mathbf{s}}$ is surjective. This proves (2) of Theorem 1.10.

Corollary 4.7 implies that $\hat{\mathrm{A}}^{\mathbf{s}_1}(\mathbb{R})$ and $\hat{\mathrm{A}}^{\mathbf{s}_2}(\mathbb{R})$ are conjugate subgroups in $\mathrm{Diff}^1(S^1)$, only if $\mathrm{s}_1 \in D_d \mathrm{s}_2$. Similarly, $\hat{\mathrm{A}}^{\mathbf{s}_1}_+(\mathbb{R})$ and $\hat{\mathrm{A}}^{\mathbf{s}_2}_+(\mathbb{R})$ are conjugate subgroups in $\mathrm{Diff}^1_+(S^1)$, if and only if $\mathrm{s}_1 \in C_d \mathrm{s}_2$. This proves (1) of the theorem.

Finally, every finite dihedral or cyclic group H is the stabilizer of infinitely many s $\in \overline{S}$, and hence there are infinitely many s $\in \overline{S}$ so that $\mathrm{Aff}^{s}(\mathbb{R}) \simeq \mathrm{Aff}_{+}(\mathbb{R}) \rtimes H$. This proves part (3). Property (4) follows from the definition of $RAFF$. This completes the proof of Theorem 1.10.

3 The relation $fgf^{-1} = g^{\ell}$: the central technical result

In this section we analyze the relation $fgf^{-1} = g^{\ell}$ near a common fixed point of f and g. If f and g are real-analytic, then they can be locally conjugated into one of the ramified affine groups described at the beginning of Section 2 [1, 10, 2]. This gives a local characterization of diffeomorphisms of S^1 satisfying this relation about each common fixed point, and to obtain a global characterization, it is a matter of gluing together these local ones. This was the way Ghys [6] proved that every solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ is metabelian. To prove Theorems 1.1 and 1.3, we adapt the arguments in [10] to the C^r setting, where additional hypotheses on f and g are required.

The initial draft of this paper contained a completely different proof of the local characterization of [1, 10, 2] that does not rely on vector fields and works for C^r diffeomorphisms as well, under the right assumptions. In the C^{ω} and C^{∞} case, this original proof gives identical results as the vector fields proof, but in the general C^r setting, the proof using vector fields gives sharper results. At the end of this section, we outline the alternate proof method. The main idea behind this method is to study the implications of the relation $fgf^{-1} = g^{\ell}$ for the Schwarzian derivative of g near a common fixed point at 0.

We now state the main technical result of this section. Let $[q, q_1)$ be a half open interval, let $r \in [2, \infty] \cup \{\omega\}$, and let $f, g \in \mathrm{Diff}^{r}([q, q_1))$ be diffeomorphisms. Assume that g has no fixed points in (q, q_1).

Standing Assumptions We assume that either (A), (B), (C) or (D) holds:

(A) $r = \omega$, and there exists an *integer* $\ell > 1$ such that $fgf^{-1} = g^{\ell}$:

(B) $r \in [2, \infty)$, and there is an *integer* $\ell > 1$ such that $fgf^{-1} = g^{\ell}$; and $f'(q) \geq \left(\frac{1}{\ell}\right)^{\frac{1}{r-1}}$.

(C) $r = \infty$, $f'(q) < 1$, and for some *integer* $\ell > 1$, $fgf^{-1} = g^{\ell}$:

(D) $r \in \{\infty, \omega\}$, g is not infinitely flat, and there is a C^{∞} flow $g^t \colon [q, q_1) \to [q, q_1)$ such that:

(1) $g^1 = g$, and

(2) $fgf^{-1} = g$ for some positive *real number* $\neq 1$.

Assumptions (A), (B) and (C) will arise in the proof of Theorems 1.1 and 1.3, and assumption (D) will arise in the proof of Theorem 1.9. The main technical result that we will use in these proofs is the following.

Proposition 3.1 *Assume that either* (A), (B), (C) *or* (D) *holds. Then there is a C^r diffeomorphism $: (q; q_1) \to (-\infty; \infty) \subset \mathbb{R}P^1$ such that for all $p \in (q; {}^{-1}(0))$:*

(1) $(p) = h(p)^s$; *where $h\colon [q; {}^{-1}(0)) \to [-\infty; 0)$ is a C^r diffeomorphism, s is an integer satisfying* $1 \leq s < r$, *and* $\in \{\ 1\}$,

(2) $g(p) = (p) + 1$ *and* $f(p) = (p)$.

We start with a lemma describing which values of $f'(q)$ and $g'(q)$ can occur.

Lemma 3.2 *Assume that one of assumptions* (A)–(D) *holds. Then* $g'(q) = 1$, *and either*

(1) $g^{(i)}(q) = 0$ *for* $2 \leq i \leq r$ *(in particular, neither assumption* (A) *nor* (D) *can hold in this case), or*

(2) $f'(q) = (\frac{1}{})^{\frac{1}{s}}$ *for some integer* $1 \leq s < r$, *and*

$$g^{(i)}(q) = 0 \text{ for } 2 \leq i \leq s; \text{ and } g^{(s+1)}(q) \neq 0.$$

Proof Since $fg = g f$,

$$f'(g(p))\, g'(p) = (g)'(f(p))\, f'(p).$$

When $p = q$, we thus have $g'(q) = (g)'(q)$. But $(g)'(q) = (g'(q))$, and so $g'(q) = 1$.

Suppose that $f'(q) = \neq 1$. Then there is an interval $[q; p)$ on which f is C^r conjugate to the linear map $x \mapsto x$ ([14], Theorem 2). So in local coordinates, identifying q with 0,

$$\begin{aligned} f(x) &= x; \text{ and} \\ g(x) &= x + ax^{s+1} + o(x^{s+1}) \text{ for some } s \geq 1. \end{aligned}$$

Then

$$\begin{aligned} fgf^{-1}(x) &= x + (\frac{a}{s})x^{s+1} + o(x^{s+1}); \text{ and} \\ g(x) &= x + ax^{s+1} + o(x^{s+1}). \end{aligned}$$

So either

(1) $a = 0$, and therefore $g^{(i)}(q) = 0$ for $2 \le i \le r$, or

(2) $a \neq 0$, in which case $\lambda = (\frac{1}{\mu})^{\frac{1}{s}}$, and

$$\begin{aligned} g^{(i)}(q) &= 0 \text{ for } 2 \le i \le s \\ &\neq 0 \text{ for } i = s+1. \end{aligned}$$

Now suppose that $f'(q) = 1$. Then in a neighborhood of q, we can write

$$f(x) = x + bx^{k+1} + o(x^{k+1}) \quad \text{and} \quad g(x) = x + ax^{s+1} + o(x^{s+1})$$

for some $k, s \ge 1$. If $k = s$, then

$$\begin{aligned} [f,g](x) &:= fgf^{-1}g^{-1}(x) = x + o(x^{s+1}) \\ &= g^{\lambda-1}(x) = x + (\lambda - 1)ax^{s+1} + o(x^{s+1}). \end{aligned}$$

So $a = 0$, and hence $g^{(i)}(0) = 0$ for $2 \le i \le r$.

If $k \neq s$, then we use the following well known result (see, eg, [13]):

Lemma 3.3 *If* $f(x) = x + bx^{k+1} + o(x^{k+1})$ *and* $g(x) = x + ax^{s+1} + o(x^{s+1})$, *and if* $s > k \ge 1$, *then*

$$[f,g](x) = x + (s-k)abx^{s+k} + o(x^{s+k}).$$

Assume that $s > k$. (If $s < k$, then the proof is similar). It follows from Lemma 3.3 that

$$\begin{aligned} x + (s-k)abx^{s+k} + o(x^{s+k}) &= [f,g](x) \\ &= g^{\lambda-1}(x) = x + (\lambda - 1)ax^{s+1} + o(x^{s+1}). \end{aligned}$$

So either

(1) $k \ge 2$, and therefore $a = 0$ and $g^{(i)}(q) = 0$, for $2 \le i \le r$, or

(2) $k = 1$, and therefore $b = \frac{\lambda-1}{s-1}$.

But if $k = 1$, then

$$\begin{aligned} x + (s-1)2abx^{s+1} + o(x^{s+1}) &= [f^2,g](x) \\ &= f^2gf^{-2}g^{-1}(x) = g^{\lambda^2-1}(x) \\ &= x + (\lambda^2 - 1)ax^{s+1} + o(x^{s+1}). \end{aligned}$$

So $b = \frac{\lambda^2-1}{2(s-1)} = \frac{\lambda-1}{s-1}$, which is impossible, since $\lambda \neq 1$. Therefore $g^{(i)}(q) = 0$, for $2 \le i \le r$. □

Lemma 3.4 *Assume that* (A), (B), (C) *or* (D) *holds. Then there is a neighborhood* $[q,p_1) \subseteq [q,q_1)$ *and a* C^r *map* $A\colon [q,p_1) \to \mathbb{R}$ *such that for all* $p \in [q,p_1)$,

(1) *$A(p) = \sigma H(p)^s$, where $H\colon [q,p_1) \to [0,\infty)$ is a C^r diffeomorphism, $1 \le s < r$ is an integer, and $\sigma \in \{\pm 1\}$;*

(2)
$$Af(p) = \frac{1}{\lambda}A(p);$$

(3)
$$Ag(p) = \frac{A(p)}{1 - A(p)}.$$

Consequently, $f'(q) = (\frac{1}{\lambda})^{\frac{1}{s}}$ for some integer $1 \le s < r$, and $g^{(s+1)}(q) \neq 0$.

Before giving a proof of Lemma 3.4, we will show how this lemma implies Proposition 3.1.

Lemma 3.5 *Assume that* (A), (B), (C) *or* (D) *holds. Let A, H, s, and σ be given by Lemma 3.4. For $p \in [q,p_1)$, let*

$$h(p) = \frac{-1}{H(p)}, \qquad \phi_0(p) = \frac{-1}{A(p)}.$$

Then ϕ_0 extends to a C^r map $\phi\colon (q,q_1) \to (-\infty,\infty)$ satisfying the conclusions of Proposition 3.1.

Proof Lemma 3.4 implies that for all $p \in (q,p_1)$, $\phi_0 g(p) = \phi_0(p) + 1$ and $\phi_0 f(p) = \lambda\phi_0(p)$. Since ϕ_0 has been defined in a fundamental domain for g, we can now extend this map to a C^r diffeomorphism ϕ from (q,q_1) to $(-\infty,\infty)$ as follows. Since g has no fixed points in (q,q_1), given any $p \in (q,q_1)$, there is some $j \in \mathbb{Z}$ such that $g^j(p) \in (q,p_1)$. Let $\phi(p) = \phi_0(g^j(p)) - j$ (which is easily seen to be independent of choice of j). By construction, $\phi g(p) = \phi(p) + 1$ for all $p \in (q,q_1)$. Since $f(p) = (g^{-\lambda j}) f g^j(p)$, we also have:

$$\phi f(p) = (\phi(g^{-\lambda j})\phi_0^{-1})(\phi_0 f \phi_0^{-1})(\phi_0 g^j(p)) = \lambda(\phi(p) + j) - \lambda j = \lambda\phi(p).$$

Hence the conclusions of Proposition 3.1 hold. □

Proof of Lemma 3.4 We say that a C^2 function $c\colon [a,b) \to [a,b)$ is a C^2 *contraction* if c' is positive on $[a,b)$ and $c(x) < x$, for all $x \in (a,b)$. Since g has no fixed points in (q,q_1), either g or g^{-1} is a C^2 contraction. We will assume until the end of the proof that g is a C^2 contraction. Replacing g by g^{-1} does not change the relation $fgf^{-1} = g^\lambda$.

Since g has no fixed points in (q,q_1), there is a a unique C^1 vector field X_0 on $[q,q_1)$ that generates a C^1 flow g^t such that $g|_{[q,q_1)} = g^1$ (Szekeres, see [11] for a discussion).

Lemma 3.6 *For all $j \in \mathbb{N}$ and $x \in [q,q_1)$, $f^{-j}gf^j(x) = g^{\frac{1}{\lambda^j}}(x)$.*

Proof We will use the following result of Kopell:

Lemma 3.7 ([8] Lemma 1) *Let $g \in \mathrm{Diff}^2[q,q_1)$ be a C^2 contraction that embeds in a C^1 flow g^t, so that $g = g^1$. If $h \in \mathrm{Diff}^1[q,q_1)$ satisfies $hg = gh$, then $h = g^t$ for some $t \in \mathbb{R}$.*

It follows from the relation $f^jgf^{-j} = g^{\lambda^j}$ that $f^{-j}gf^j$ commutes with g, and therefore Lemma 3.7 implies that $f^{-j}gf^j = g^t$ for some $t \in \mathbb{R}$. This relation also implies that $(f^{-j}gf^j)^{\lambda^j} = g$. So $f^{-j}gf^j = g^{\frac{1}{\lambda^j}}$. □

Let $\lambda = f'(q)$. We may assume, by Lemma 3.2, that $\lambda \neq 1$, and therefore there is an interval (q,p_1) on which f has no fixed points, and a C^r diffeomorphism $H\colon [q,p_1) \to [0,\infty)$ such that $HfH^{-1}(x) = \lambda x$ ([14], Theorem 2). The diffeomorphism H is unique up to multiplication by a constant. Let $F = HfH^{-1}$ and let $G = HgH^{-1}$. Since we have assumed that g is a contraction, we have $g([q,p_1)) \subset [q,p_1)$.

Let X be the push-forward of the vector field X_0 to $[0,\infty)$ under H, and let G^t be the semiflow generated by X_0, so that $G = G^1$.

Lemma 3.8 *If $F'(0) \le (\frac{1}{\ })^{\frac{1}{r-1}}$ and G is r–flat at 0, then $X(x) = 0$ on $[0,\infty)$.*

Proof We will show that for all $x \in [0,\infty)$,

$$\lim_{t \to 0} \frac{G^t(x) - x}{t} = 0.$$

Since the limit exists, it is enough to show that it converges to 0 for a subsequence $t_i \to 0$. We will use the subsequence $t_i = \frac{1}{\lambda^i}$. Writing $\lambda = F'(0)$ as before, we have

$$F(x) = \lambda x \quad \text{and} \quad G(x) = x + R(x),$$

where $R(x)/x^r \to 0$ as $x \to 0$, and therefore:

$$G^{t_i}(x) = F^{-i}GF^i(x) = x + \frac{1}{\lambda^i}R((\lambda^i x)).$$

So

$$\begin{aligned} 0 \le \lim_{i \to \infty} \frac{|G^{t_i}(x) - x|}{t_i} &= \lim_{i \to \infty} \lambda^i \frac{1}{\lambda^i}|R(\lambda^i x)| \\ &= \lim_{i \to \infty} (\lambda^{r-1})^i x^r \frac{|R((\lambda^i x))|}{(\lambda^i x)^r} \\ &\le \lim_{i \to \infty} x^r \frac{|R((\lambda^i x))|}{(\lambda^i x)^r} = 0, \end{aligned}$$

since $\lambda^{r-1} \geq \frac{1}{\lambda}$. $\square$

Corollary 3.9 *Under any assumption* (A)–(D)*,* g *is not* r–*flat at* q*, and therefore* $f'(q) = (\frac{1}{\lambda})^{\frac{1}{s}}$*, for some integer* $1 \leq s < r$*.*

Proof Clearly g cannot be infinitely flat if (A) or (D) holds. Under assumption (C), $f'(q) < (\frac{1}{\lambda})^{\frac{1}{k}}$, for some $k > 0$ and f, g are C^k, so (C) reduces to (B). By Lemma 3.8, under assumption (B), if g is r–flat at q, then the semiflow G^t is tangent to the trivial vector field, $X(x) = 0$. But then $G = id$, and therefore $g = id$ on $[q, p_1)$, contradicting the assumption that g has no fixed points in (q, q_1). $\square$

Lemma 3.10 *If* $F'(0) = (\frac{1}{\lambda})^{\frac{1}{s}}$ *for some integer* $1 \leq s < r$*, then for some* $a < 0$*,* $X(x) = ax^{s+1}$ *on* $[0, \infty)$*.*

Proof As in the proof of Lemma 3.8, it is enough to show that for all $x \in [0, \infty)$, and for $t_i = \frac{1}{\lambda^i}$,

$$\lim_{i \to \infty} \frac{G^{t_i}(x) - x}{t_i} = ax^{s+1}$$

for some $a \in \mathbb{R}$. If $F'(0) = (\frac{1}{\lambda})^{\frac{1}{s}}$ for some integer $1 \leq s < r$, then by Lemma 3.2,

$$F(x) = (\frac{1}{\lambda})^{\frac{1}{s}} x \quad \text{and} \quad G(x) = x + ax^{s+1} + R(x)$$

for some $a \in \mathbb{R}$, where $R(x)/x^{s+1} \to 0$ as $x \to 0$. The value of a depends on the choice of linearizing map h for $f|_{[q,p_1)}$. For all $i \in \mathbb{N}$,

$$\begin{aligned} G^{t_i}(x) = F^{-i}GF^{i}(x) &= \lambda^{\frac{i}{s}} G\left(\frac{x}{\lambda^{\frac{i}{s}}}\right) \\ &= x + ax^{s+1}\frac{1}{\lambda^i} + \lambda^{\frac{i}{s}} R\left(\frac{x}{\lambda^{\frac{i}{s}}}\right). \end{aligned}$$

So

$$\begin{aligned} \lim_{i \to \infty} \frac{G^{t_i}(x) - x}{t_i} &= \lim_{i \to \infty} \lambda^{i}\left(ax^{s+1}\frac{1}{\lambda^i} + \lambda^{\frac{i}{s}} R\left(\frac{x}{\lambda^{\frac{i}{s}}}\right)\right) \\ &= \lim_{i \to \infty} ax^{s+1} + R\left(\frac{x}{\lambda^{\frac{i}{s}}}\right)\frac{\lambda^{\frac{i(s+1)}{s}}}{x^{s+1}} x^{s+1} = ax^{s+1} \end{aligned}$$

Since G is a contraction on $[0, \infty)$, it follows that $a \leq 0$, and since $G \neq id$, we must have $a < 0$. $\square$

Corollary 3.11 *If one of* (A)–(D) *holds, then $f'(q) = (\frac{1}{\lambda})^{\frac{1}{s}}$ for some integer $1 \le s < r$, and, after a suitable rescaling of the linearizing map H,*

$$G(x) = \frac{x}{(1 + x^s)^{\frac{1}{s}}}.$$

Proof Choose H so that $a = -1/s$. Solving the differential equation $\frac{\partial}{\partial t}G^t(x) = aG^t(x)^{s+1}$ with initial condition $G^0(x) = x$, we obtain $G^t(x) = x/(1 + tx^s)^{\frac{1}{s}}$. Since $G(x) = G^1(x)$, the conclusion follows. □

To complete the proof of Lemma 3.4 assuming that g is a contraction, let s, H be given by Corollary 3.11, and let $\sigma = -1$. Then Corollary 3.11 implies that $A(p) = \sigma(H(p)^s)$ satisfies the desired conditions. If g is not a contraction, we replace g by g^{-1} in the proof. Setting $\sigma = 1$, we obtain the desired conclusions. □

3.1 Idea behind an alternate proof of Proposition 3.1

Suppose that f and $g \neq id$ are C^r diffeomorphisms, defined in a neighborhood of 0 in $\mathbb{R}$, both fixing the origin, and satisfying the relation:

$$fgf^{-1} = g^\lambda;$$

for some $\lambda > 1$. In this context, the conclusion of Proposition 3.1 can be reformulated as follows: f and g are conjugate, via $-1/h$, to the maps

$$x \mapsto (\frac{1}{\lambda})^{\frac{1}{s}}x \quad \text{and} \quad x \mapsto \frac{x}{(1 - \sigma x^s)^{\frac{1}{s}}}$$

for some integer $1 \le s \le r$ and some $\sigma \in \{\pm 1\}$. The proof of Proposition 3.1 uses vector fields; here we sketch an alternative proof of this reformulation, using the Schwarzian derivative. This sketch can be made into a complete proof of Proposition 3.1 under assumptions (A), (C) and (D), but gives a weaker result in case (B): for this proof we will need both $r \ge 2s + 1$ and $f'(q) \ge (\frac{1}{\lambda})^{1/(s-1)}$, for some $s \ge 1$.

For simplicity, assume that $r = \omega$ and that $\lambda = 2$. First note that, since g is not infinitely flat, Lemma 3.2 implies that $f'(0) \in \{(\frac{1}{2})^{\frac{1}{s}} \mid s \ge 1\}$. After conjugating f and g by an analytic diffeomorphism, we may assume, then, that:

$$f(x) = \frac{x}{2^{\frac{1}{s}}};$$

for some $s \geq 1$.

Let $F(x) = f(x^{\frac{1}{s}})^s = x/2$ and let $G(x) = g(x^{\frac{1}{s}})^s$. Rewriting the relation $FGF^{-1} = G^2$, we obtain:

$$\frac{1}{2}G(2x) = G^2(x);$$

rearranging and iterating this relation, we obtain:

$$G(x) = 2^k G^{2^k}\left(\frac{x}{2^k}\right); \tag{5}$$

for all $k \geq 1$.

Recall that the Schwarzian derivative of a C^3 function H is defined by:

$$S(H)(x) = \frac{H'''(x)}{H'(x)} - \frac{3}{2}\left(\frac{H''(x)}{H'(x)}\right)^2;$$

and has the following properties:

(1) $S(H)(x) = 0$ for all x iff G is Möbius, and
(2) for any C^3 function K, $S(H \circ K)(x) = K'(x)^2 S(H)(K(x)) + S(K)(x)$.

Combining these properties with (5), we will show that $S(G) = 0$, which implies that G is Möbius. Lemma 3.2 implies that $G'(0) = 1$ and $G''(0) \neq 0$, so we have $G(x) = \frac{x}{1-ox}$ for $o \in \{\pm 1\}$. Writing $g(x) = G(x^s)^{1/s}$, we obtain the desired result.

The first thing to check is that G is C^3. To obtain this, we use a slightly stronger version of Lemma 3.2 (whose proof is left as an exercise), which states that, if g is not infinitely flat, then

$$g(x) = x + ax^{s+1} + bx^{2s+1} + \cdots$$

Performing the substitution $G = g(x^{1/s})^s$ in this series, one finds that G is C^3. (This requires that g be at least C^{2s+1}, in contrast to the proof of Proposition 3.1, which requires only C^{s+1}).

Equation (5) implies that

$$S(G)(x) = \frac{1}{2^{2k}} S(G^{2^k})(\frac{x}{2^k});$$

for all $k \geq 1$. Thus, by the cocycle condition (2) of the Schwarzian, we have:

$$S(G)(x) = \frac{1}{2^{2k}} \sum_{i=1}^{2^k} S(G)(G^{i-1}(\frac{x}{2^k}))\left((G^{i-1})'(\frac{x}{2^k})\right)^2 \tag{6}$$

$$= \frac{1}{2^{2k}} \sum_{i=1}^{2^k} S(G)(x_i)\left(\prod_{j=1}^{i-1} G'(x_j)\right)^2 \tag{7}$$

where $x_i := G^{i-1}(\frac{x}{2^k})$.

Fix x, and assume without loss of generality that $G^j(x) \to 0$ as $j \to \infty$. Since G is C^3 and $G'(0) = 1$, there is a constant $C > 0$ such that $|G'(x_i)| \le 1 + \frac{C}{2^k}$, and $|S(G)(x_i)| \le C$, for all i between 1 and 2^k and all $k \ge 1$. Combined with (6), this gives us a bound on the Schwarzian of G at x:

$$\begin{aligned}
|S(G)(x)| &\le \frac{C}{2^{2k}} \sum_{i=1}^{2^k} \left(1 + \frac{C}{2^k}\right)^{2(i-1)} \\
&\le \frac{C}{2^{2k}} \left(\frac{1-(1+\frac{C}{2^k})^{2^{k+1}}}{1-(1+\frac{C}{2^k})^2} \right) \\
&\le \frac{1}{2^k} \left(\frac{e^{2C}-1}{2+\frac{C}{2^k}} \right);
\end{aligned}$$

for all $k \ge 1$. Hence $S(G)(x) = 0$, for all x, which implies that G is Möbius.

4 Further properties of ramified covers: proofs of Proposition 2.3, Lemma 2.4 and Proposition 2.5

The next lemma describes a useful normal form for ramified covering maps.

Lemma 4.1 *Let $\phi\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ be a ramified covering map over 0, where $\phi^{-1}(0) = \{x_1, \dots, x_d\}$. Let $\mathbf{s} = (s(x_1), \dots, s(x_d); o_1, \dots, o_d)$ be the signature of ϕ. Then given any $x_i \in \phi^{-1}(0)$, there is a neighborhood U of x_i and an analytic diffeomorphism $h\colon U \to \mathbb{R}$ such that for all $x \in U$,*

$$\phi(x) = h(x)^{s(x_i)}.$$

Proof In local coordinates at x_i, identifying x_i with 0, we can write

$$\begin{aligned}
\phi(x) &= ax^s(1 + O(x)) \\
&= ax^s g(x)
\end{aligned}$$

where $a > 0$, $s = s(x_i)$ and $g(x) = (1 + O(x))$. Let $h(x) = a^{\frac{1}{s}} x g(x)^{\frac{1}{s}}$. Then $h(x)$ is analytic in a neighborhood of 0, and $\phi(x) = h(x)^s$. □

This lemma motivates the following definition.

Definition A C^r *ramified cover over* $p \in S^1$ is a map $\phi\colon S^1 \to S^1$ satisfying:

(1) $\pi^{-1}(p) = \{q_1, q_2, \ldots, q_d\}$, where $q_1 < q_2 < \ldots < q_d$;

(2) the restriction of π to $\pi^{-1}(S^1 \setminus \{p\})$ is a regular C^r covering map onto $S^1 \setminus \{p\}$ of degree $d \geq 1$;

(3) for all $1 \leq i \leq d$, there are neighborhoods U_i of q_i and V of p, and C^r charts $h_i\colon U_i \to \mathbb{R}$ and $k_i\colon V \to \mathbb{R}$ with $h_i(q_i) = 0$ and $k_i(p) = 0$, such that

$$k_i \circ \pi \circ h_i^{-1}(x) = x^{s_i}$$

for some integer $s_i > 0$.

Remark By Lemma 4.1, a ramified cover is a C^ω ramified cover.

We define the signature of a C^r ramified cover in the obvious way.

Definition Let π_1 and π_2 be C^r ramified covering maps of degree d over p_1 and p_2, respectively. Fix an orientation preserving identification between $\pi_1^{-1}(p)$ and $\pi_2^{-1}(p)$ and between $E(\pi_1)$ and $E(\pi_2)$. Suppose that $f \in \mathrm{Diff}^r(S^1)$ satisfies $f(p_1) = p_2$, and let $\sigma \in D_d$. We say that $\hat{f} \in \mathrm{Diff}^r(S^1)$ is a (π_1, π_2, σ)*–ramified lift of* f if:

(1) $\hat{f}(q) = \sigma(q)$, for all $q \in \pi_1^{-1}(p_1)$,

(2) $\hat{f}(e) = \sigma(e)$, for all $e \in E(\pi_1)$, and

(3) the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xrightarrow{\hat{f}} & S^1 \\ \pi_1 \big\downarrow & & \big\downarrow \pi_2 \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Lemma 4.2 *Let π_1, π_2 and f be as above. Suppose that $\sigma \in D_d$ satisfies*

$\sigma(\mathrm{s}(\pi_1)) = \mathrm{s}(\pi_2)$, *if* $f \in \mathrm{Diff}^r_+(S^1)$, *or*

$\sigma(\mathrm{s}(\pi_1)) = I(\mathrm{s}(\pi_2))$, *if* $f \in \mathrm{Diff}^r_-(S^1)$,

where $I\colon S_d \to S_d$ is the involution that reverses the sign of the last d coordinates.

Then there exists a unique (π_1, π_2, σ)–ramified lift of f. We denote this lift by $\hat{f}(\pi_1, \pi_2, \sigma)$, or by $\hat{f}(\pi, \sigma)$, if $\pi_1 = \pi_2 = \pi$.

Furthermore, we have that if $\sigma \in C_d$, then $\hat{f}(\pi_1, \pi_2, \sigma) \in \mathrm{Diff}^r_+(S^1)$.

Proof Suppose first that f preserves orientation. Since the restriction of π_1 to $\pi_1^{-1}(S^1 \setminus \{p_1\})$ and the restriction of π_2 to $\pi_2^{-1}(S^1 \setminus \{p_2\})$ are both regular C^r covering maps of degree d, for any $\sigma \in D_d$ there is a unique C^r diffeomorphism $\hat f_0\colon S^1 \setminus \pi_1^{-1}\{p_1\} \to S^1 \setminus \pi_2^{-1}\{p_2\}$ such that $\hat f_0(e) = \sigma(e)$ for all $e \in E(\pi_1)$, and the diagram (3) commutes on the restricted domains. The condition $s(\pi_2) = \sigma(s(\pi_1))$ implies that $\hat f_0$ extends to a unique homeomorphism $\hat f$ such that $\hat f(q) = \sigma(q)$, for all $q \in \pi_1^{-1}(p_1)$ and such that the diagram in (3) commutes. It remains to show that $\hat f$ is a C^r diffeomorphism.

It suffices to show that $\hat f$ is a C^r diffeomorphism at each $q \in \pi_1^{-1}(p_1)$. By Lemma 5.6, there are local coordinates near q and $\hat f(q)$, identifying both of these points with 0, such that

$$(\hat f(x))^{s(\hat f(q))} = f(x^{s(q)})$$

for some integers $s(\hat f(q))$ and $s(q)$. Since $s(\pi_2) = \sigma(s(\pi_1))$, we have $s(\sigma(q)) = s(q)$. Let $j = s(q) = s(\hat f(q))$. Since f is C^r and has a fixed point at 0,

$$f(x) = a_1 x + a_2 x^2 + \dots + x^r + o(x^r)$$

and we can assume that the coordinates have been chosen so that $a_1 > 0$. So near $x = 0$,

$$\begin{aligned} \hat f(x) &= (a_1 x^j + a_2 x^{2j} + \dots + x^{rj} + o(x^{rj}))^{\frac{1}{j}} \\ &= x\,(a_1 + a_2 x^j + \dots + x^{(r-1)j} + o(x^{(r-1)j}))^{\frac{1}{j}}; \end{aligned}$$

where the root is chosen so that $\hat f'(0) > 0$. Since $a_1 > 0$ and $r \geq 2$, $\hat f$ is a C^r diffeomorphism at 0. Similarly, we see that if f is analytic, then $\hat f$ is analytic. Finally, we note that since f is orientation preserving, if $\sigma \in C_d$, then $\hat f$ must also be orientation preserving.

Now suppose that $f \in \mathrm{Diff}^r_-(S^1)$, and that $\sigma(s(\pi_1)) = I(s(\pi_2))$. Let $\bar\pi_1 = f \circ \pi_1$. Setting $\hat f$ to be the $(\bar\pi_1; \pi_2; \sigma)$–lift of the identity map, we obtain the desired conclusions. □

Lemma 4.3 *Let f_1 and f_2 be C^r diffeomorphisms of S^1, both with a fixed point at p, let $\pi\colon S^1 \to S^1$ be a C^r ramified covering map over p with signature* s*, and let $\sigma_1, \sigma_2 \in D_d$. Suppose that σ_1 and σ_2 satisfy σ_i(s) = s if $f_i \in$* Diff$^r_+(S^1)$*, and σ_i(s) = I(s) if $f_i \in$* Diff$^r_-(S^1)$*. Then*

$$\hat f_2(\pi; \sigma_2) \circ \hat f_1(\pi; \sigma_1) = \widehat{f_2 \circ f_1}(\pi; \sigma_2 \circ \sigma_1).$$

Proof The map $\hat f_2(\pi; \sigma_2) \circ \hat f_1(\pi; \sigma_1)(q)$ satisfies:

(1) $\hat f_2(\psi;\sigma_2)\circ\hat f_1(\psi;\sigma_1)(q) = \sigma_2\sigma_1(q)$, for all $q\in\psi^{-1}(p)$,

(2) $\hat f_2(\psi;\sigma_2)\circ\hat f_1(\psi;\sigma_1)(e) = \sigma_2\sigma_1(e)$, for all $e\in E(\psi)$, and

(3) the following diagram commutes:

$$\begin{array}{ccccc} S^1 & \xrightarrow{\hat f_1(\psi;\sigma_1)} & S^1 & \xrightarrow{\hat f_2(\psi;\sigma_2)} & S^1 \\ \downarrow{\psi} & & \downarrow{\psi} & & \downarrow{\psi} \\ S^1 & \xrightarrow{f_1} & S^1 & \xrightarrow{f_2} & S^1 \end{array}$$

By Lemma 4.2, we must have $\hat f_2(\psi;\sigma_2)\circ\hat f_1(\psi;\sigma_1) = \widehat{f_2\circ f_1}(\psi;\sigma_2\sigma_1)$. □

The following proposition is a C^r version of Proposition 2.3.

Proposition 4.4 *Suppose that G is a group, and that $\rho\colon G\to \mathrm{Diff}^r_+(S^1)$ is a representation with global fixed point p. Let $\psi\colon S^1\to S^1$ be a C^r ramified cover over p with signature vector* s. *Then for every homomorphism $h\colon G\to \mathrm{Stab}_{D_d}(\mathrm{s})$, there is a unique representation*

$$\hat\rho = \hat\rho(\psi;h)\colon G\to \mathrm{Diff}^r(S^1)$$

such that, for all $\gamma\in G$, $\hat\rho(\gamma)$ is the $(\psi;h(\gamma))$–ramified lift of $\rho(\gamma)$. If h takes values in $\mathrm{Stab}_{C_d}(\mathrm{s})$, then $\hat\rho$ takes values in $\mathrm{Diff}^r_+(S^1)$.

Proof This follows immediately from the previous two lemmas. □

The following lemma is a C^r version of Lemma 2.4.

Lemma 4.5 *Let G be a group, and let $\rho\colon G\to \mathrm{Diff}^\omega(S^1)$ be a representation with global fixed point p. Let $\psi_1,\psi_2\colon S^1\to S^1$ be C^r ramified covers over $p\in S^1$, with $\mathrm{s}(\psi_1)=\sigma(\mathrm{s}(\psi_2))$, for some $\sigma\in D_d$.*

Then for every homomorphism $h\colon G\to \mathrm{Stab}_{D_d}(\mathrm{s})$, the representation $\tilde\rho(\psi_1;h)$ is conjugate to $\tilde\rho(\psi_2;\sigma h\sigma^{-1})$ in $\mathrm{Diff}^r(S^1)$, where $(\sigma h\sigma^{-1})(\gamma):=\sigma h(\gamma)\sigma^{-1}$. If ρ takes values in $\mathrm{Diff}^\omega_+(S^1)$, if $\sigma\in C_d$, and if h takes values in $\mathrm{Stab}_{C_d}(\mathrm{s})$, then $\tilde\rho(\psi_1;h)$ and $\tilde\rho(\psi_2;\sigma h\sigma^{-1})$ are conjugate in $\mathrm{Diff}^r_+(S^1)$.

Proof This lemma follows from the diagram below, which commutes by Proposition 4.4 and Lemma 4.2. (Here $\widehat{id}=\widehat{id}(\psi_1;\psi_2;\sigma^{-1})$).

$$\begin{tikzcd}
 & S^1 \arrow[rr, "\hat{\phi}(\pi_2;\, h^{-1})"] \arrow[dd, "\pi_2"] & & S^1 \arrow[dd, "\pi_2"] \\
S^1 \arrow[rr, "\hat{\phi}(\pi_1;\, h)"] \arrow[dd, "\pi_1"] \arrow[ur, "h"] & & S^1 \arrow[dd, "\pi_1"] \arrow[ur, "h"] & \\
 & S^1 \arrow[rr] & & S^1 \\
S^1 \arrow[rr] \arrow[ur, "id"] & & S^1 \arrow[ur, "id"] &
\end{tikzcd}$$

$\square$

Consider two lifts $\hat{f}(\pi_1;\,\cdot)$, $\hat{f}(\pi_2;\,\cdot)$ of the same diffeomorphism f (or, more generally, of conjugate diffeomorphisms). For purely topological reasons, if these lifts are conjugate by a map with rotation number 0, then $s(\pi_1)$ and $s(\pi_2)$ have the same length $2d$, and the final d entries in these vectors must agree. (More generally, if the conjugacy has nonzero rotation number, then the final d entries of the first vector must lie in the D_d–orbit of the final d entries of the second). We now examine the first d entries of both vectors. We show that, under appropriate regularity assumptions on f and on the conjugacy, these entries must also agree, so that $s(\pi_1) = s(\pi_2)$. The next lemma is the key reason for this.

Lemma 4.6 *Let* $c\colon [0,\infty) \to [0,\infty)$ *be a* C^2 *contraction. Suppose that, for some integers* $m, n > 0$, *the maps* $v_1(x) = c(x^m)^{1/m}$ *and* $v_2(x) = c(x^n)^{1/n}$ *are conjugate by a* C^1 *diffeomorphism* $h\colon [0,\infty) \to [0,\infty)$. *Then* $m = n$.

Proof Since c is a C^2 contraction, the standard distortion estimate (see, eg [8]) implies that for all $x, y \in [0,\infty)$, there exists an $M \geq 1$, such that for all $k \geq 0$,

$$\frac{1}{M} \leq \frac{(c^k)'(x)}{(c^k)'(y)} \leq M. \tag{8}$$

Assume without loss of generality that $n > m$ and suppose that there exists a C^1 diffeomorphism $h\colon [0,\infty) \to [0,\infty)$ such that $hv_1(x) = v_2(h(x))$, for all

$x \in [0, \infty)$. Let $H(x) = h(x^{1/m})^n$. Note that the C^1 function $H\colon [0,\infty) \to [0,\infty)$ has the following properties:

(1) $H'(x) \geq 0$, for all $x \in [0,\infty)$, and $H'(x) = 0$ iff $x = 0$;

(2) for all $k \geq 0$, $H \circ c^k = c^k \circ H$.

Then (2) implies that for every $x \in [0,\infty)$:

$$H'(x) = H'(c^k(x)) \frac{(c^k)'(x)}{(c^k)'(H(x))}$$

for all $k \geq 0$. But (8) implies that $(c^k)'(x)/(c^k)'(H(x))$ is bounded independently of k, so that $H'(x) = \lim_{k \to \infty} H'(c^k(x)) = 0$, contradicting property (1). □

Corollary 4.7 *Let G and H be infinite subgroups of $\hat{A}^{\mathbf{s}_1}(\mathbb{R})$ and $\hat{A}^{\mathbf{s}_2}(\mathbb{R})$, respectively, for some $\mathbf{s}_1, \mathbf{s}_2 \in \overline{S}$. If there exists $\varphi \in \mathrm{Diff}^\infty(S^1)$ such that $\varphi G \varphi^{-1} = H$, then $\mathbf{s}_1 = \mathbf{s}_2$.*

If $G < \hat{A}^{\mathbf{s}_1}_+(\mathbb{R})$ and $H < \hat{A}^{\mathbf{s}_2}_+(\mathbb{R})$, with $\mathbf{s}_1, \mathbf{s}_2 \in \overline{S}_+$, and there exists $\varphi \in \mathrm{Diff}^\infty_+(S^1)$ such that $\varphi G \varphi^{-1} = H$, then $\mathbf{s}_1 = \mathbf{s}_2$.

Proof Let G, H and φ be given. Note that $\mathbf{s}_1$ and $\mathbf{s}_2$ must have the same length $2d$, since the global finite invariant sets of G and H must be isomorphic. Let g, h be elements of G and H with rotation number 0 such that $h = \varphi g \varphi^{-1}$. Since dilations have twice as many fixed points in $\mathbb{R}P^1$ as translations, if g is a ramified lift of a translation, then so is h. Assume that $g = \hat{S}(\sigma_{\mathbf{s}_1}; id)$ and $h = \hat{T}(\sigma_{\mathbf{s}_2}; id)$, where $S\colon x \mapsto x + s$ and $T\colon x \mapsto x + t$ are translations with $s, t > 0$. Let $q_1, \dots, q_d$ and $\varphi(q_1), \dots, \varphi(q_d)$ be the preimages of ∞ under $\sigma_{\mathbf{s}_1}$ and $\sigma_{\mathbf{s}_2}$, respectively. In a neighborhood of q_i, the map g is conjugate to $x \mapsto (S(x)^{m_i})^{1/m_i}$ and in a neighborhood of $\varphi(q_i)$, h is conjugate to $x \mapsto (T(x)^{n_i})^{1/n_i}$, where $m_i = s(q_i)$ and $n_i = s(\varphi(q_i))$. Since S is a C^2 contraction in a neighborhood of ∞ and T is conjugate to S, it follows from Lemma 4.6 that $m_i = n_i$ for $1 \leq i \leq d$, which implies that $\mathbf{s}_1 = \mathbf{s}_2$.

Suppose instead that g is a ramified lift of a map in $A(\mathbb{R})$ conjugate to the dilation $D\colon x \mapsto ax$, for some $a > 1$. Since g must have d fixed points with derivative a, so must h, and so h is also a ramified lift of a map in $A(\mathbb{R})$ conjugate to D. Around ∞, the map D is a C^2 contraction, and the same proof as above shows that $\mathbf{s}_1 = \mathbf{s}_2$.

The proof in the orientation-preserving case is analogous. □

Proof of Proposition 2.5 Let $\rho_n\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^{\omega}(S^1)$ be the standard representation. Suppose that $\hat{\rho}_n(\pi_{\mathbf{s}_1}, h_1)$ and $\hat{\rho}_n(\pi_{\mathbf{s}_2}, h_2) \in V$ are conjugate by $\phi \in \mathrm{Diff}^1(S^1)$, where $\mathbf{s}_1, \mathbf{s}_2 \in S$. It follows from Corollary 4.7 that $\mathbf{s}_1 = \mathbf{s}_{\mathbf{s}}$.

We next show $h_1 = h_2$. Let $\gamma \in \mathrm{BS}(1,n)$ and let $k = \rho_n(\gamma)$. Let $k_1 = \hat{k}(\pi_{\mathbf{s}_1}, h_1(\gamma))$ and $k_2 = \hat{k}(\pi_{\mathbf{s}_2}, h_2(\gamma))$ Then for all $q \in \pi_{\mathbf{s}_1}^{-1}(\infty)$, we have:

$$h_1(\gamma)(q) = \phi k_1(q) = k_2(\phi(q)) = h_2(\gamma)(\phi(q));$$

and for all $e \in E(\pi_1)$,

$$h_1(\gamma)(e) = \phi k_1(e) = k_2(\phi(e)) = h_2(\gamma)(\phi(e)).$$

Since $\phi(\pi_{\mathbf{s}_1}^{-1}(\infty)) = \pi_{\mathbf{s}_1}^{-1}(\infty)$ and $\phi(E(\pi_1)) = E(\pi_1)$, it follows that $h_1(\gamma) = h_2(\gamma)$. So $h_1 = h_2$. Recall that each element $\hat{\rho}_n(\pi_{\mathbf{s}}, h)$ of V is given by a signature vector $\mathbf{s} \in \overline{S}$ and a representative h of a conjugacy class in $\mathrm{Hom}(\mathrm{BS}(1,n), \mathrm{Stab}_{D_d}(\mathbf{s}))$. So $h_1 = h_2$. □

5 Proof of Theorems 1.1 and 1.3

The construction behind this proof is very simple. We are given a C^r representation ρ of $\mathrm{BS}(1,n)$. Using elementary arguments, we are reduced to the case where $f = \rho(a)$ and $g = \rho(b)$ have a common finite invariant set, the set of periodic orbits of g. Assume that the rotation numbers of f and g are both 0. Using the results from Section 3, we obtain a local characterization of f and g on the intervals between the common fixed points. On each of these intervals, f is conjugate to the dilation $x \mapsto nx$ and g is conjugate to the translation $x \mapsto x + 1$. Gluing together the conjugating maps gives us a C^r ramified covering map over ∞. Hence ρ is a C^r ramified lift of the standard representation. Proposition 2.2 implies that there is a rational ramified cover with the same signature as the the given C^r ramified cover. Lemma 4.5 implies that ρ is C^r conjugate to a ramified lift of ρ_n under the rational ramified cover. It remains to handle the case where the rotation numbers of f and g are not 0, but this is fairly simple to do, since the elements of the standard representation embed in analytic vector fields. We now give the complete proof.

Let $\rho\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^r(S^1)$ be a representation, where $r \in [2, \infty]$, or $r = \omega$. If $r < \infty$, assume that $\lambda(\rho) < \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$. If $r = \infty$, we assume that $\lambda(\rho) < 1$.

Let $f = \rho(a)$ and $g = \rho(b)$, where $aba^{-1} = b^n$. Since g is conjugate to g^n, it follows that $\tau(g) = \tau(g^n) = n\tau(g)$, where $\tau(h)$ denotes the rotation number of $h \in \mathrm{Homeo}(S^1)$. Hence g has rational rotation number.

Lemma 5.1 *f preserves the set of periodic points of g.*

Proof This follows from the relation $fg = g^n f$. If $g^k(q) = q$, then $g^{nk}(f(q)) = fg^k(q) = f(q)$. So $f(q)$ is also periodic for g. □

Suppose that $\rho(f)$ is irrational. Then by Lemma 5.1, the periodic points of g are dense in S^1, which implies that $g^k = id$, for some $k \leq n+1$. This implies that conclusion (1) of Theorem 1.1 holds.

Suppose, on the other hand, that $\rho(f)$ is rational. Choose l so that g^l and f^l are both orientation-preserving and both have rotation number 0. Then f^l leaves $\mathrm{Fix}(g^l)$ invariant. Choose $p \in \mathrm{Fix}(g^l)$. Any accumulation of $\{f^{ln}(p)\}$ must be a fixed point for f^l and for g^l. We have shown:

Lemma 5.2 *f^l and g^l have a common fixed point.*

Note that the fixed points for f^l are isolated; if f is not analytic, then $(\ ) < 1$, which implies that the fixed points for f^l are hyperbolic. Let $w_1 < w_2 < \cdots < w_k$ be the set of fixed points of f^l. We will see that if g^l is not the identity map, then the set of fixed points for g^l is exactly equal to the set of sinks for f^l.

Lemma 5.3 *If $g^l(w_i) = w_i$ and $(f^l)'(w_i) > 1$, then $g^l = id$ on $[w_{i-1}, w_{i+1}]$.*

Proof Suppose that $(f^l)'(w_i) = \lambda > 1$, and let $\psi\colon [w_i, w_{i+1}) \to [0, \infty)$ be a C^1 linearizing diffeomorphism such that $\psi f^l \psi^{-1}(x) = \lambda x$ for all $x \in [0, \infty)$. Let $F = \psi f^l \psi^{-1}$, and let $G = \psi g^l \psi^{-1}$. If $g^l \neq id$ on $[w_i, w_{i+1})$, then there is a point $x_0 \in [0, \infty)$ such that $G(x_0) \neq x_0$. Let x_0 be any such point. We may assume that $G^k(x_0) \to c$ as $k \to \infty$, for some $c < \infty$, because this will be true for either G or G^{-1}. Since $GF^{-k} = F^{-k}G^{n^k}$ for all $k \in \mathbb{N}$, it follows that

$$G'(F^{-k}(x_0)) = \frac{(F^{-k})'(G^{n^k}(x_0))}{(F^{-k})'(x_0)} (G^{n^k})'(x_0),$$

for all $k \in \mathbb{N}$. But since $G'(0) = 1$ (by Lemma 3.2), this means that $(G^{n^k})'(x_0) \to 1$, as $k \to \infty$ (or $k \to -\infty$), for every point x_0 that is not fixed by G. Since G is not the identity, this is not possible. Hence $g = id$ on $[w_i, w_{i+1}]$. A similar argument shows that $g = id$ on $[w_{i-1}, w_i]$, □

Corollary 5.4 *If g^l has a fixed point in the interval (w_i, w_{i+1}), then $g^l = id$ on $[w_i, w_{i+1}]$. That is, $\partial\mathrm{Fix}(g^l) \subset \mathrm{Fix}(f^l)$.*

Proof Suppose that $g^l(p) = p$ for some $p \in (w_i, w_{i+1})$, and suppose that $f^{kl}(p) \to w_i$ as $k \to -\infty$. By Lemma 5.1, $f^{lk}(p)$ is periodic for g^l for all $k \in \mathbb{Z}$. Since g^l is an orientation preserving circle diffeomorphism with a fixed point, $f^{lk}(p)$ is a fixed point of g^l for all k. By continuity, w_i is a common fixed point for f^l and g^l. Since $(f^l)'(w_i) > 1$, Lemma 5.3 implies that $g^l = id$ on $[w_i, w_{i+1}]$. Similarly, if $f^{kl}(p) \to w_{i+1}$ as $k \to -\infty$, then $g = id$ on $[w_i, w_{i+1}]$. □

This has the immediate corollary:

Corollary 5.5 *f^l fixes every component of $S^1 \setminus$ Fix(g^l).*

Remark Corollary 5.5 also follows from Theorem 1.6. We have given a different proof here since we will need Lemma 5.3 for the proof of Lemma 5.6.

Let $-\infty \leq q_1 < q_2 < \cdots < q_d < \infty$ be the elements of ∂Fix(g^l).

Lemma 5.6 *On each interval $(q_{i-1}, q_i]$, either $g^l = id$, or there is a C^r map $\psi_i\colon (q_{i-1}, q_i] \to (-\infty, \infty]$ such that*

(1) *ψ_i conjugates f^l to the map $x \mapsto n^l x$, and conjugates g^l to the map $x \mapsto x + 1$;*

(2) *$\psi_i|_{(q_{i-1}, q_i)}$ is a C^r diffeomorphism onto $(-\infty, \infty)$*

(3) *For all p in a neighborhood of q_i,*

$$\psi_i(p) = o_i h(p)^s$$

where h is a C^r orientation-preserving diffeomorphism onto a neighborhood of ∞, $1 \leq s < r$, and $o_i \in \{\pm 1\}$.

Proof This follows from Proposition 3.1. Note that we can apply Proposition 3.1 in this setting since we know that if $g^l \neq id$ on $(q_{i-1}, q_i]$, then $(f^l)'(q_i) \leq 1$ (by Lemma 5.3). By our assumptions on $\psi(\ )$, if $2 \leq r < \infty$, then $(f^l)'(q_i) \leq (\frac{1}{n})^{\frac{1}{r-1}}$, and if $r = \infty$, then $(f^l)'(q_i) < 1$. Therefore one of the assumptions (A)–(C) of Proposition 3.1 will hold. □

Corollary 5.7 *Either $g^l = id$, or ∂Fix$(g^l) =$ Fix$(g^l) = \{q_1, \ldots, q_d\}$.*

Proof Assume that ∂Fix$(g^l) = \{q_1, \ldots, q_d\} \neq$ Fix(g^l), but $g^l \neq id$. Then there is an interval $[q_{i-1}, q_i]$ on which $g^l = id$ but where $g^l \neq id$ on $[q_i, q_{i+1}]$. By Lemma 3.4, either g^l or g^{-l} is C^r conjugate in a neighborhood $[q_i, p)$ to the map $x \mapsto x/(1 - x^s)^{\frac{1}{s}}$ for some integer $1 \leq s < r$. But this map is not r–flat at $x = 0$, so g^l is not C^r at q_i, a contradiction. □

Corollary 5.8 *If $g^l \neq id$, then the map $\psi\colon S^1 \to \mathbb{R}P^1$ defined by:*

$$\psi(p) = \psi_i(p); \quad \text{for } p \in (q_{i-1}; q_i]$$

is a C^r ramified covering map over ∞, f^l is a ψ–ramified lift of $x \mapsto n^l x$, and g^l is a ψ–ramified lift of the map $x \mapsto x + 1$.

Proof Let $q_i \in \mathrm{Fix}(g^l)$. Applying Lemma 5.6 to the interval $[q_i; q_{i+1})$, we obtain a map $\bar\psi_{i+1}\colon [q_i; q_{i+1}) \to [-\infty; \infty)$ which is a C^r diffeomorphism on $(q_i; q_{i+1})$, and which is a power of a C^r diffeomorphism in a (right) neighborhood of q_i; $\bar\psi_{i+1}(p) = h(p)^s$ for p near q_i, for some C^r diffeomorphism h and some integer $1 \leq s < r$. Similarly, on the interval $(q_{i-1}; q_i]$ there is a map $\psi_i\colon (q_{i-1}; q_i] \to (\infty; -\infty]$ which is a power of a diffeomorphism in a (left) neighborhood of q_i; $\psi_i(p) = h'(p)^{s'}$ near q_i. We will show that $s = s'$, and that the diffeomorphisms h and h' glue together to give a C^r diffeomorphism in a neighborhood of q_i. This will prove that the map

$$\tilde\psi_i(p) = \begin{cases} \psi_i(p); & \text{for } p \in (q_{i-1}; q_i] \\ \bar\psi_{i+1}(p); & \text{for } p \in [q_i; q_{i+1}) \end{cases}$$

is the restriction to $(q_{i-1}; q_{i+1})$ of a C^r ramified covering map over ∞. By construction, the restrictions of f^l and g^l to $(q_{i-1}; q_{i+1})$ are $\tilde\psi_i$–ramified lifts of the maps $x \mapsto n^l x$ and $x \mapsto x + 1$ respectively.

The diffeomorphism $1/h$ maps q_i to 0, and conjugates g^l to the map $x \mapsto x/(1 + x^s)^{1/s}$. Similarly, $1/h'$ conjugates g^l to $x \mapsto x/(1 + x^{s'})^{1/s'}$. Since g is C^r, we must have $s = s'$. Both $1/h$ and $1/h'$ are linearizing maps for f^l at q_i, and it is not hard to see that they define a C^r diffeomorphism H in a neighborhood of q_i. Therefore h and h' glue together to give a C^r diffeomorphism $1/H$ in a neighborhood of q_i.

It remains to show that $\psi_i = \psi_{i+1}$ on $(q_i; q_{i+1})$. Since the restriction of both of these maps to $(q_i; q_{i+1})$ are diffeomorphisms which linearize f^l, they are the same up to a constant multiple. There is a unique point $x_0 \in (q_{i-1}; q_i)$ satisfying $f^l(x_0) = g^{n^l - l}(x_0)$; – this is the point $x_0 = g^l(y)$, where y is the unique fixed point for f^l in $(q_i; q_{i+1})$. Both ψ_i and ψ_{i+1} send the point x_0 to the same point $1 \in \mathbb{R}$. So we have $\psi_i = \psi_{i+1}$ on $(q_i; q_{i+1})$. □

It follows from Lemma 4.5 that the representation of $\mathrm{BS}(1; n^l)$ generated by f^l and g^l is C^r conjugate to an element of V. In the remainder of this section, we will show that the diffeomorphisms f and g are C^r ramified lifts of the generators of the standard action of $\mathrm{BS}(1; n)$ on S^1, hence the representation they generate is also C^r conjugate to an element of V. We begin with some lemmas about ramified lifts of flows on S^1.

Lemma 5.9 *Let $\varphi : S^1 \to S^1$ be a C^r flow with a fixed point at p, and let $\pi : S^1 \to S^1$ be a C^r ramified covering map over p. Let $F = \widehat{\varphi^1}(\pi; id)$ be the π–ramified lift of the time-1 map φ^1 with rotation number zero. Then F embeds as the time-1 map of a C^r flow F^t on S^1, and for all $t \in \mathbb{R}$, $F^t = \widehat{\varphi^t}(\pi; id)$.*

Proof By Lemma 4.2, given any $t \in \mathbb{R}$ there is a unique $(\pi; id)$-ramified lift of φ^t, $F^t := \widehat{\varphi^t}(\pi; id)$. Lemma 4.3 imples that $F^t \circ F^s = F^{s+t} = F^s \circ F^t$ for all $s, t \in \mathbb{R}$.

Let X be the C^{r-1} vector field that generates φ, and let $\hat{X}$ be the lift of X under π. This vector field is clearly C^{r-1} on $S^1 \setminus \pi^{-1}(p)$ and clearly generates the flow F^t on S^1. In a neighborhood of $q_i \in \pi^{-1}(p)$, $\hat{X}$ takes the form

$$\hat{X}(q) = d\pi_{(q)}^{-1} X(\pi q);$$

and π takes the form $\pi(x) = x^s$. A straightforward calculation shows that the vector field $\hat{X}$ is C^{r-1}. Similarly, $\hat{X}$ is analytic if X and π are. This completes the proof. □

Lemma 5.10 *Let $F : S^1 \to S^1$ be the time-1 map of a C^r flow F^t, where $r \geq 2$. Suppose that F is not r–flat, and $\rho(F) = 0$. If G is a C^r orientation preserving diffeomorphism such that $FG = GF$, and if $\rho(G) = 0$, then $G = F^t$ for some $t \in \mathbb{R}$.*

Proof Since $\rho(F) = 0$ and F is not r–flat, F has a finite set of fixed points. Let $q_1 < \dots < q_d$ be the elements of $\mathrm{Fix}(F)$. If $FG = GF$, then G permutes the fixed points of F, and since G is orientation preserving and has rotation number zero, $G([q_i, q_{i+1}]) = [q_i, q_{i+1}]$ for all $q_i \in \mathrm{Fix}(F)$. By Lemma 3.7, on $[q_i, q_{i+1})$, $G = F^{t_i}$ for some $t_i \in \mathbb{R}$, and on $(q_i, q_{i+1}]$, $G = F^{s_i}$ for some s_i. Clearly, $t_i = s_i$. So for $1 \leq i \leq d$,

$$G|_{[q_i, q_{i+1}]} = F^{t_i}, \text{ for some } t_i \in \mathbb{R}.$$

If $F'(q_i) \neq 1$ for some $q_i \in \mathrm{Fix}(F)$, then since G is C^1 at q_i, it follows that $t_i = t_{i-1}$. If $F'(q_i) = 1$, then in local coordinates, identifying q_i with 0,

$$F(x) = F^1(x) = x + ax^k + o(x^k)$$

for some $a \neq 0$ and $k \leq r$. Therefore

$$G(x) = \begin{cases} x + t_{i-1}ax^k + o(x^k); & \text{for } x \in (q_{i-1}, q_i] \\ x + t_i ax^k + o(x^k); & \text{for } x \in [q_i, q_{i+1}). \end{cases}$$

Since $k \leq r$ and G is C^r, $t_i = t_{i-1}$. □

Corollary 5.11 *Let $\psi\colon S^1 \to S^1$ be a C^r ramified covering map over ∞, and let $F = \hat{k}(\psi; id)$ be the $(\psi; id)$–ramified lift of $k \in \mathcal{A}(\mathbb{R})$, $k \neq id$. Let $s(\psi) = (s_1, \ldots, s_d; o_1, \ldots, o_d)$, where $s_i \leq r-1$ for $1 \leq i \leq d$. By Lemma 5.9, F embeds as the time-1 map of a C^r flow F^t. If $H\colon S^1 \to S^1$ is a C^r orientation preserving diffeomorphism such that $FH = HF$, and if $\rho(H) = 0$, then $H = F^t$ for some $t \in \mathbb{R}$.*

Proof By Lemma 5.10, it is enough to show that F is not r–flat. In coordinates identifying a fixed point with 0,

$$F(x) = \frac{x}{(b + ax^s)^{\frac{1}{s}}},$$

where either $a \neq 1$ or $b \neq 0$, which is clearly not r–flat, if $s < r$. □

Proposition 5.12 *Let $F \in \mathrm{Diff}^r(S^1)$ be a diffeomorphism such that F^l is orientation-preserving and $\rho(F^l) = 0$, for some $l > 0$. Suppose that $F^l = \hat{k}^l(\psi; id)$ is a C^r ramified lift of $k^l \neq id$, where $k \in \mathcal{A}_+(\mathbb{R})$, and suppose that $s(\psi) = (s(q_1), \ldots, s(q_d); o_1, \ldots, o_d)$, where $s(q_i) \leq r-1$ for $1 \leq i \leq d$. Then either F is a ψ–ramified lift of k or F is a ψ–ramified lift of $-k$.*

Proof Let $\sigma \in D_d$ be such that $\sigma(q) = F(q)$ for all $q \in \psi^{-1}(\infty)$, and $\sigma(e) = F(e)$ for all $e \in E(\psi)$.

Lemma 5.13 $\sigma \in \mathrm{Stab}^{\#}_{D_d}(s(\psi))$.

Proof Given any $q \in \psi^{-1}(\infty)$, there is an interval $[q, p)$ and C^r diffeomorphisms $h_1\colon [q, p) \to [0, \infty)$ and $h_2\colon [F(q), F(p)) \to [0, \infty)$ such that

$$h_1 F^l h_1^{-1}(x) = [k^l(x^s)]^{\frac{1}{s}}; \quad \text{and} \quad h_2 F^l h_2^{-1}(x) = [k^l(x^t)]^{\frac{1}{t}},$$

where $s = s(q)$ and $t = s(F(q))$. We can assume that k^l is a contraction on $[0, \infty)$. (If not, then use k^{-l} and F^{-l}). Since $F^l|_{[q,p)}$ is conjugate by F to $F^l|_{[F(q),F(p))}$, Lemma 4.6 implies that $s(q) = s(F(q))$, and therefore $\sigma \in \mathrm{Stab}^{\#}_{D_d}(s(\psi))$. □

By Lemma 5.13, either $\sigma(s(\psi)) = s(\psi)$, or $\sigma(s(\psi)) = I(s(\psi))$. If $\sigma(s(\psi)) = s(\psi)$, then let $\hat{\sigma}\colon S^1 \to S^1$ be the $(\psi; \psi)$–ramified lift of the identity map: $\hat{\sigma} = \widehat{id}(\psi; \psi)$. By Lemma 4.3, $\hat{\sigma}$ commutes with F^l. So $F\hat{\sigma}^{-1}$ commutes with F^l, and by construction, $F\hat{\sigma}^{-1}$ fixes every interval $(q_i, q_{i+1}) \subset \psi^{-1}(\mathbb{R}P^1 \setminus \{\infty\})$. By Lemma 5.10, F^l embeds as the time-1 map of a C^r flow, F^t, and $F\hat{\sigma}^{-1} =$

$F^{t_0} = \widehat{\ }_{0}(\ ; id)$ for some $t_0 \ 2 \ \mathbb{R}$, where is an analytic flow with ${}^1 = k^l$. Therefore $F = \widehat{\ }_{0}(\ ; id) \ \widehat{k}^l(\ ; \ ) = \widehat{\ }_{0}(\ ; \ )$ (using Lemma 4.3). It follows that $t_0 = 1{=}l$, and therefore F is the $(\ ; \ )${rami ed lift of the map k.

If (s( )) = I(s( )), then we let $= \widehat{-id}(\ ; \ )$, the $(\ ; \ )${rami ed lift of $-id\colon x \ ! \ -x$. As above, $F \ ^{-1}$ is the $(\ ; \ )${rami ed lift of k, and therefore $F = \ ^{t} \ = \widehat{-k}(\ ; \ )$. ◻

Corollary 5.14 *f and g are C^r rami ed lifts under of the generators of the standard action of* BS$(1;n)$ *on* S^1.

Proof The standard representation $_{n^l}\colon$ BS$(1;n^l) \ ! \ $Di $^l(\mathbb{R}P^1)$ is analytically conjugate to the representation : BS$(1;n^l) \ ! \ $Di $^l(\mathbb{R}P^1)$ with generators $(a^l)\colon x \ 7! \ n^l x$ and $(b^l)\colon x \ 7! \ x + l$. So there is a C^r rami ed covering map $: S^1 \ ! \ S^1$ over p such that $f^l = \widehat{(a^l)}(\ ; id)$ and $g^l = \widehat{(b^l)}(\ ; id)$. By Proposition 5.12, either f is a rami ed lift of $_n(a)\colon x \ 7! \ nx$, or f is a rami ed lift of $- \ _n(a)\colon x \ 7! \ -nx$. Similarly, g is either a rami ed lift of $_n(b)\colon x \ 7! \ x + 1$, or a rami ed lift of $- \ _n(b)\colon x \ 7! \ -x - 1$. Since f and g satisfy the relation $fgf^{-1} = g^n$, the maps that they are lifted from must also satisfy this relation. Given this requirement, the only possibility is that f is a {rami ed lift of $_n(a)$ and g is a {rami ed lift of $_n(b)$. ◻

Since the generators $(a) = f$ and $(b) = g$ of the representation are rami ed lifts under of the generators $_n(a)$ and $_n(b)$, respectively, of $_n$, it follows that, for every $\gamma \ 2$ BS$(1;n)$, there exists a unique $h(\gamma) \ 2 \ D_d$ (or in C_d if is orientation-preserving) such that:

$$(\gamma) = \widehat{\ _n(\gamma)}(\ ; h(\gamma)):$$

Since $(\gamma_1\gamma_2) = (\gamma_1) \ (\gamma_2)$, it follows that $h\colon$ BS$(1;n) \ ! \ D_d$ (C_d) is a homomorphism. Finally, note that h must take values in Stab$_{D_d}$(s) (or Stab$_{C_d}$(s), if is orientation-preserving).

This conclude the proof of Theorems 1.1 and 1.3. ◻

Finally, we sketch the proof of Theorem 1.6.

Sketch of proof of Theorem 1.6 Let be a C^r representation of BS$(1;n)$, with $r \quad 2$, let $f = (a)$ and $g = (b)$. We may assume that f has rational rotation number. By taking powers of the elements of BS$(1;n)$, we may assume that both f and g have rotation number 0. Assume that g is not the identity map.

Let J be a component of the complement of $\mathrm{Fix}(g)$. Using a distortion estimate and the group relation one shows that J must be fixed by f, as follows. Otherwise, the f–orbit of J must accumulate at both ends on a fixed point of f. The standard C^2 distortion estimate shows that there is an $M > 1$ such that for all x, y in same component of the f–orbit of J, and for all $k \in \mathbb{Z}$,

$$\frac{1}{M} < \left|\frac{(f^k)'(x)}{(f^k)'(y)}\right| < M.$$

But, for all $k \in N$, we have that $f^k g f^{-k} = g^{n^k}$. Hence, for all $p \in J$, we have:

$$(g^{n^k})'(p) = g'(y)\frac{(f^k)'(g(y))}{(f^k)'(y)},$$

where $y = f^{-k}(p)$. Note that y and $g(y)$ lie in the same component $f^{-k}(J)$, and $g'(y)$ is uniformly bounded. This implies that for all $p \in J$ and all $k \in \mathbb{N}$, $(g^{n^k})'(p)$ is bounded, so that $g = id$ on J, a contradiction.

So f fixes each component of the complement of $\mathrm{Fix}(g)$. Let J be such a component. Since g has no fixed points on J, g embeds in a C^1 flow g^t, defined on J minus one of its endpoints, that is C^r in the interior of J (see, eg [16]). Furthermore, for all t, $fg^tf^{-1} = g^{nt}$ (this follows from Kopell's lemma). Fixing some point p in the interior of J, this flow defines a C^r diffeomorphism between the real line and the interior of J, sending $t \in \mathbb{R}$ to $g^t(p) \in J$. Conjugating by this diffeomorphism, g^t is sent to a translation by t, and f is sent to a diffeomorphism F satisfying $F(x+t) = F(x)+tn$, for all $t, x \in \mathbb{R}$. But this means that $F'(x) = n$ for all $x \in \mathbb{R}$. Up to an affine change of coordinates, g is conjugate on J to $x \mapsto x+1$ and f is conjugate to $x \mapsto nx$. □

6 Proof of Proposition 1.5

Let $\Phi\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^\omega(S^1)$ be a –ramified lift of the standard representation ρ_n with $\lambda(\Phi) = \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$, for some $r \geq 2$. Let Q be the set of all points $q \in \Phi^{-1}(\infty)$ satisfying $s(q) = r-1$; this set is nonempty since $\lambda(\Phi) = \left(\frac{1}{n}\right)^{\frac{1}{r-1}}$. Lemma 4.1 implies that, in a neighborhood of q, $\Phi\colon x \mapsto x^{r-1}$, in the appropriate coordinates identifying q with 0.

For $t \in (-1, 1)$ we deform Φ to obtain a C^{r-1+t^2} map $\Phi_t\colon S^1 \to S^1$ with the following properties:

$_0 =$ and $^{-1}(1) = {}_t^{-1}(1)$, for all t;

$_t|_{S^1 \setminus {}_t^{-1}(1)}$ is a C^1 covering map onto its image;

about each $q \in {}^{-1}(1) \setminus Q$, $_t$ is locally equal to ;

about each $q \in Q$, $_t$ is locally $x \mapsto x^{r-1+t^2}$, in the same charts identifying q with 0 described above.

A slight modi cation of the proof of Proposition 4.4 also shows that $_n$ has a lift to a C^r representation $_t\colon \mathrm{BS}(1;n) \to \mathrm{Di}^r(S^1)$ so that the following diagram commutes, for all $\gamma \in \mathrm{BS}(1;n)$:

$$\begin{array}{ccc} S^1 & \xrightarrow{\ {}_t(\gamma)\ } & S^1 \\ {}_t\big\downarrow & & {}_t\big\downarrow \\ S^1 & \xrightarrow{\ {}_n(\gamma)\ } & S^1 \end{array}$$

(One merely needs to check that the integer j in the proof of Lemma 4.2 can be replaced by the real number $r - 1 + t^2$).

Notice that $_t$ has the property that $({}_t) = \left(\frac{1}{n}\right)^{\frac{1}{r-1+t^2}}$, so that $_s$ is not C^1 conjugate to $_t$ unles $s = t$. One can further modify this construction by replacing the points of Q by intervals of length ϵ_t, extending $_t(b)$ isometrically across these intervals, and extending $_t(a)$ in an arbitrary C^r fashion to these intervals. Since $_t(b)$ is r–flat (by Lemma 3.2) on Q for $t \neq 0$ and $r-1$ flat for $t = 0$, the representation $_t$ is C^r and varies C^{r-1} continuously in t if we choose $\epsilon_t \to 0$ as $t \to 0$. In this way, one can create uncountably many deformations of . (Note that, in essence, we have deformed to obtain a "broken C^r rami ed cover" a la Theorem 1.6). □

7 Proof of Theorem 1.9

Let $r \in \{1; \infty\}$, and let $G < \mathrm{Di}^r(S^1)$ be a solvable group without in nitely flat elements. Suppose that $G^m := \{g^m : g \in G\}$ is not abelian, for any $m \in \mathbb{Z}$. We begin by showing that the group $G^2 < \mathrm{Di}^r_+(S^1)$ has a nite set of points that is globally invariant.

Lemma 7.1 *G^2 contains a non-trivial normal abelian subgroup N, such that N contains an element of in nite order. There is an integer $d > 0$, and a nite set $\{q_1; \ldots; q_d\}$, with $q_1 < q_2 < \cdots < q_d$, such that:*

(1) *for all* $f \in G^2$, $\rho(f^d) = 0$ *and* $f\{q_1, \ldots, q_d\} = \{q_1, \ldots, q_d\}$;
(2) *for all* $g \in N$, *either* $g^d = id$ *or* $\mathrm{Fix}(g^d) = \{q_1, \ldots, q_d\}$.

Proof Note that G^2 is a solvable group, and every diffeomorphism in G^2 is orientation preserving. Let

$$G^2 = G_0 > G_1 > \ldots > G_n > G_{n+1} = \{id\}$$

be the derived series for G^2, and let $N = G_n$ be the terminal subgroup in this series. Recall that N is a normal abelian subgroup of G^2. We first show that N contains an element of infinite order. We will use the following result of Ghys ([7] Proposition 6.17):

Lemma 7.2 *If* $H \subset \mathrm{Homeo}_+(S^1)$ *is solvable, then the rotation number* $\rho: H \to \mathbb{R}/\mathbb{Z}$ *is a homomorphism.*

Suppose that every diffeomorphism in N has finite order. Since $G^2 \neq N$ (because G^2 is not abelian), G_{n-1} cannot be abelian – if it were, then G_n would be trivial. Suppose that $f, h \in G_{n-1}$. By Lemma 7.2, $\rho(fhf^{-1}h^{-1}) = 0$. But $fhf^{-1}h^{-1}$ is orientation preserving and has finite order, since $fhf^{-1}h^{-1} \in G_n$. Therefore $fhf^{-1}h^{-1} = id$, and G_{n-1} is abelian, a contradiction. So N contains a diffeomorphism with infinite order.

If $\rho(g)$ is irrational, for some orientation-preserving $g \in N$, then the elements of N are simultaneously conjugate to rotations. But, since N is normal in G^2, this implies that the elements of G^2 are simultanously conjugate to rotations, which implies that G^2 is abelian, a contradiction.

Hence $\rho(g) \in \mathbb{Q}/\mathbb{Z}$, for every $g \in N$. Note that every $g \in N$ either has finite order, or a finite set of periodic points: if $\mathrm{Fix}(g^l)$ is infinite, for some integer $l \neq 0$, then there is a point $q \in \mathrm{Fix}(g^l)$ that is an accumulation point for a sequence $\{q_i\} \subset \mathrm{Fix}(g^l)$. But this implies that g^l is infinitely flat at q, and therefore $g^l = id$.

Hence there exists $g \in N$ with infinite order and a finite fixed set, $\mathrm{Fix}(g) = \{q_1, \ldots, q_d\}$. If $h \in N$ is another element of N, then, since h commutes with g, it follows that $h(\{q_1, \ldots, q_d\}) = \{q_1, \ldots, q_d\}$, and so $\rho(h^d) = 0$. If the set of fixed points for h^d is infinite, then $h^d = id$, and if $\mathrm{Fix}(h^d)$ is finite, then $\mathrm{Fix}(h^d) = \{q_1, \ldots, q_d\}$.

Finally, let $f \in G^2 \setminus N$ and pick $g \in N$ satisfying $\mathrm{Fix}(g) = \{q_1, \ldots, q_d\}$. Then there exists a $\overline{g} \in N$ such that $fgf^{-1} = \overline{g}$. This implies that $f(\mathrm{Fix}(g)) = \mathrm{Fix}(\overline{g})$; that is, $f(\{q_1, \ldots, q_d\}) = \{q_1, \ldots, q_d\}$. It follows that $\rho(f^d) = 0$. This completes the proof. □

Let $\{q_1, \dots, q_d\}$ be given by the previous lemma, labelled so that $-\infty \le q_1 < q_2 < \cdots < q_d < \infty$, and let $l = 2d$. We will begin by working with the group G^l. Note that every $g \in G^l$ is orientation-preserving, has zero rotation number, and fixes every point in the set $\{q_1, \dots, q_d\}$. Throughout this section, we will be working on the intervals (q_i, q_{i+1}), where we adopt the convention that $q_{d+1} = q_1$.

Let M be a normal abelian subgroup of G^l which contains an element of infinite order. For the rest of the proof, fix a diffeomorphism $g \in M$ which has infinite order.

Lemma 7.3 *Let* $C(g) = \{f \in G^l \mid gf = fg\}$*. Then* $C(g) \neq G^l$*.*

Proof A proof of this lemma is essentially contained in [5]. This lemma is implied by the following theorem, which is classical.

Theorem 7.4 (Hölder's Theorem) *If a group of homomorphisms acts freely on* $\mathbb{R}$*, then it is abelian.*

If $f \in C(g)$, then $\mathrm{Fix}(f) = \mathrm{Fix}(g)$. So on every interval (q_i, q_{i+1}), $1 \le i \le d$, no element of $C(g)$ has a fixed point. By Theorem 7.4, the restriction of the action of $C(g)$ to each interval (q_i, q_{i+1}) is abelian. Since $f(q_i) = q_i$ for all $f \in C(g)$ and for all $q_i \in \mathrm{Fix}(g)$, $C(g)$ is an abelian subset of G^l. But G^l is not abelian, so $C(g) \neq G^l$. □

Lemma 7.5 *Let* $f \in G^l \setminus C(g)$*. Then for every interval* $(q_{i-1}, q_i]$ *there is a positive real number* λ_i *and a* C^r *map* $\phi_i \colon (q_{i-1}, q_i] \to \mathbb{R}P^1$ *with the following properties:*

1. $\phi_i g(p) = \phi_i(p) + 1$ *and* $\phi_i f(p) = \lambda_i \phi_i(p)$*;*
2. $\phi_i|_{(q_{i-1}, q_i)}$ *is a* C^r *diffeomorphism onto* $(-\infty, \infty)$*;*
3. *there is an orientation-preserving* C^r *diffeomorphism* h_i *from a neighborhood of* q_i *to a neighborhood of* ∞ *and integers* $s_i \in \{1, \dots, r-1\}$*,* $o_i \in \{\pm 1\}$*, such that, for all* p *in this neighborhood:*

$$\phi_i(p) = o_i h_i(p)^{s_i}$$

The same conclusions hold, with the same λ_i*,* o_i *and* $\phi_i|_{(q_{i-1}, q_i)}$*, but different local (orientation-reversing) diffeomorphism* h_i *and integer* s_i*, when* q_i *is replaced by* q_{i-1} *and the interval* $(q_{i-1}, q_i]$ *is replaced by* $[q_{i-1}, q_i)$*.*

Remark To ensure that the conditions $\lambda_i > 0$ (as opposed to $\lambda_i \neq 0$) hold in Lemma 7.5, it is necessary that we chose l to be even.

Proof We use the following fact, proved by Takens:

Theorem 7.6 ([15], Theorem 4) *Let $h\colon [0,1) \to [0,1)$ be a C^∞ diffeomorphism with unique fixed point $0 \in [0,1)$. If h is not infinitely flat, then there exists a unique C^∞ vector field X on $[0,1)$ such that $h = h^1$, where h^t is the flow generated by X.*

For each $1 \leq i \leq d$, Let $g_i^t\colon (q_{i-1}, q_i] \to (q_{i-1}, q_i]$ be the flow given by this theorem with $g_i^1 = g|_{(q_{i-1}, q_i]}$. If $f \in G^l \setminus C(g)$, then since $g \in M$, we have $fgf^{-1} \in M$, and therefore $fgf^{-1} \in C(g)$. By Lemma 3.7, for $1 \leq i \leq d$, we must have

$$fgf^{-1} = g_i^{\lambda_i}$$

on $(q_{i-1}, q_i]$, for some $\lambda_i \in \mathbb{R} \setminus \{0\}$, $\lambda_i \neq 1$. Note that, because l is even, λ_i must be positive, for all i. So assumption (D) of Section 3 holds in the interval $(q_{i-1}, q_i]$ for each $q_i \in \{q_1, \ldots, q_d\}$.

The same reasoning can be applied to $[q_{i-1}, q_i)$, using a possibly different flow $\tilde{g}_i^t$ and constant $\tilde{\lambda}_i$ with

$$fgf^{-1} = \tilde{g}_i^{\tilde{\lambda}_i}.$$

Since $g_i^{\lambda_i}$ and $\tilde{g}_i^{\tilde{\lambda}_i}$ coincide on (q_i, q_{i+1}), it is not hard to see that we must have $\lambda_i = \tilde{\lambda}_i$. Now the result follows from Proposition 3.1, as in the proof of Lemma 5.6 and Corollary 5.8. □

Corollary 7.7 *For every $f \in G^l \setminus C(g)$, there is a positive real number $\lambda = \lambda(f) \neq 1$ such that $\lambda_i = \lambda$, for all $1 \leq i \leq d$, where λ_i is given by Lemma 7.5. For every i, $s_i = s_{i+1}$, where addition is mod d.*

Proof Let $1 \leq i \leq d$. As in the proof of Corollary 5.8, we have that g is conjugate in a left neighborhood of q_i to $x \mapsto x/(1 + x^{s_i})^{1/s_i}$ and g is conjugate in a right neighborhood of q_i to $x \mapsto x/(1 + x^{s_{i+1}})^{1/s_{i+1}}$. Since g is C^∞, we must have $s_i = s_{i+1}$. But then f is conjugate in a left neighborhood of q_i to $x \mapsto x/\lambda_i^{1/s_i}$ and f is conjugate in a right neighborhood of q_i to $x \mapsto x/(\lambda_{i+1})^{1/s_{i+1}}$. It follows that $\lambda_i = \lambda_{i+1}$. Set λ to be this common value. □

The proof of the next corollary is identical to the proof of Corollary 5.8.

Corollary 7.8 *For every* $f \in G^l \setminus C(g)$*, the map* $\pi : S^1 \to \mathbb{R}P^1$ *defined by:*

$$\pi(p) = \pi_i(p); \quad \text{for } p \in (q_{i-1}, q_i]$$

is a C^r *ramified cover with signature* $s = (s_1, \ldots, s_d; o_1, \ldots, o_d)$*. The diffeomorphism* f *is a* π*–ramified lift of the map* $x \mapsto \lambda(f)x$*, and* g *is a* π*–ramified lift of the map* $x \mapsto x + 1$*.*

Corollary 7.9 g *embeds in a unique* C^r *flow* g^t*, with* $g = g^1$*. The elements of* $C(g)$ *belong in the flow for* g *and, for each* $f \in G^l \setminus C(g)$*, lie in the ramified lift under* π_f *of the translation group* $\{x \mapsto x + \beta \mid \beta \in \mathbb{R}\}$*. That is, for any* $h \in C(g)$*, there exist real numbers* β*,* t *such that* $h = g^t$ *is a* π_f*–ramified lift of the map* $x \mapsto x + \beta$*.*

Proof This corollary follows directly from Corollary 7.8, Lemma 5.9 and Lemma 5.10. □

Lemma 7.10 *For any* $f_1, f_2 \in G^l \setminus C(g)$*, there exists a real number* γ *such that* f_2 *is a* π_{f_1}*–ramified lift of the map* $x \mapsto \lambda(f_2)x + \gamma$*.*

Proof The proof is expressed in a series of commutative diagrams.

Lemma 7.11 *There exists an* $\alpha \in \mathbb{R}$ *such that* g^α *is the* $(\pi_{f_1}, \pi_{f_2}, id)$*–ramified lift of the identity map.*

Proof The following diagram shows that if $\widehat{id}$ is the $(\pi_{f_1}, \pi_{f_2}, id)$–ramified lift of the identity map on $\mathbb{R}P^1$, then $\widehat{id} \circ g = g \circ \widehat{id}$:

$$\begin{tikzcd}
 & S^1 \arrow[rr, "g"] \arrow[dd, "\pi_{f_2}"] & & S^1 \arrow[dd, "\pi_{f_2}"] \\
S^1 \arrow[ur, "\widehat{id}"] \arrow[rr, "g"] \arrow[dd, "\pi_{f_1}"] & & S^1 \arrow[ur, "\widehat{id}"] \arrow[dd, "\pi_{f_1}"] & \\
 & \mathbb{R}P^1 \arrow[rr, "x \mapsto x+1"] & & \mathbb{R}P^1 \\
\mathbb{R}P^1 \arrow[ur, "id"] \arrow[rr, "x \mapsto x+1"] & & \mathbb{R}P^1 \arrow[ur, "id"] &
\end{tikzcd}$$

Since g embeds in a flow g^t that is a ramified lift of an affine flow, it follows from Corollary 7.9 that there exists an such that $h = g$. □

Lemma 7.12 *For every $t_0 \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that g^{t_0} is the $(\ _{f_1}; \ _{f_2}; id)$–ramified lift of the map $x \mapsto x - y$.*

Proof Let be given by the previous lemma. Let y be the real number such that $g^{t_0 -}$ is a $\ _{f_1}$-ramified lift of $x \mapsto x - y$. The proof follows from the following diagram:

$$\begin{array}{ccccc} S^1 & \xrightarrow{g^{t_0 -}} & S^1 & \xrightarrow{h = g} & S^1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ \mathbb{R}P^1 & \xrightarrow{x \mapsto x - y} & \mathbb{R}P^1 & \xrightarrow{id} & \mathbb{R}P^1 \end{array}$$

The composition of the maps on the top row is $g \ \ g^{t_0 -} = g^{t_0}$. The composition of the maps on the bottom row is $x \mapsto x - y$. □

Lemma 7.13 *For all $t \in \mathbb{R}$, $f_2 g^t f_2^{-1} = g^{\ (f_2)t}$.*

Proof The proof follows from the following diagram:

$$\begin{array}{ccccccc} S^1 & \xrightarrow{f_2^{-1}} & S^1 & \xrightarrow{g^t} & S^1 & \xrightarrow{f_2} & S^1 \\ {\scriptstyle f_2}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_2}\downarrow \\ \mathbb{R}P^1 & \xrightarrow{x \mapsto \frac{1}{\ (f_2)} x} & \mathbb{R}P^1 & \xrightarrow{x \mapsto x + t} & \mathbb{R}P^1 & \xrightarrow{x \mapsto \ (f_2) x} & \mathbb{R}P^1 \end{array}$$

The composition of the maps on the bottom row gives $x \mapsto x + \ (f_2)t$. By uniqueness (Lemma 5.9), $f_2 g^t f_2^{-1} = g^{\ (f_2)t}$ for all $t \in \mathbb{R}$. □

Let be given by Lemma 7.11, and let y be given by Lemma 7.12, with $t_0 = \ (f_2)\ $. From the following diagram:

$$
\begin{tikzcd}
& S^1 \arrow[rr, "g"] \arrow[dd, "f_1"] & & S^1 \arrow[dd, "f_2"] \\
S^1 \arrow[ur, "f_2"] \arrow[rr, "h = g"] \arrow[dd, "f_1"] & & S^1 \arrow[ur, "f_2"] \arrow[dd, "f_2"] & \\
& \mathbb{R}P^1 \arrow[rr, "x \mapsto x - y"] & & \mathbb{R}P^1 \\
\mathbb{R}P^1 \arrow[ur, "F"] \arrow[rr, "id"] & & \mathbb{R}P^1 \arrow[ur, "x \mapsto \lambda(f_2)x"] &
\end{tikzcd}
$$

it follows that $F(x) = \lambda(f_2)x + y$, completing the proof of Lemma 7.10. □

Proposition 7.14 *Fix $f \in G' \setminus C(g)$. Then for each $h \in G$, there exists $F \in \mathrm{Aff}(\mathbb{R})$ such that h is a λ_f–ramified lift of F.*

Proof By Corollary 7.9 and Lemma 7.10, we have that for each $h \in G$, there exists $k \in \mathrm{Aff}_+(\mathbb{R})$, so that h' is a λ_f–ramified lift of k'. Therefore, by Proposition 5.12, h is a λ_f–ramified lift of either k or $-k$. □

By Lemma 4.5, this completes the proof of Theorem 1.9. □

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