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On the dynamics of isometries

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Abstract

We provide an analysis of the dynamics of isometries and semicontractions of metric spaces. Certain subsets of the boundary at infinity play a fundamental role and are identified completely for the standard boundaries of CAT(0)–spaces, Gromov hyperbolic spaces, Hilbert geometries, certain pseudoconvex domains, and partially for Thurston's boundary of Teichmüller spaces. We present several rather general results concerning groups of isometries, as well as the proof of other more specific new theorems, for example concerning the existence of free nonabelian subgroups in CAT(0)–geometry, iteration of holomorphic maps, a metric Furstenberg lemma, random walks on groups, noncompactness of automorphism groups of convex cones, and boundary behaviour of Kobayashi's metric.

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1 Introduction

The notion of a metric space was introduced in 1906 by M Fréchet. Although a systematic study of metric spaces from the point of view of point-set topology was subsequently undertaken, the most natural morphisms, isometries and semicontractions, seem to have received less attention. To be sure, there are special topics that have inspired deep investigations: Euclidean and hyperbolic geometry, extensions of the contraction mapping principle, iteration of holomorphic maps as well as, in more recent years, CAT(0)–spaces and Gromov hyperbolic groups. But in contrast to the category of topological vector spaces and continuous linear operators, a basic general text on metric spaces and semicontractions seems to be absent. Note that there are a number of contexts in for example geometry, topology, complex analysis in one and several variables, Lie theory, ergodic theory and group theory, where metrics and semicontractions arise. Some of these will be recalled in more detail later on.

This paper presents a general and unified theory of the dynamics of semicontractions and (groups of) isometries. It studies and exploits (generalized) halfspaces and their limits, *the stars at infinity*. These subsets are of fundamental importance for the dynamics of isometries and provide moreover a convenient framework for asymptotical geometric information, and should therefore be of interest to the subjects of Riemannian and metric geometry. Even though halfspaces are classical in the definition of Dirichlet fundamental domains and appear particularly in the literature on Kleinian groups, it seems they have not been systematically considered previously. The stars relate well to standard concepts such as Tits geometry of CAT(0)–spaces, Thurston's boundary of Teichmüller space, hyperbolicity of metric spaces, strict pseudoconvexity, the face lattice of convex domains, rank 1 isometries, etc.

In the theory of word hyperbolic groups, the study of how the group acts on its boundary plays an important role. Our generalizations of hyperbolic phenomena bringing in the stars and their incidence geometry, are perhaps also interesting in light of Mostow's proof of strong rigidity in the higher rank case.

Several of the results obtained are new even in areas which have been much studied, for example CAT(0)–geometry or boundary behaviour in complex domains and holomorphic maps. Let us highlight a few of these results:

Theorem 1 *Let X be a proper* CAT(0)*–space. Assume g_n is a sequence of isometries such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$. Then for any $\eta \in \overline{X}$ with $\angle(\eta, \xi^-) > \pi/2$ we have that*

$$g_n \eta \to \{\zeta : \angle(\xi^+, \zeta) \leq \pi/2\}$$

and the convergence is uniform outside neighborhoods of $S(\xi^-)$.

This is completely new in that it equally well deals with parabolic isometries. Previously, mainly iterates of a single hyperbolic isometry could be treated, in which case a lemma of Schroeder [6] generalized by Ruane [33] to include also singular CAT(0)–spaces actually gives more information. For several other new corollaries on groups acting on CAT(0)–spaces, see section 6. Next, the following describes a novel phenomenon for simple random walks on *any* finitely generated nonamenable group:

Theorem 2 *Let* Γ *be a nonamenable group generated by a finite set* S *and consider the random walk defined by the uniform distribution on* $S \cup S^{-1}$. *For almost every trajectory there is a time after which every finite collection of halfspaces defined by the trajectory intersect nontrivially.*

For more discussion and explanations, see subsection 4.4. Every holomorphic map is in a sense a semicontraction and taking advantage of this we will obtain the following new Wolff–Denjoy theorem:

Theorem 3 *Let* X *be a bounded* C^2*–domain in* $\mathbb{C}^n$ *which is complete in the Kobayashi metric satisfying the boundary estimate (6) in subsection 8.2. Let* $f : X \to X$ *be a holomorphic map. Then either the orbit of* f *stays away from the boundary or there is a unique boundary point* ξ *such that*

$$\lim_{m\to\infty} f^m(z) = \xi$$

for any $z \in X$.

Examples include real analytic pseudoconvex domains in which case the theorem for $n = 2$ was proved by Zhang and Ren in [36]. Finally, we mention:

Theorem 4 *Any polyhedral cone with noncompact automorphism group has simplicial diameter at most 3.*

Here one should note that in dimension 2 a rather complete result concerning which convex sets have infinite automorphism group can be found in de la Harpe's paper [16].

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Part I General theory

2 Halfspaces and stars at infinity

2.1 Definitions

Let X be a metric space. For a subset W of X we let

$$d(x, W) = \inf_{w \in W} d(x, w).$$

Fix a base point x_0. We define the *halfspace* defined by the subset W and the real number C to be

$$H(W, C) = H^{x_0}(W, C) := \{z : d(z, W) \leq d(z, x_0) + C\}.$$

We use the notation $H(W) := H(W, 0)$ and for two points x and y in X we let $H_y^x = \{z : d(z, y) \leq d(z, x)\}$, so $H_y^{x_0} = H(\{y\}, 0)$. Note that the latter sets define halfspaces in the more standard sense when X is a Euclidean or real hyperbolic space. See Figure 1.

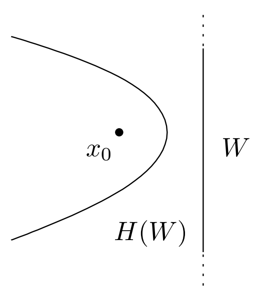

Figure 1: The halfspace defined by a line W in $\mathbb{R}^2$ is the region containing W bounded by a parabola.

Let X be a complete metric space. By a *bordification* of X we here mean a Hausdorff topological space $\overline{X}$ with X embedded as an open dense subset. The *boundary* is $\partial X = \overline{X} \setminus X$. If $\overline{X}$ is compact we refer to it as a *compactification.* We define $d(x, \xi) = \infty$ for any $x \in X$ and $\xi \in \partial X$ (which is consistent with the completeness of X) and extend the definition of $d(x, W)$ for $W \subset \overline{X}$ in the expected way.

A metric space is *proper* if every closed ball is compact. Recall that every proper metric space X has a (typically nontrivial) metrizable Isom(X)–compactification

$\overline{X}^h$ by horofunctions: X is embedded into the space of continuous functions $C(X)$ with the topology of uniform convergence on bounded sets via

$$x \mapsto d(x, \cdot) - d(x, x_0)$$

The closure of the image now defines the compactification, see [6], [4], and [8] for more details.

Example Another general compactification, *the end compactification*, was introduced by Freudenthal. Here let X be path connected, proper metric space and define the following equivalence relation on the set of proper rays from x_0. Two rays are equivalent if for any compact set K in X, the two rays are eventually contained in the same path connected components of $X \setminus K$. The equivalence classes of proper rays union X with the natural topologization constitute the compactification of X. See [8] for more details.

Let $\mathcal{V}_\xi$ denote the collection of open neighborhoods in $\overline{X}$ of a boundary point ξ. The *star based at* x_0 of a point $\xi \in \partial X$ is

$$S^{x_0}(\xi) := \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V)},$$

where the closures are taken in $\overline{X}$, and the *star* of ξ is

$$S(\xi) := \overline{\bigcup_{C \geq 0} \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V, C)}}.$$

The latter definition in particular removes an a priori dependence of x_0 as will be clear later on. Note also that because of the monotonicity built into the definition of H, we may restrict $\mathcal{V}_\xi$ to some fundamental system of neighborhoods of ξ.

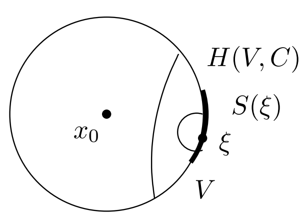

Figure 2: The definition of $S(\xi)$

We introduce the *star-distance*: Let s be the largest metric on ∂X taking values in $[0, \infty]$ such that $s(\xi, \eta) = 0$ if $S(\xi) = S(\eta)$, and $s(\xi, \eta) = 1$ if at least one of $\xi \in S(\eta)$ or $\eta \in S(\xi)$ holds. More explicitly, $s(\xi, \eta)$ equals the minimum number k such that there are points γ_i with $\gamma_0 = \xi$, $\gamma_k = \eta$, and $s(\gamma_i, \gamma_{i+1}) = 1$ for all i. See Figure 2.

Example Let X be the Euclidean space $\mathbb{R}^n$ and ∂X the visual sphere at infinity being the space of geodesic rays from the origin. Then $S(\xi)$ as well as $S^{x_0}(\xi)$ is the hemishpere in ∂X centered at ξ. For a generalization of this example, see Proposition 25. Hence the visual sphere has stardiameter 2. Indeed, for all points $\eta \in S(\xi)$ different from ξ we have $s(\eta, \xi) = 1$ and for the points ζ outside this star $s(\zeta, \xi) = 2$.

Example Let X be a proper and path connected metric space and ∂X the space of ends as defined above. Consider two nonequivalent proper rays denoted η and ξ (with an abuse of notation). For any two small disjoint neighborhoods of these points, there is a compact set K which separate these two neighborhoods in the sense that they are in different path components of $X \setminus K$. Therefore any path between the two neighborhoods must pass through K, and it follows that η is not in $S(\xi)$. Hence $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ and $s(\xi, \eta) = \infty$ for any two distinct boundary points. See section 5 for further examples with this kind of "trivial" or "hyperbolic" star geometry.

It does not seem clear whether, or when, $\xi \in S(\eta)$ implies $\eta \in S(\xi)$. Let

$$S^\vee(\xi) = \{\eta : \xi \in S(\eta)\},$$

and we say that the bordification is *star-reflexive* when $S(\xi) = S^\vee(\xi)$ for all ξ. The examples below turn out to have this property.

The *face* of a subset A of ∂X is the intersection of all stars containing A. The face of the empty set is defined to be the empty set. We define for a subset $A \subset \partial X$ the sets

$$S(A) = \bigcap_{a \in A} S(a).$$

and similarily for $S^\vee(A)$.

By the notation $x_n \to S$, where x_n is a sequence of points and S a set, we mean that for any neighborhood U of S we have $x_n \in U$ for all sufficiently large n.

2.2 Some lemmas

Lemma 5 *For any $\xi \in \partial X$, the sets $\overline{H(V)}$ for $V \in \mathcal{V}_\xi$ contain V and $\xi \in S^{x_0}(\xi) \subset S(\xi) \subset \partial X$. If ∂X is compact, then for every neighborhood U of $S^{x_0}(\xi)$ there is a neigborhood V of ξ such that $\overline{H(V)} \subset U$.*

Proof Note that $V \subset \overline{H(V)}$. Indeed, first observe that $V \cap X \subset H(V)$ because $d(v,V)=0$ for any $v \in V$. Secondly, note that for any $v \in V$ and any open neighborhood U of v, $U \cap V$ is again an open neighborhood and every open set in $\overline{X}$ has to intersect X. Finally, $S^{x_0}(\xi)$ is nonempty because ξ is contained in every V, and $S^{x_0}(\xi) \subset S(\xi) \subset \partial X$ since $d(V,x_0)$ is unbounded for $V \in \mathcal{V}_\xi$.

Let U be a neighborhood of $S^{x_0}(\xi)$. We may assume that U is open and so U^c is compact. Consider a fundamental system of neighborhoods of ξ. Suppose for any V in this system it holds that $\overline{H(V)} \cap U^c \neq \emptyset$. Becuase of the monotonicity of halfspaces we hence have a decreasing, nested system of closed sets $\overline{H(V)} \cap U^c$ inside U^c. By compactness we get $\bigcap \overline{H(V)} \cap U^c \neq \emptyset)$. This is a contradiction to $S^{x_0}(\xi) \subset U$, and proves the last assertion of the lemma. □

Note that if $z_n \to \xi$ and $d(z_n, y_n) < C$ then every limit point of y_n belongs to $S^{x_0}(\xi)$. A priori, $S^{x_0}(\xi)$ depends on x_0 although in the examples below this turns out not to be the case. On the other hand:

Lemma 6 *The sets $S(\xi)$ are independent of the base point x_0. If $z_n \to \xi \in \partial X$, $d(z_n, y_n) < C$ and $y_n \to \eta$, then $S(\xi) = S(\eta)$. Moreover, ξ and η belong to the same stars.*

Proof The first statement follows from

$$H^{x_0}(W, C - d(x,x_0)) \subset H^x(W,C) \subset H^{x_0}(W, C + d(x,x_0)),$$

and because of the increasing union over $C \geq 0$ in the definition of $S(\xi)$. The other two claims hold for similar reasons. □

Lemma 7 *Assume that $\overline{X}$ is sequentially compact and that $S(\xi) = S^{x_0}(\xi)$ for every $\xi \in \partial X$. Let ξ_n and η_n be two sequences in ∂X converging to ξ and η, respectively. If $s(\xi_n, \eta_n) > 0$ for all n, then*

$$s(\xi, \eta) \leq \liminf_{n \to \infty} s(\xi_n, \eta_n).$$

Proof By the assumption we can work with the S^{x_0}–stars. It is enough to consider $s(\xi_n, \eta_n) = 1$ for all n, because of the sequential compactness and the way s is defined. Moreover, we may suppose that $\xi_n \in S(\eta_n)$ for all n. Hence $\xi_n \in \overline{H(V)}$ for every neighborhood V of η_n. Given a neighborhood U of η, there is a N such that U is also a neighborhood of η_n for $n \geq N$. We therefore have that $\xi_n \in \overline{H(U)}$ for all $n \geq N$, and hence also $\xi \in \overline{H(U)}$. Because U was arbitrary, we have that $\xi \in S(\eta)$ and so $s(\xi, \eta) \leq 1$ as required. □

3 Dynamics of isometries

3.1 Definitions

Let X be a metric space. A *semicontraction* f is a map $f : X \to X$ such that

$$d(f(x), f(y)) \le d(x, y)$$

for every x, $y \in X$. An *isometry* is here an isomorphism in this categroy, which means it is a distance preserving bijection.

A subset D of semicontractions is called *bounded* (resp. *unbounded*) if Dx_0 is a bounded (resp. an unbounded) set. A single semicontraction f is called *bounded* (resp. *unbounded*) if $\{f^n\}_{n>0}$ is bounded (resp. unbounded). Note that these definitions are independent of x_0.

If the action of the isometries of X extends to an action by homeomorphisms of $\overline{X}$ we call the bordification an *Isom*(X)*–bordification.* Note that when X is a proper metric space, the horofunction compactification $\overline{X}^h$ is a (almost always nontrivial) metrizable Isom(X)–compactification (see the previous section).

Under the assumption that $\overline{X}$ is an Isom(X)–bordification, the isometries of X act on the stars $S(\xi)$ as can be seen from:

$$\begin{aligned} gH(W,C) &= \{z : d(g^{-1}z, W) \le d(g^{-1}z, x_0) + C\} \\ &= \{z : d(z, gW) \le d(z, gx_0) + C\}, \end{aligned}$$

which is included in $H(gW, C + d(x_0, gx_0))$ and contains $H(gW, C - d(x_0, gx_0))$. Hence we have $gS(\xi) = S(g\xi)$ and it is plain that g preserves star distances. Note that we also have an action on the faces.

3.2 A contraction lemma

The following observation lies behind the construction of Dirichlet fundamental domains (see eg [32]): For any isometry g it holds that

$$g(H_x^{g^{-1}y}) = H_{gx}^y.$$

This leads to a contraction lemma, which in spite of its simplicity and fundamental nature, we have not been able to locate in the literature:

Lemma 8 *Let g_n be a sequence of isometries such that $g_nx_0 \to \xi^+$ and $g_n^{-1}x_0 \to \xi^-$ in a bordification $\overline{X}$ of X. Then for any neighborhoods V^+ and V^- of ξ^+ and ξ^- respectively, there exists $N > 0$ such that*

$$g_n(X \setminus H(V^-)) \subset H(V^+)$$

for all $n \geq N$.

Proof Given neighborhoods V^+ and V^- as in the statement, by assumption there is an N such that $g_nx_0 \in V^+$ and $g_n^{-1}x_0 \in V^-$ for every $n \geq N$. For any $z \in X$ outside $H(V^-)$, so $d(z,v) > d(z,x_0)$ for every $v \in V^-$, we have

$$d(g_nz, V^+) \leq d(g_nz, g_nx_0) = d(z,x_0) < d(z, g_n^{-1}x_0) = d(g_nz, x_0)$$

for every $n \geq N$. □

Here is a version of the contraction phenomenon when the isometries act on the boundary:

Proposition 9 *Assume that $\overline{X}$ is an Isom(X)–compactification. Let g_n be a sequence of isometries such that $g_nx_0 \to \xi^+$ and $g_n^{-1}x_0 \to \xi^-$ in $\overline{X}$. Then for any $z \in \overline{X} \setminus S^{x_0}(\xi^-)$,*

$$g_nz \to S^{x_0}(\xi^+).$$

Moreover, the convergence is uniform outside neighborhoods of $S^{x_0}(\xi^-)$.

Proof Since z does not belong to $S^{x_0}(\xi^-)$ there is some neighborhood V^- of ξ^- such that $z \notin \overline{H(V^-)}$. As the latter is a closed set, there is an open neighborhood U of z disjoint from $\overline{H(V^-)}$. Given a neighborhood V^+ of ξ^+ we therefore have for all sufficiently large n that $g_n(U \cap X) \subset H(V^+)$ for all $n > N$. Since g_n are homeomorphisms we have that $g_nz \subset \overline{H(V^+)}$ as required. The conclusion now follows in view of Lemma 5. □

In some cases, for example if $z \in S(\eta)$ for some $\eta \notin S(\xi^-)$, one can say more in view that the isometries preserve stars.

3.3 Individual semicontractions

Let f be a semicontraction of a complete metric space X and let $\overline{X}$ be a bordification of X. The *limit set* of the f–orbit of x_0 is

$$L^{x_0}(f) := \overline{\{f^n(x_0)\}_{n>0}} \cap \partial X,$$

which necessarily is empty if f is bounded.

Proposition 10 *Let g be an (unbounded) isometry and $\overline{X}$ an Isom(X)–bordification. Then g fixes the star and the face of every point $\xi \in L^{x_0}(g)$, that is, $S(g\xi) = S(\xi)$ and $F(g\xi) = F(\xi)$. Moreover, the subset $F(g)$ defined below, is also fixed by g.*

Proof Since by continuity

$$g\xi = g(\lim_{k\to\infty} g^{n_k}x_0) = \lim_{k\to\infty} g^{n_k}(gx_0)$$

we have that $S(g\xi) = S(\xi)$ in view of Lemma 6. If $\xi \in S(\eta)$, then $g\xi \in S(g\eta)$ and again we have $\xi \in S(g\eta)$. Since g is a bijection, the final part of the proposition follows. □

Let $a_n = d(f^n(x_0), x_0)$. A subsequence $n_i \to \infty$ is called *special* for f if $a_{n_i} \to \infty$ and there is a constant $C \geq 0$ such that $a_{n_i} > a_m - C$ for all i and $m < n_i$. Note that being special clearly passes to subsequences and by the triangle inequality it is independent of x_0 (see (1) below). Moreover, special subsequences are invariant under the shift $\{n_i\} \mapsto \{n_i + N\}$, where N is some fixed integer.

Let $A^{x_0}(f)$ denote the limit points of $f^n(x_0)$ along the special subsequences. The *characteristic set* $F(f)$ of f is the face of $A^{x_0}(f)$. (It may of course happen that $A^{x_0}(f) = \emptyset$, in which case $F(f) := \emptyset$.)

Theorem 11 *Assume that X is proper and that $\overline{X}$ is a sequentially compact bordification of X. To any semicontraction f, the subset $F(f) \subset \partial X$ is canonically associated to f. It holds that $F(f) = \emptyset$ if and only if f is bounded, and that if $F(f) \neq \emptyset$, then every f–orbit accumulates only at ∂X. Moreover, for any $x_0 \in X$*

$$L^{x_0}(f) \subset S^{\vee}(F(f)).$$

If in addition $\overline{X}$ is star-reflexive, then

$$L^{x_0}(f) \subset S(F(f)).$$

Proof From the triangle inequality we get

$$|d(g^k x, x) - d(g^k x_0, x_0)| \leq 2d(x, x_0), \tag{1}$$

which implies in view of Lemma 6 that $F(f)$ is independent of x_0. By completeness, if f is bounded, then $F(f) = \emptyset$. The converse is proved below. Calka's theorem [9] asserts that if there is a bounded subsequence of the orbit, then in fact the whole orbit is bounded.

Now suppose f is unbounded and let n_i be a special sequence for f (it is obvious, see [23], that special subsequences exist if and only if f is unbounded) and such that $f^{n_i}(x_0)$ converges to some point $\xi \in \partial X$. Observe that for any positive $k < n_i$ it holds that

$$d(f^{n_i}(x_0), f^k(x_0)) \leq d(f^{n_i-k}(x_0), x_0) = a_{n_i-k} < a_{n_i} + C = d(f^{n_i}(x_0), x_0) + C.$$

Now suppose we have a convergent sequence $f^{k_j}x_0 \to \eta \in \partial X$, which means that given a neighborhood V of η, we can find j large so that $f^{k_j}x_0 \in V$. Now from the above inequality we get that for all large enough i

$$f^{n_i}x_0 \in H(\{f^{k_j}x_0\}, C) \subset H(V, C).$$

Therefore $\xi \in \overline{H(V,C)}$ and since V was an arbitrary neighborhood we have $\xi \in S(\eta)$. (Note that in particular this means that $A^{x_0}(f)$ and $F(f)$ are nonempty.) Finally, assuming star-reflexivity we have showed that $\eta \in S(\xi)$ for every special limit point ξ. □

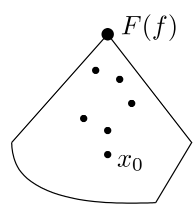

Figure 3: The orbit $f^n(x_0)$

Under some extra assumptions it is possible to prove that actually

$$L^{x_0}(f) \subset F(f).$$

4 Groups of isometries

4.1 Generalizations of Hopf's theorem on ends

The following extends Hopf's theorem that the number of ends of a finitely generated group is either 0, 1, 2, or ∞:

Proposition 12 *Assume that $\overline{X}$ is a sequentially compact Isom(X)–bordification. Let G be a group of isometries fixing a finite set $F \subset \partial X$, that is, $GF = F$. If F is not contained in two stars, then G is bounded.*

Proof By passing to a finite index subgroup (which does not effect the boundedness) we can assume that G fixes F pointwise. Now suppose there is a sequence g_n in G such that $g_n^{\pm 1}x_0 \to \xi^{\pm} \in \partial X$. Then F must be contained in $S(\xi^+) \cup S(\xi^-)$ since otherwise there is is a point in F which on the one hand should be contracted towards $S(\xi^+)$ under g_n, but on the other hand it is fixed by G. ☐

To see how this implies Hopf's theorem: If two boundary points belong to different ends, then their stars are disjoint. So if one has a finitely generated group with finite number of ends, then applying the proposition with F being the set of ends, one obtains that the number of ends must be at most two.

By the same method of proof:

Proposition 13 *Assume that $\overline{X}$ is a sequentially compact Isom(X)–bordification. Let G be a group of isometries which fixes some collection of stars S_i in the sense that $GS_i = S_i$ for every i. Suppose that for any two arbitrary stars, there is always an i such that S_i is disjoint from these two stars. Then G is bounded.*

These two statements can be useful to rule out the existence of compact quotients of certain Riemannian manifolds or complex domains.

4.2 Commuting isometries and free subgroups

The proof of Proposition 10 in fact shows the following:

Proposition 14 *Let g be an isometry and $\overline{X}$ an Isom(X)–bordification. Suppose that $g^{n_i}x_0 \to \xi \in \partial X$ and let $Z(g)$ denote the centralizer of g in Isom(X). Then $Z(g)S(\xi) = S(\xi)$, $Z(g)F(\xi) = F(\xi)$, and $Z(g)F(g) = F(g)$ (when it exists).*

Proposition 15 *Assume that $\overline{X}$ is compact. Let g and h be two isometries such that $g^{\pm n_k} \to \xi^{\pm} \in \partial X$, $h^{\pm m_l} \to \eta^{\pm} \in \partial X$ for some subsequences n_k and m_l. Assume that $S(\xi^+) \cup S(\xi^-)$ and $S(\eta^+) \cup S(\eta^-)$ are disjoint. Then the group generated by g and h contains a noncommutative free subgroup.*

Proof By a compactness argument (similar to that in the proof of Lemma 5) we can find large enough K such that

$$H(\{g^{n_k}x_0\}_{k>K}) \cup H(\{g^{-n_k}x_0\}_{k>K})$$

and

$$H(\{h^{m_l}x_0\}_{l>K}) \cup H(\{h^{-m_l}x_0\}_{l>K})$$

are disjoint. From the contraction observations in subsection 3.2 and the usual freeness criterion [17], the proposition is proved. □

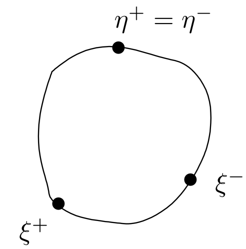

Figure 4: Example of the situation in Proposition 15

By a similar proof one has:

Proposition 16 *Assume that $\overline{X}$ is compact. Let g and h be two isometries such that $g^{\pm n_k} \to \xi^\pm \in \partial X$, $h^{\pm m_l} \to \eta^\pm \in \partial X$ for some subsequences n_k and m_l. Assume that $S(\xi^+)$, $S(\eta^+)$ and $S(\xi^-) \cup S(\eta^-)$ are disjoint. Then the group generated by g and h contains a noncommutative free semigroup.*

4.3 A metric Furstenberg lemma

The following can be viewed as an analog of the so-called Furstenberg's lemma:

Lemma 17 *Assume that $\overline{X}$ is a metrizable Isom(X)–compactification such that $S(\xi) = S^{x_0}(\xi)$ for every $\xi \in \partial X$. Let $g_n \in Isom(X)$ and μ, ν be two probability measures on ∂X. Suppose that $g_n\mu \to \nu$ (in the standard weak topology). Then either g_n is bounded or the support of ν is contained in two stars.*

Proof We assume that g_n is unbounded and by compactness we select a subsequence so that $g_n x_0 \to \xi^+$, $g_n^{-1}x_0 \to \xi^-$, and $g_n\xi^- \to \xi$. We then have that $g_n S(\xi^-) \to S(\xi)$ in view of the proof of Lemma 7. Indeed for any $\eta \in S(\xi^-)$, we have $g_n\eta \in S(g_n\xi^-)$ and as in the lemma we conclude that any limit of $g_n\eta$ belongs to $S(\xi)$.

Write $\mu = \mu_1 + \mu_2$ where $\mu_1(\partial X \setminus S(\xi^-)) = 0$ and $\mu_2(S(\xi^-)) = 0$ by letting $\mu_1(A) := \mu(A \cap S(\xi^-))$. By compactness we can further assume that $g_n\mu_i \to \nu_i$

and $\nu = \nu_1 + \nu_2$. Since μ_1 is supported on $S(\xi^-)$, it follows that ν_1 is supported on $S(\xi)$. Suppose that f is a continuous function vanishing on $S(\xi^+)$. Then

$$\int f(\eta)d\nu_2 = \lim_{n\to\infty}\int f(\eta)d(g_n\mu_2) = \lim_{n\to\infty}\int f(g_n\eta)d\mu_2 = 0$$

by the dominated convergence theorem in view of Proposition 9. Hence we have shown that $\operatorname{supp}\nu \subset S(\xi)\cup S(\xi^+)$ as required. □

Furstenberg's lemma, which deals with matrices acting on projective spaces, has found several beautiful applications since its first appearance in [13]. For example it is the key lemma in Furstenberg's proof of Borel's density theorem, which in turn is a fundamental tool in the theory of discrete subgroups in Lie groups.

Our lemma here might be useful for analyzing amenable groups of isometries (let $\mu = \nu$ be an invariant measure).

4.4 Random walks

Let (X,d) be a proper metric space and $\overline{X}$ a metrizable $\operatorname{Isom}(X)$–compactification. Let (Ω,ν) be a measure space with $\nu(\Omega)=1$ and L a measure preserving transformation. Given a measurable map $w:\Omega\to \operatorname{Isom(X)}$ we let

$$u(n,\omega) = w(\omega)w(L\omega)...w(L^{n-1}\omega).$$

Let $a(n,\omega) = d(x_0, u(n,\omega)x_0)$ and assume that

$$\int_\Omega a(1,\omega)d\nu(\omega) < \infty.$$

For a fixed ω we call a subsequence $n_i\to\infty$ *special* for ω if there are constants C and K such that $a(n_i,\omega) > a(m, L^{n_i-m}\omega) - C$ for all i and $m < n_i - K$. Let $F(\omega)$ denote the face of all limit points of $u(n,\omega)x_0$ in ∂X along special subsequences.

Theorem 18 *Suppose that*

$$\liminf_{n\to\infty}\frac{1}{n}\int_\Omega a(n,\omega)d\nu(\omega) > 0. \tag{2}$$

Then for a.e. ω,

$$u(n,\omega)x_0 \to \{\eta : F(\omega)\subset S(\eta)\}.$$

Proof Proposition 4.2 in [24] guarantees that special subsequences exist for a.e. ω. From this point on, the theorem is proved in the same way as Theorem 11. □

Note that the theorem is not true in general without the condition (2), since for example a.e. trajectory of the standard random walk on the ordinary lattices $\mathbb{Z}^n$ has no asymptotic direction.

We now specialize to the case when $u(n,\cdot)$ is a random walk and describe a related result more in terms of boundary theory.

Let $\mathcal{S}$ be the space of closed nonempty subsets of ∂X with Hausdorff's topology. Denote by $\Phi(\omega)$ the closure of

$$\{\xi : \exists C \text{ s.t. } \xi \in \overline{H(u(k,\omega)x_0, C)} \cap \partial X \text{ for all but finitely many } k\}.$$

This set may a priori be empty. The next result will guarantee that it is not empty for a.e. ω and hence we have an a.e. defined map $\omega \to \mathcal{S}$. This map is measurable since assigning to a point its halfspace-closure and the operation of intersecting closed subsets are continuous.

Theorem 19 *Let μ be a probability measure on a discrete group of isometries Γ. In the case $(\Omega,\nu) = \prod_{-\infty}^{\infty}(\Gamma,\mu)$ with L being the shift, and under assumption (2), the measure space $(\mathcal{S},\Phi_*(\nu))$ is a μ–boundary of Γ.*

Proof Proposition 4.2 in [24] guarantees that for a.e. ω there is a $K>0$ and an infinite sequence n_i such that

$$a(n_i,\omega) > a(n_i-k, L^k\omega)$$

for all $K<k<n_i$. This means that

$$\begin{aligned} d(u(n_i,\omega)x_0, u(k,\omega)x_0) &\leq d(u(n_i-k,L^k\omega)x_0,x_0) = a(n_i-k,L^k\omega) \\ &< a(n_i,\omega) = d(u(n_i,\omega)x_0,x_0) \end{aligned}$$

for all $K<k<n_i$ and all i. This means that all the limits of $u(n_i,\omega)x_0$ belong to $\overline{H(u(k,\omega)x_0)}$ for all $k>K$, in particular $\Phi(\omega)$ is nonempty and belongs to $\mathcal{S}$.

Consider the path space $\Gamma^{\mathbb{Z}_+}$ with the induced probability measure $\mathbb{P}$ from the random walk defined by μ starting at e. Note that Γ naturally acts on $\mathcal{F}$. The map Ψ gives rise to a map Π defined on the path space rather than Ω.

Note that if $\{n_i\}$ is special for ω, then $\{n_i-1\}$ is special for $L\omega$ (cf [24, page 117]). Special subsequences are moreover independent of the base point x_0. This implies that $w(\omega)\Psi(L\omega) = \Psi(\omega)$. Now see [21, 1.5]. □

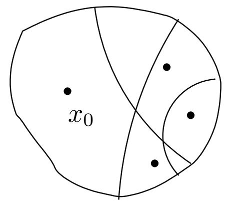

Figure 5: Halfspaces of the random walk intersect.

In some situations, eg, when X is δ–hyperbolic (under some reasonable conditions), the μ–boundary obtained in the theorem is in fact isomorphic to the Poisson boundary, see [21].

The result stated in the introduction follows from the proof of Theorem 18 and the well-known fact that $A > 0$ for simple random walks on finitely generated nonamenable groups.

4.5 Proper actions

An isometric action of a group Γ is (*metrically*) *proper* if for every $x \in X$ and every closed ball B centered at x, the set $\{g \in \Gamma : gx \in B\}$ is finite. A pair of stars S_1 and S_2 are *maximal* if the only union of two stars containing them is $S_1 \cup S_2$.

Lemma 20 *Assume that $\overline{X}$ is a Hausdorff Isom(X)–compactification and that ∂X is not the union of two stars. Suppose that g and h are two unbounded isometries generating a proper action and that $h^{\pm n_j} x_0 \to \xi^{\pm}$ with $S(\xi^+)$ and $S(\xi^-)$ disjoint and maximal. If g fixes $S(\xi^-)$, then $h^k = gh^l g^{-1}$ for two nonzero integers k and l, and g fixes a star contained in $S(\xi^+)$.*

Proof (Compare [26].) Since $\overline{X}$ is a compact Hausdorff space we can find two disjoint neighborhoods U^+ and U^- of $S(\xi^+)$ and $S(\xi^-)$ respectively, so that $E := \overline{X} \setminus (U^+ \cup U^-)$ is nonempty and not contained in X. Since g is a homeomorphism fixing $S(\xi^{\pm})$ we can moreover suppose that

$$hU^- \cap U^+ = \emptyset. \tag{3}$$

Because h^{-n_j} contracts toward $S(\xi^-)$ (Proposition 9) and g is a homeomorphism fixing $S(\xi^-)$ we have that

$$gh^{-n_j}(E) \subset U^-$$

for all large j. In view of (3) we can find a $k = k(j)$ such that $h^{k(j)}gh^{-n_j}E \cap E$ is nonempty. Let $g_j = h^{k(j)}gh^{-n_j}$. Note that

$$g_jS(\xi^-) = S(\xi^-) \tag{4}$$

and since $g_jS(\xi^+) = h^{k(j)}gS(\xi^+)$, $gS(\xi^+) \cap gS(\xi^-) = \emptyset$, and $k(j) \to \infty$,

$$g_jS(\xi^+) \to S(\xi^+). \tag{5}$$

In view of (4), (5), and the assumptions on $S(\xi^\pm)$ we have that if $g^\pm_{j_k}x_0 \to \eta^\pm \in \partial X$, then either $S(\eta^\pm) = S(\xi^\pm)$ or $S(\eta^\pm) = S(\xi^\mp)$. In either case this contradicts that $g_jE \cap E$ is nonempty for all large j. Therefore g_j is bounded and by properness we have $g_j = g_i$ for many i, j different. This means that $h^k = gh^lg^{-1}$ for two nonzero integers k and l. Hence

$$h^kS(g\xi^+) = gh^lg^{-1}gS(\xi^+) = gS(\xi^+) = S(g\xi^+)$$

and we conclude that $S(g\xi^+) \subset S(\xi^+)$, since $gS(\xi^-)$ equals all of $S(\xi^-)$. □

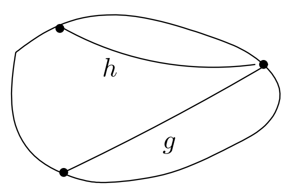

Figure 6: g and h generate a nonproper action.

It is instructive to compare Lemma 20 with the case of a Baumslag–Solitar group $< g, h : h^k = gh^lg^{-1} >$ acting on its Cayley graph.

An *axis* of an isometry is an invariant geodesic line on which the isometry acts by translation. We say that an isometry h fixes an endpoint of a geodesic line c if there is a $C > 0$ such that $d(h(c(t)), c(t)) < C$ for all $t > 0$ or all $t < 0$.

Proposition 21 *Let g, h be two isometries generating a group which acts properly on a complete metric spaces X. Assume that g has an axis c and that h fixes an endpoint of c. Then $[h, g^N] = 1$ for some $N > 0$.*

Proof Letting $x_0 = c(0)$ we have that:

$$d(x_0, g^{-n}hg^nx_0) = d(g^nx_0, hg^nx_0) = d(c(nd_g), hc(nd_g)) < C$$

for all $n > 0$ (or $n < 0$). As the action of the group is proper, we must then have that for some $m \neq n$

$$g^{-m}hg^m = g^{-n}hg^n$$

or in other words there is a number $N > 0$ such that $h = g^{-N}hg^N$. □

An isometry g is called *strictly hyperbolic* if $L(g) = \{\xi^+\}$ and $L(g^{-1}) = \{\xi^-\}$ for two distinct hyperbolic boundary points ξ^+ and ξ^-, which by definition means that $S(\xi^+) = \{\xi^+\}$ and $S(\xi^-) = \{\xi^-\}$.

By the contraction lemma a strictly hyperbolic isometry can have no further fixed points apart from its two limit points. (Note that if one knows that ξ^+ and ξ^- are hyperbolic limit points, then it follows that the limit set cannot be larger.)

Examples include pseudo-Anosov elements of mapping class groups, hyperbolic isometries of a δ–hyperbolic space, and Ballmann's rank 1 isometries (see [4], [5]) of a CAT(0)–space, see Proposition 30 below. From Lemma 20 and in view of Proposition 15 one has (the star-reflexivity guarantees maximality of any two hyperbolic boundary points):

Proposition 22 *Assume that $\overline{X}$ is a Hausdorff star-reflexive Isom(X)–compactification. The fixed point sets of two strictly hyperbolic isometries which together generate a proper action either coincide or are disjoint. In the latter case, the group generated by the two isometries contains a noncommutative free subgroup.*

Part II Examples and applications

5 Hyperbolicity

A boundary point ξ is called *hyperbolic* if $S(\xi) = \{\xi\}$. A bordification $\overline{X}$ is called *hyperbolic* if all boundary points are hyperbolic. A complete metric space X is *asymptotically hyperbolic* if all stars in $\overline{X}^h$ are disjoint. It is known that visibility spaces and Gromov's δ–hyperbolic spaces (due to P Storm) are asymptotically hyperbolic.

Recall the following standard notation:

$$(x|z)_{x_0} := \frac{1}{2}(d(x,x_0) + d(z,x_0) - d(x,z)),$$

and note that

$$(x|z) \geq \frac{1}{2}d(z,x_0)$$

if and only if $x \in H_z^{x_0}$, which gives some insight to the relation between hyperbolicity and halfspaces.

The following axiom is known to hold for the usual boundary of visibility spaces and Gromov's δ–hyperbolic spaces, as well as for the end-compactification, and Floyd's boundary construction (compare [26]):

HB For any $\xi \in \partial X$, there is a family of neighborhoods $\mathcal{W}$ of ξ in $\overline{X}$, such that the collection of open sets

$$\{x : (x|W) > R\} \cup W,$$

where $W \in \mathcal{W}$, $R > 0$, and $(z|W) := \sup_{w \in W \cap X}(z|w)$, is a fundamental system of neighborhoods of ξ in $\overline{X}$.

Proposition 23 *Every bordification $\overline{X}$ which satisfies **HB** is hyperbolic, indeed $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

Proof Given U a neighborhood of ξ in $\overline{X}$ and $C > 0$. By definition we may find R and $W \in \mathcal{W}$ such that $\{z : (z|W) > R - C/2\} \subset U$ and by making W smaller we can also arrange so that $R < d(W, x_0)/2$ $(d(x_0, \xi) = \infty)$. Now

$$\begin{aligned}
H(W,C) &= \{z : d(z,W) \leq d(z,x_0) + C\} \\
&= \{z : 0 \leq \sup_w (d(z,x_0) - d(z,w)) + C\} \\
&= \{z : \inf_w d(w,x_0) \leq \sup_w (d(z,x_0) - d(z,w)) + \inf_w d(w,x_0) + C\} \\
&\subset \{z : d(W,x_0) \leq \sup_w (d(z,x_0) + d(w,x_0) - d(z,w)) + C\} \\
&= \{z : (z|W) > R - C/2\} \subset U,
\end{aligned}$$

which proves the proposition, because $\mathcal{W}$ is a fundamental system of neighborhoods and C plays no role. □

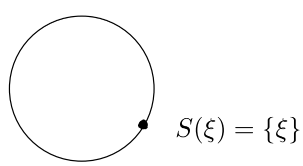

Figure 7: Hyperbolicity

For spaces with hyperbolic bordifications, our theory provides alternative proofs of (mostly) well-known facts, see eg [8] for the theory of ends, [32] for classical hyperbolic geometry, [14] for word hyperbolic groups, and [26] for non-locally compact spaces.

6 Nonpositive curvature

Let X be a complete CAT(0)–space [8]. Recall that the angular metric is $\angle(\xi,\xi') = \sup_{p\in X} \angle_p(\xi',\xi)$, where ξ,ξ' are points in the standard visual boundary ∂X of X.

6.1 Stars and Tits geometry

The following lemma and its proof can essentially be found in [6]:

Lemma 24 *Let c and c' be two geodesic rays emanating from x_0 and let $\xi = [c]$ and $\xi' = [c']$ be the corresponding boundary points. Let p_i denote the projection of $c'(i)$ onto c. If $\angle(\xi,\xi') > \pi/2$ then p_i stays bounded as $i \to \infty$. If $\angle(\xi,\xi') < \pi/2$, then p_i is unbounded. In the case $\angle(\xi,\xi') = \pi/2$ then $\{p_i\}$ is bounded if and only if x_0, c, and c' define a flat sector.*

Proof First recall the basic angle property of projections [8, Proposition II.2.4]: $\angle_{p_i}(c'(i),\xi) \geq \pi/2$ and $\angle_{p_i}(c'(i),x_0) \geq \pi/2$ (when $p_i \neq x_0$).

If p_i is bounded we may assume $p_i \to p$ (along some subsequence), because the points p_i are restricted to a compact subset of c. Then by the upper semicontinuity of angles ([8, Proposition II.9.2]) we have:

$$\angle(\xi',\xi) \geq \angle_p(\xi',\xi) \geq \limsup_i \angle_{p_i}(c'(i),\xi) \geq \pi/2.$$

If p_i is unbounded, then in view of [8, Proposition II.9.8] we have

$$\begin{aligned}\angle(\xi,\xi') &= \lim_{i\to\infty}(\pi - \angle_{p_i}(c'(i),x_0) - \angle_{c'(i)}(p_i,x_0)) \\ &\leq \pi/2 - \lim_{i\to\infty} \angle_{c'(i)}(p_i,x_0) \leq \pi/2.\end{aligned}$$

It remains to analyze the case $\angle(\xi',\xi) = \pi/2$. If p_i is a bounded sequence then as above

$$\pi/2 \geq \angle_p(\xi',\xi) \geq \limsup_i \angle_{p_i}(c'(i),\xi) \geq \pi/2$$

and then [8, Corollary II.9.9] shows that x_0, c, and c' define a flat sector. The converse is trivial: $p_i = x_0$. □

Proposition 25 *Assume X is a complete CAT(0)–space and $\overline{X}$ is the visual bordification. Then $S(\xi) = S^{x_0}(\xi) = \{\eta : \angle(\eta,\xi) \leq \pi/2\}$ for every $\xi \in \partial X$.*

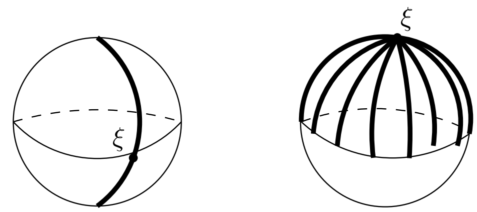

Figure 8: The minimal and maximal star in $\mathbb{R} \times \mathbb{H}^2$

Proof Consider two rays c_1 and c_2 from x_0 representing ξ and η respectively. Assume that the projections of $c_2(i)$ onto c_1 are unbounded. Since by definition projections realize the shortest distance, we then have that for any neighborhood V of ξ and for every large enough i (so that $p_i \in V$) that $d(c_2(i), V) \leq d(c_2(i), p_i) \leq d(c_2(i), x_0)$. In the case $\angle(\eta, \xi) = \pi/2$ and c_1, c_2, and x_0 define a flat sector, then by Euclidean geometry V contains a point ξ' with $\angle(\xi', \eta) < \pi/2$. In view of Lemma 24 we hence have $\{\eta : \angle(\eta, \xi) \leq \pi/2\} \subset S^{x_0}(\xi)$.

Assume $\angle(\xi, \eta) > \pi/2$ and given $C > 0$. By definition there is a point y such that $\angle_y(\xi, \eta) > \pi/2$. By continuity ([8, Proposition II.9.2.(1)]) we can find neigborhoods V of ξ and U of η in $\overline{X}$ such that $\angle_y(z, w) \geq \pi/2 + \theta$ for every $z \in U$, $w \in V$ and some $\theta > 0$. Further we make V smaller (if necessary) so that $d(y, V)|\cos(\pi/2+\theta)| \geq d(x_0, y) + C'$ for some $C' > C$. For any $w \in V \cap X$, $z \in U \cap X$ we have by the cosine inequality (ie, comparison with the Euclidean cosine law):

$$\begin{aligned} d(z,w)^2 &\geq d(y,z)^2 + d(y,w)^2 - 2d(y,z)d(y,w)\cos\angle_y(z,w) \\ &\geq d(y,z)^2 + d(y,w)^2 + 2d(y,z)d(y,w)|\cos(\pi/2+\theta)| \\ &\geq d(y,z)^2 + (d(x_0,y)+C')^2 + 2d(y,z)(d(x_0,y)+C') \\ &= (d(x_0,y)+C'+d(y,z))^2 \end{aligned}$$

which implies that $d(z, w) > d(z, x_0) + C'$ by the triangle inequality. Therefore $d(z, V) > d(z, x_0) + C$ for all $z \in U \cap X$ and it follows that $\eta \notin S(\xi)$ as desired. □

We will also have use for:

Lemma 26 *Let X be a proper* CAT(0)*–space and assume that ξ is a hyperbolic point in ∂X. Then ξ can be joined to any other boundary point by a geodesic line in X.*

Proof Assume that there is no such geodesic between ξ and $\eta \in \partial X \setminus \{\xi\}$. By [4, Theorem 4.11] it then holds that $\angle(\xi, \eta) \leq \pi$. In fact, there is a (midpoint) $\zeta \in \partial X$ with $\angle(\xi, \zeta) \leq \pi/2$ ([4, page 39]), which contradicts that $S(\xi) = \{\xi\}$ in view of Proposition 25. □

6.2 Corollaries

All results of the general theory specialized to the CAT(0)–setting (with the help of Proposition 25) seem to be new except Theorem 18, Propositions 14 and 21. Moreover, in view of Propositions 25 and 9 (or their proofs in the non-proper case) we have:

Theorem 27 *Let X be a complete* CAT(0)*–space. Let g_n be a sequence of isometries such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$. Then for any $\eta \in \overline{X}$ with $\angle(\eta, \xi^-) > \pi/2$ we have that*

$$g_n \eta \to \{\zeta : \angle(\xi^+, \zeta) \leq \pi/2\}$$

(in the sense that $\limsup \angle(\xi^+, g_n \eta) \leq \pi/2$ *when X is not proper). Assuming that X is proper, the convergence is uniform outside neighborhoods of $S(\xi^-)$.*

Applied to the special case of iterates of a single isometry $g_n := h^{k_n}$, the theorem partially extends (since it also deals with parabolic isometries) a lemma of Schroeder [6] generalized by Ruane [33] to include also singular CAT(0)–spaces. Let us emphasize that this theorem gives information also about the dynamics of parabolic isometries of general CAT(0)–spaces.

Combining Propositions 25 and 15 yields the following result which generalizes the main theorem in [33] (because no group is here assumed to act cocompactly and properly):

Theorem 28 *Let X be a proper* CAT(0)*–space. If g and h are two unbounded isometries with limit points ξ^-, ξ^+ and η^-, η^+ respectively (not necessarily all distinct), with $Td(\{\xi^\pm\}, \{\eta^\pm\}) > \pi$, then the group generated by g and h contains a noncommutative free subgroup.*

The following proposition generalizes [5, Lemma 4.5] and [34, Theorem 8] (by weakening the hypothesis):

Proposition 29 *Let X be a complete* CAT(0)*–space and g a hyperbolic isometry with an axis c. Assume that h is an isometry which fixes one endpoint of c and that g and h generate a group acting properly. Then h commutes with some power of g and h fixes both endpoints of c.*

Proof First note that the notion of fixing an endpoint of a geodesic coincide with the usual one for CAT(0)–spaces and the standard ray boundary ∂X. From Proposition 21 we have $h = g^{-N} h g^N$. Therefore

$$h(c(\pm\infty)) = \lim_{n\to\infty} h g^{\pm nN} x_0 = \lim_{n\to\infty} g^{\pm nN} h x_0 = c(\pm\infty). \qquad \square$$

An isometry is called a rank 1 isometry if it is hyperbolic with an axis which does not bound a flat halfplane [4]. The usefulness of this notion was demonstrated by Ballmann and collaborators.

Proposition 30 *Rank 1 isometries are strictly hyperbolic.*

Proof Recall that g fixes the stars of its limit points. So unless $S(\xi^\pm) = \{\xi^\pm\}$ this would contradict the contraction lemma for rank 1 isometries [4, Lemma III.3.3]. $\square$

Actually, the converse is also true since strictly hyperbolic isometries clearly cannot be elliptic, and also not parabolic (look at preserved horoballs) and that the axis cannot bound a flat halfplane in view of Proposition 25. We obtain the following theorem which sheds some light on the question [4, Question III.1.1], see also [4, Theorem III.3.5]:

Theorem 31 *Suppose that a group Γ acts properly by isometry on a proper* CAT(0)*–space. If the limit set contains at least three points, one of which is hyperbolic, then Γ contains noncommutative free subgroups.*

Proof Let ξ be a hyperbolic boundary point. Then together with any other boundary point it does not bound a flat halfplane in view of Lemma 26 and Proposition 25. Therefore given a sequence g_n in Γ which we can assume that $g_n x_0 \to \xi$ and $g_n^{-1} x_0 \to \eta$ for some other boundary point eta, which we moreover suppose is different from ξ. Indeed, if $\xi = \eta$ then by the basic contraction lemma and since the limit set contains at least three points, we can find such g_n.

The lemma [4, Lemma III.3.2] now guarantees the existence of a rank 1 isometry g with hyperbolic limitpoints say $\xi^\pm$. By assumption there is another limit point η for the group different from $\xi^\pm$, take h_n for which $h_n x_0 \to \eta$. Since the boundary is star-reflexive, by the basic contraction lemma we have that some h_N moves one of $\xi^\pm$ say ξ^+. Moreover it moves a neighborhood of ξ^+ into a neighborhood of the star of η. Now consider the sequence $h_N g^n$ (in

n), this has one limit point outside both $\xi^{\pm}$ and the other, the hyperbolic point ξ^{-}. Apply again Ballmann's lemma to obtain another rank 1 isometry h. Since it does not have have ξ^{+} as limit point, the theorem is proved in view of Proposition 22. □

In the end it might be more powerful to use ping-pong arguments with the halfspaces directly without pushing it to the boundary. For example, one can in this way extend the main theorem in [3] somewhat: the condition of no-fake-angles can be removed and the translation lengths do not necessarily have to be *strictly* greater than the length of S.

7 Hilbert's geometry on convex sets

Let X be a bounded convex domain in $\mathbb{R}^n$ and ∂X the usual boundary. The Hilbert metric on this domain is a complete metric and is defined as follows. For any two distinct points x and y draw the chord through these points. Now $d(x,y)$ is the logarithm of the projective cross-ratio of x, y, and the two endpoints of the chord. We refer to [16] or [31] for more information, note in particular that semicontractions of Hilbert's metric arise in several situations, for example in potential theory. Recall that in this context the *star* of a boundary point ξ, Star(ξ), is the intersection of ∂X with the union of all hyperplanes which are disjoint from X but contain ξ. We have:

Proposition 32 *Assume that X is a bounded convex domain equipped with Hilbert's metric and let $\overline{X}$ be the closure in $\mathbb{R}^n$. Then $S(\xi)=S^{x_0}(\xi)=$Star(ξ) for every $\xi\in\partial X$.*

Proof The inclusion $S(\xi)\subset$Star(ξ) follows from the inclusion

$$H(W,C)\subset\{z:(z|W)\geq\frac{1}{2}d(W,x_0)+C'\}$$

proved in Proposition 23 using the same terminology, together with the proof of Theorem 5.2 in [25]. The other inclusion follows because given W and ζ it is simple to see that we can approximate ζ with a point arbitrary far from x_0 but staying on finite Hilbert distance to W (the Hilbert metric remains bounded near a line segment of the boundary in the direction parallel to this line segment). In particular, $\bigcap\overline{H(V,C)}$ is independent of C and equals $S^{x_0}(\xi)$. □

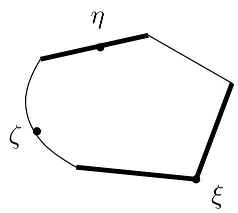

Figure 9: Examples of stars for Hilbert's metric

For these metric spaces, it seems that several results obtained in this paper cannot be found in the literature. For example, we obtain from combining Propositions 12 and 32:

Theorem 33 *Any polyhedral cone with noncompact automorphism group has simplicial diameter at most 3.*

.

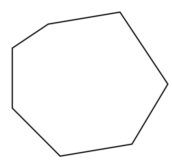

Figure 10: A convex set with compact automorphism group

The simplicial diameter is the smallest number of simplices required to connect any two points. In dimension 2 a rather complete result concerning which convex sets have nonfinite automorphism group can be found in [16].

The literature on symmetric or homogeneous cones is vast. For recent works on cones where the automorphism group admits a cocompact lattices, see the works of Y Benoist. eg [7]. Hilbert's metric can also be a tool in the study of Coxeter groups via the Tits cone such as in [30].

8 Several complex variables

Let X be a bounded domain in $\mathbb{C}^N$. We denote by d_X the Kobayashi distance and by F_X the corresponding infinitesimal metric on X. The metric space (X, d_X) is not always complete (pseudoconvexity X is for example a necessary condition), but when it is, (X, d_X) is in addition proper and geodesic. We refer to [27] for more details.

8.1 Points of strict pseudoconvexity

Here is a relation between strictly pseudoconvex points and stars:

Theorem 34 *Let X be a bounded domain in $\mathbb{C}^n$ with C^2–smooth boundary equipped with Kobayashi's metric. If ξ_1 and ξ_2 are two distinct boundary points at which X is strictly pseudoconvex, then $s(\xi_1, \xi_2) \geq 2$.*

Proof Combining [27, Theorem 4.5.8] with an estimate due to Forstneric–Rosay, cf [27, Corollary 4.5.12], one has for some constant C and fixed x_0,

$$d(z_1, z_2) \geq C + d(z_1, x_0) + d(z_2, x_0)$$

for all z_1 (resp. z_2) sufficiently close to ξ_1 (resp. ξ_2). Hence $\xi_1 \notin S(\xi_2)$ and $\xi_2 \notin S(\xi_1)$. □

Corollary 35 *Let X be a strictly pseudoconvex bounded domain with C^2–boundary equipped with Kobayashi's metric. Then $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

8.2 From metric to distance estimates

Let X be a bounded $C^{1+\alpha}$–smooth $(\alpha > 0)$ domain in $\mathbb{C}^N$ which is complete in the Kobayashi metric and fix some $x_0 \in X$. Euclidean distances are denoted by δ. Assume that for some $\varepsilon > 0$ and $c_1 > 0$

$$F_X(z; v) \geq c_1 \frac{||v||}{|\delta(z, \partial X)|^{\varepsilon}} \tag{6}$$

for all $z \in X$ and $v \in \mathbb{C}^N$. Examples include bounded pseudoconvex domains with real analytic boundary [10] and C^2–strictly pseudoconvex domains, see Theorem E.3, and section X.10.4 in [20].

Lemma 36 *Let γ be a minimizing geodesic between two points z and w in X. Then*

$$\delta(z, w) \leq C(2d_X(x_0, \gamma) + \varepsilon^{-1})e^{-\varepsilon d_X(x_0, \gamma)} \tag{$*$}$$

for some $C > 0$ depending only on X and x_0.

Proof Let m be a point on γ of minimal distance $r := d_X(x_0, \gamma)$ to x_0 and denote by $\gamma_1 : [0, a] \to X$ the (reparametrized) piece of γ going from m to z. Because of the minimality of r and the triangle inequality we have

$$d_X(x_0, \gamma_1(t)) \geq r$$
$$d_X(x_0, \gamma_1(t)) \geq t - r$$

for all t. The following estimate is known (in the case of C^2–smoothness see [27, Theorem 4.5.8] or [20, X.10.4], and in the more general case it is a consequence of [12, Proposition 2.5]): there is a constant c_3 such that

$$d_X(x_0, z) \leq c_3 - \log \delta(z, \partial X) \tag{7}$$

for all $z \in Z$.

In the case $a > 2r$, we have from the above estimates, since γ_1 is a unit speed geodesic that:

$$\begin{aligned}\delta(m, z) &\leq \int_0^a ||\dot\gamma_1(t)|| dt \leq c_1 \int_0^a \delta(\gamma_1(t), \partial X)^\varepsilon F_X(\gamma_1(t); \dot\gamma_1(t)) dt \\ &= c_1 \int_0^a \delta(\gamma_1(t), \partial X)^\varepsilon dt \leq c_4 \int_0^a e^{-\varepsilon d_X(x_0, \gamma_1(t))} dt \\ &\leq c_4 \int_0^{2r} e^{-\varepsilon r} dt + c_4 \int_{2r}^a e^{-\varepsilon(t-r)} dt < c_4 2r e^{-\varepsilon r} + c_4 \varepsilon^{-1} e^{-\varepsilon r}.\end{aligned}$$

In the case, $a \leq 2r$ we make the same estimate but without decomposing the integral. By a symmetric argument with w instead of z, the lemma is proved in view of the triangle inequality. □

Theorem 37 *The closure $\overline{X}$ is a hyperbolic compactification of X, indeed $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

Proof First note that the right hand side of $(*)$ in Lemma 36 tends to 0 if $d_X(x_0, \gamma) \to \infty$ (x_0 is fixed). Now recall the simple and standard fact that (for any geodesic space)

$$(z_1|z_2)_{x_0} := \frac{1}{2}(d_X(z_1, x_0) + d_X(z_2, x_0) - d_X(z_1, z_2)) \leq d_X(x_0, \gamma)$$

for any geodesic segment joining z_1 and z_2, see eg [25]. This means that the condition **HB** and the assertion follows from Proposition 23. □

We record the following result which is formulated in a more traditional style but which we have not been able to find in the literature.

Theorem 38 *Given two distinct boundary points ξ_1, $\xi_2 \in \partial X$ there exists constants $\kappa > 0$ and $c \in \mathbb{R}$ depending only on X, ξ_1 and ξ_2 such that*

$$d_X(z_1, z_2) \geq c + \log \frac{1}{\delta(z_1, \partial X)} + \log \frac{1}{\delta(z_2, \partial X)}.$$

Proof In view of the proof of Theorem 37 and Lemma 36 we have that for some neighborhoods of ξ_1 and ξ_2 there is a constant R such that for any z_1 and z_2 in these neighborhoods respectively,

$$(z_1|z_2)_{x_0} \leq 2R$$

which spelled out reads

$$\begin{aligned} d(z_1, z_2) &\geq R - d(z_1, x_0) - d(z_2, x_0) \\ &\geq c + \log \frac{1}{\delta(z_1, \partial X)} + \log \frac{1}{\delta(z_2, \partial X)} \end{aligned}$$

in view of (7). □

8.3 Convex domains

Recall that the face Face(ξ) is the intersection of all hyperplanes which contains ξ but avoids the interior of the convex set. The following result is due to Abate (see [1] or [2, Corollary 2.4.25]):

Theorem 39 *Let X be a convex C^2–smooth bounded domain in $\mathbb{C}^N$ and given ξ_1, $\xi_2 \in \partial X$ such that $\xi_1 \notin F(\xi_2)$ (and hence also $\xi_2 \notin F(\xi_1)$). Then there exists $\kappa > 0$ and $c \in \mathbb{R}$ such that*

$$d_X(z_1, z_2) \geq c + \log \frac{1}{\delta(z_1, \partial X)} + \log \frac{1}{\delta(z_2, \partial X)}$$

for any $z_1 \in X \cap \{w : \delta(w, F(\xi_1)) < \kappa\}$ and $z_2 \in X \cap \{w : \delta(w, F(\xi_2)) < \kappa\}$.

Corollary 40 *Let X be a convex C^2–smooth bounded domain in $\mathbb{C}^N$. Then $S(\xi) \subset \mathrm{Star}(\xi)$ for any $\xi \in \partial X$.*

Proof This is deduced similarily to Theorem 34. □

What are the stars for a general bounded pseudoconvex (Kobayashi hyperbolic) domain with Kobayashi's metric? Note here that Hilbert's metric is an analogous metric and Teichmüller metric is another example.

8.4 Iteration of holomorphic maps on bounded domains

The iteration of holomorphic self-maps of bounded domains in $\mathbb{C}$ was studied by Wolff, Denjoy, Valiron and Heins. In several variables perhaps the first works were done by H Cartan and Hervé. From 1980 and onwards there have appeared many papers by several authors including Vesentini, Abate, Vigue, D Ma, X-J Huang, Zhang, Ren, and Mellon. Most often the main tool is an appropriately generalized Schwarz–Pick lemma.

Since holomorphic maps semicontract Kobayashi distances,

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in X$, we can canonically associate the set $F(f)$ to any holomorphic map (in view of Theorem 11). From Theorem 34, we obtain the following:

Theorem 41 *Let X be a C^2 bounded domain in $\mathbb{C}^n$, $f : X \to X$ a holomorphic map, and d the Kobayashi distance. Assume that (X, d) is complete. Then $F(f)$ contains at most one point of strong pseudoconvexity.*

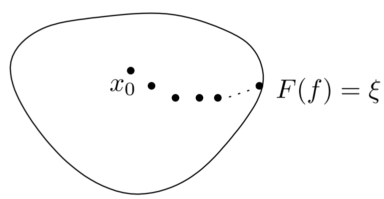

Figure 11: Wolff–Denjoy result: orbit of f

To see the relevance of this to iterations of holomorphic maps, recall that (Theorem 11) if the stars are only singletons, then for any z either $f^m(z)$ stays away from ∂X ($F(f) = \emptyset$), or $\lim_{m\to\infty} f^m(z) = \xi$ for some $\xi \in \partial X$ ($F(f) = \{\xi\}$). In particular, we can formulate the following new Wolff–Denjoy result (the case of $n = 2$ and real analytic boundary, was proved by Zhang and Ren in [36]):

Theorem 42 *Let X be a bounded domain in $\mathbb{C}^n$ such as in subsection 8.2. Let $f : X \to X$ be a holomorphic map. Then either the orbit of f stays away from the boundary or there is a unique boundary point ξ such that*

$$\lim_{m\to\infty} f^m(z) = \xi$$

for any $z \in X$.

Finally we remark that that in view of [23] it would be interesting to identify the horofunction boundary for domains with Kobayashi's metric.

9 Teichmüller spaces and mapping class groups

Let M be a closed surface of genus $g \geq 2$ and denote by $\mathcal{T}$ the associated Teichmüller space $\mathcal{T}$. It is known that $\mathcal{T}$ can be embedded as a bounded domain in $\mathbb{C}^N$ and that it is pseudoconvex but not strictly pseudoconvex. By a theorem of Royden, one also knows that Kobayashi's metric coincides with Teichmüller's metric.

9.1 Stars in the Thurston boundary

Let $\mathcal{S}$ be the set of homotopy classes of simple closed curves on M. Denote by $i(\alpha,\beta)$ the minimal number of intersection of representatives of $\alpha,\beta \in \mathcal{S}$. Let $\mathcal{MF}$ (resp. $\mathcal{PMF}$) be the set of (resp. projective equivalence classes of) measured foliations, which coincides with the closure of the image of the embedding

$$\alpha \mapsto i(\alpha,\cdot)$$

of $\mathcal{S}$ into $\mathbb{R}^{\mathcal{S}}$ (resp. $P\mathbb{R}^{\mathcal{S}}$). The intersection number i extends to a bihomogeneous continuous function on $\mathcal{MF}\times\mathcal{MF}$. A foliation $F \in \mathcal{PMF}$ is called *minimal* if $i(F,\alpha) > 0$ for every $\alpha \in \mathcal{S}$. The Teichmüller space $\mathcal{T}$ of M is embedded into $P\mathbb{R}^{\mathcal{S}}$ by the hyperbolic length function. (Below, V_ϕ stands for the vertical foliation and h_ϕ the horizontal length associated to a quadratic differential ϕ, see [22] for more details.)

Lemma 43 *Let ϕ_n be the quadratic differential corresponding (in the Teichmüller embedding with reference point x_0) to $x_n \in \mathcal{T}$. Assume $\phi_n \to \phi_\infty$, a norm one quadratic differential, and $x_n \to F$ in $\mathcal{PMF}$. Whenever $\beta_n \in \mathcal{S}$ such that $Ext_{x_n}(\beta_n) < D$ and $\beta_n \to H$ in $\mathcal{PMF}$, it holds that*

$$i(V_{\phi_\infty},H) = 0 = i(F,H).$$

Proof A proof analysis shows that this is proved in [28]: Denote by ψ_n the terminal differential corresponding to the Teichmüller map from x_0 to x_n. Since

$$h_{\psi_n}(\beta_n) \leq Ext_{x_n}(\beta_n)^{1/2} < D^{1/2},$$

$x_n \to \infty$ in $\mathcal{T}$, and in view of the stretching of the Teichmüller map ($h_{\psi_n} = e^{d(x_0,x_n)}h_{\phi_n}$) we see that

$$\lim_{n\to\infty} i(V_{\phi_n},\beta_n) = \lim_{n\to\infty} h_{\phi_n}(\beta_n) = 0.$$

Since β_n is a sequence in $\mathcal{S}$ converging to H in $\mathcal{PMF}$, there is a sequence λ_n of bounded positive scalars such that $\lambda_n\beta_n \to H$ in $\mathcal{MF}$. By continuity and homogeneity of i we have $i(V_{\phi_\infty}, H) = 0$.

For the second equality note that it is known that $x_n \to F$ in $\mathcal{PMF}$ implies that there is a sequence $r_n \to 0$ such that $i(r_n x_n, \cdot) \to i(F, \cdot)$ in $\mathcal{MF}$. From definitions we also have

$$i(x_n, \beta_n) \leq A_g^{1/2} Ext_{x_n}(\beta_n)^{1/2} < A_g^{1/2} D^{1/2},$$

which by the same argument as before now also shows the second equality. □

The following result can be viewed as a slight generalization of Lemma 1.4.2 in [22] although formulated in a very different way (note that there seems that there is a misprint in their statement) and is obtained by almost the same proof.

Theorem 44 *Let X be the Teichmüller space of a compact surface and equipped with the Teichmüller metric d. Let $\overline{X}$ be the Thurston compactification $X \cup \mathcal{PMF}$. For $F \in \mathcal{PMF}$, a minimal foliation, we have*

$$S(F) \subset \{G : i(F, G) = 0\}.$$

Proof Given $y_n \to G \in S^{x_0}(F)$, select $x_n \to F$ such that $d(y_n, x_n) \leq d(y_n, x_0) + C$ for all n and some $C > 0$. From continuity and Mumford compactness, it is a fact that sequences β_n as in Lemma 43 corresponding to x_n always exist. Assume now that F is minimal. It is then known (due to Rees) that, $i(F, G) = 0$ if and only if G is minimal and equivalent to F. Hence V_{ϕ_0}, F and H as in Lemma 43 are all equivalent minimal foliations. Fix these. Note that $\lambda_n \to 0$ here because of the minimality. Let θ_n (resp. ψ_n) denote the initial (resp. terminal) quadratic differential of the Teichmüller map from x_0 to y_n. We have

$$\begin{aligned} i(V_{\theta_n}, \lambda_n\beta_n) &= \lambda_n h_{\theta_n}(\beta_n) \\ &= \lambda_n e^{-d(y_n, x_0)} h_{\psi_n}(\beta_n) \\ &\leq \lambda_n e^{-d(y_n, x_0)} D e^{d(y_n, x_n)} \to 0, \end{aligned}$$

where the last inequality follows from Kerckhoff's formula for Teichmüller distances. Thus $i(V_{\theta_0}, H) = 0$, which implies what we want, since $i(F, G) = 0$ is an equivalence relation for minimal foliations and because of Lemma 43. Finally since the set on the right in the theorem is closed, we have $i(F, G) = 0$ for all $G \in S(F)$. □

The set of uniquely ergodic foliations is denoted by $\mathcal{UE}$ and is a subset of full Lebesgue measure in the Thurston boundary.

Corollary 45 *Every point $F \in \mathcal{UE}$ is a hyperbolic boundary point, indeed, $S(F) = S^{x_0}(F) = \{F\}$.*

Conjecture 46 *For any $F \in \mathcal{PMF}$, it holds that*

$$S(F) = \{G : i(F, G) = 0\}.$$

The conjecture would imply that the Thurston boundary equipped with the star distance restricted to $\mathcal{S}$ is the 1–skeleton of the curve complex (it is interesting to here recall the important result of Masur–Minsky that this complex is Gromov hyperbolic).

9.2 Mapping class groups

Although the arguments in this paper provide (especially if all stars of the Teichmüller spaces can be identified) an alternative explanation of some theorems on the mapping class groups of surfaces obtained notably in [18] and [29], it might however be preferable to study the action directly on the Thurston boundary (or the curve complex) as is done in those works.

It is a standard fact that the only fixed points of a pseudo-Anosov element are two uniquely ergodic foliations and so from Corollary 45 we get:

Proposition 47 *Pseudo-Anosov elements of the mapping class groups are strictly hyperbolic.*

Hence Proposition 22 applies and gives known facts. Theorem 44 and the fundamental contraction property gives a new, more analytic approach, as well as some additional information, to Theorems 7.3.A and 7.3.B in [19]:

Theorem 48 *Suppose that g_n is a sequence of elements in the mapping class groups for which $g_n^{\pm 1} x_0$ converge to two minimal foliations $F^{\pm}$ in $\overline{X}$, the Thurston compactification. Then for z outside a neighborhood of $\{G : i(F^-, G) = 0\}$ in $\overline{X}$, $g_n z$ converges uniformly to $\{G : i(F^+, G)\}$.*

10 Infinite groups

This section contains some remarks and questions concerning infinite groups from the point of view of stars and dynamics of isometries.

Free subgroups Let Γ be a group with a left invariant word metric $||\cdot||$. So $||g|| = d(g,e)$ where e is the identity element in Γ. Moreover, $||xy^{\pm 1}|| = \min\{d(x,y), d(x,y^{-1}\}$. We may formulate the following freeness criterion:

Lemma 49 *Let g and h be two elements of order at least 3 in Γ and let Λ be the subgroup generated by g and h. If for any $a \in \Lambda$ at least one of $||ag^{\pm 1}||$ and $||ah^{\pm 1}||$ is strictly greater than $||a||$, then $\Lambda \cong F_2$.*

Proof The statement $||ag|| > ||a||$ is equivalent to that $a \notin H^e_{g^{-1}}$. So $||ag^{\pm 1}|| > ||a||$ means that $a \notin H^e_g \cup H^e_{g^{-1}}$. Therefore the hypothesis implies that $H^e_g \cup H^e_{g^{-1}}$ and $H^e_h \cup H^e_{h^{-1}}$ are disjoint inside the invariant set Λ. In view of the first observation in subsection 3.2 and the ping-pong lemma [17] applied to the unions of the halfspaces associated to $g^{\pm 1}$ and $h^{\pm 1}$ respectively, and intersected by Λ, the lemma now follows. □

There is a similar criterion for free semigroups.

Random walks Let Γ be a finitely generated group, μ the uniformly distributed probability measure on a finite generating set A, X the Cayley graph associated to A and $\overline{X}^h$ the horofunction compactification. If Γ is nonamenable, then Theorem 19 provides a (probably often nontrivial) μ–boundary for (Γ, μ). In particular we have that if the the random walk has a linear rate of escape, then, from some time on, all the halfspaces defined by the points of the random walk intersect. This is a nontrivial phenomenon, which for example can be seen from thinking about random walks on $\mathbb{Z}^n$.

Associated incidence geometries Let Γ denote a finitely generated group with a boundary $\partial\Gamma$. Consider the incidence geometry (cf [35]) defined by the points and the stars in $\partial\Gamma$, and acted upon by Γ. In general or for some specific group, what can this geometry be? In view of section 3, for example torsion, subexponential growth, or amenability of Γ implies strong restrictions.

As we have seen above, examples of metric spaces and associated incidence geometry are:

- Gromov hyperbolic spaces – trivial
- Hilbert's geometry – the face lattice

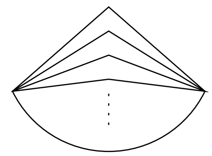

Figure 12: Incidence geometry of the Baumslag–Solitar group BS$(1,n)$

- CAT(0)–space – Tits geometry
- mapping class groups – the curve complex (conjectural).

It may also be interesting to extend the existing theory of convergence groups to a more general setting where one has nontrivial incidence geometry mixed in. A first instance of what we essentially have in mind can be found in [11].

Rigidity theory The following philosophy, vaguely formulated, lies behind the Mostow–Margulis rigidity theory. Any proper homomorphism of one group to another gives rise to an incidence preserving map at infinity of the groups. Incidence preserving maps at infinity must be of a very special kind. Can one make this more precise and when does it (or a part of it) hold? The most simple cases are homomorphisms from $\mathbb{Z}^n$ into the isometry group of some hyperbolic space. Another instance (but now with trivial incidence geometry) is the Floyd–Cannon–Thurston maps in Kleinian group theory.

Group cohomology The following is relevant for (the first) L^2–cohomology: Consider the class of harmonic (ie, satisfying the mean-value property) functions on the vertices of an oriented Cayley graph such that its differential (which is a function on the edges, the difference of the values on the two vertices) is square summable. These functions are called Dirichlet harmonic functions. Compactify the graph in the Stone–Cech way relative to this family of functions. The group does not admit nonconstant Dirichlet harmonic functions if and only if this compactification is the one-point compactification (an easy consequence of the maximum principle). What is the star geometry of this compactification? For many groups it seems that the compactification should be hyperbolic.

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