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On the quantum sl_2 invariants of knots and integral homology spheres

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Abstract We will announce some results on the values of quantum sl_2 invariants of knots and integral homology spheres. Lawrence's universal sl_2 invariant of knots takes values in a fairly small subalgebra of the center of the h-adic version of the quantized enveloping algebra of sl_2. This implies an integrality result on the colored Jones polynomials of a knot. We define an invariant of integral homology spheres with values in a completion of the Laurent polynomial ring of one variable over the integers which specializes at roots of unity to the Witten-Reshetikhin-Turaev invariants. The definition of our invariant provides a new definition of Witten-Reshetikhin-Turaev invariant of integral homology spheres.

AMS Classification 57M27; 17B37

Keywords Quantum invariant, colored Jones polynomial, universal invariant, Witten-Reshetikhin-Turaev invariant

1 Introduction

The purpose of this note is to announce some new results on the values of quantum sl_2 invariants of knots and integral homology spheres. We give a fairly small subalgebra of the center of the quantized enveloping algebra $U_h(sl_2)$ of sl_2 in which Lawrence's universal sl_2 invariant of knots takes values (Theorem 2.1). This implies a formula for the colored Jones polynomials of a knot (Theorem 3.1). We define an invariant $I(M)$ of integral homology spheres M with values in a completion of the Laurent polynomial ring $\mathbb{Z}[q,q^{-1}]$ which specializes at roots of unity to the sl_2 Witten-Reshetikhin-Turaev (WRT) invariants. (Theorem 4.1). The definition of $I(M)$ leads to a new definition of WRT invariant of integral homology spheres since we do not use the WRT invariant in defining $I(M)$. The invariant $I(M)$ is as strong as the totality of the WRT invariants at various roots of unity, and also as the Ohtsuki series (Theorem 4.8). The proofs, details and some generalizations will appear in separate papers [2].

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2 Lawrence's universal sl_2-invariant of links

Let us recall the definition of the *universal sl_2-invariant* introduced by Lawrence [6]. She actually introduced the invariant for more general Lie algebra, but we will consider only the sl_2 case.

2.1 The algebra $U_h(sl_2)$

Let $U_h = U_h(sl_2)$ denote the quantized enveloping algebra of the Lie algebra sl_2, i.e., the h-adically complete $\mathbb{Q}[[h]]$-algebra topologically generated by H, E, F with the relations

$$HE = E(H+2),\ HF = F(H-2),\ EF - FE = \frac{\exp(hH/2) - \exp(-hH/2)}{\exp(h/2) - \exp(-h/2)}.$$

It is useful to introduce the elements

$$v = \exp(h/2)$$

and

$$K = v^H = \exp(hH/2).$$

For each $n \in \mathbb{Z}$, set

$$[n] = (v^n - v^{-n})/(v - v^{-1}) \in \mathbb{Z}[v, v^{-1}].$$

and for $n \geq 0$ set

$$[n]! = [1][2]\dots[n].$$

The algebra U_h has the structure of topological Hopf algebra given by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H, \tag{1}$$

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \epsilon(E) = 0, \quad S(E) = -EK^{-1}, \tag{2}$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \epsilon(F) = 0, \quad S(F) = -KF. \tag{3}$$

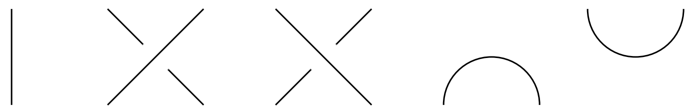

Figure 1: The fundamental tangle diagrams

The Hopf algebra U_h has a universal R-matrix

$$R = v^{\frac{1}{2}H\otimes H} \sum_{n\geq 0} v^{n(n-1)/2} \frac{(v-v^{-1})^n}{[n]!}(E^n \otimes F^n),$$

and a ribbon element

$$r = K^{-1} \sum S(R_{(2)})R_{(1)},$$

where we write $R = \sum R_{(1)} \otimes R_{(2)}$.

2.2 Definition of the universal sl_2 invariant

Let T be an oriented tangle diagram in $\mathbb{R}\times[0,1]$. By small isotopy we may assume that the critical points of the strings composed with the "height function" onto $[0,1]$ are nondegenerate. We may also assume that on each crossing the two intersecting strings are not critical. As is well known, T can be obtained from the *fundamental tangle diagrams* depicted in Figure 1 by composition (i.e., pasting vertically) and tensor product (i.e., pasting horizontally) up to isotopy of $\mathbb{R}\times[0,1]$.

Let T be a tangle diagram as above consisting of l strings $K_1,\ldots,K_l$ and m circle components $K'_1,\ldots,K'_m$. Choose a base point b_i of each K'_i disjoint from the critical points and crossings.

We define below two elements J'_T and J_T by pretending for simplicity that we have $R = R_{(1)}\otimes R_{(2)}$ with $R_{(1)},R_{(2)}\in U_h$, though R actually is an (infinite) sum of elements of the form $a\otimes b\in U_h\otimes U_h$. The precise definition of J'_T follows from multilinearity. We put elements of U_h on the two strings near each crossing and on the string near each right-directed critical points as depicted in Figure 2. Then apply the antipode S to each element put on up-oriented strings. For each K_i, let $J_{(K_i)}$ be the expression obtained by reading from left to right the elements on K_i in the opposite orientation. Similarly, let $J'_{(K'_i,b_i)}$ be the similarly obtained expression for K'_i reading from the base points b_i. Then the expression

$$J'_T = \sum J_{(K_1)}\otimes\cdots\otimes J_{(K_l)}\otimes J_{(K'_1,b_1)}\otimes\cdots\otimes J_{(K'_l,b_l)}$$

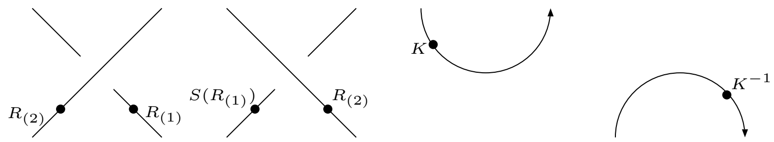

Figure 2: How to put elements of U_h on the strings

defines an element of $U_h^{\hat{\otimes}(l+m)}$, the h-adic completion of the $(l+m)$-fold tensor product of U_h.

Let I denote the h-adic closure of the $\mathbb{Q}[[h]]$-submodule of U_h generated by the commutators $xy - yx$ for $x, y \in U_h$. For $x \in U_h$, let $[x]_I$ denote the coset $x + I \in U_h/I$. For a tangle diagram T as above, set $J_{(K'_i)} = [J_{(K'_i, b_i)}]_I$, which does not depend on the choice of the base point of K'_i. Then the expression

$$J_T = \sum J_{(K_1)} \otimes \cdots \otimes J_{(K_l)} \otimes J_{(K'_1)} \otimes \cdots \otimes J_{(K'_l)}$$

defines an element of $U_h^{\hat{\otimes} l} \hat{\otimes} (U_h/I)^{\hat{\otimes} m}$, where $\hat{\otimes}$ denote the h-adically completed tensor product. It is well known that J_T is invariant under isotopy of diagrams (fixing endpoints) and framed Reidemeister moves, and defines an invariant of framed tangles with total order on the set of components. J_T is called the *universal sl_2 invariant* of T.

In the case of string knot K (i.e., a 1-string string link), the universal sl_2 invariant J_K of K is contained in U_h. It is well known that J_K is contained in the center $Z(U_h)$ of U_h. It is also well known that $Z(U_h)$ is as a complete $\mathbb{Q}[[h]]$-algebra topologically freely generated by the element

$$c = FE + \frac{vK + v^{-1}K^{-1} - v - v^{-1}}{(v - v^{-1})^2}.$$

2.3 Integrality of the universal sl_2 invariant

Let C denote the well known central element of U_h

$$C = (v - v^{-1})^2 FE + vK + v^{-1}K^{-1} \in Z(U_h)$$

and set

$$\sigma_n = \prod_{i=1}^{n} (C^2 - (v^i + v^{-i})^2) \in Z(U_h).$$

We also set

$$q = v^2 \in \mathbb{Q}[[h]],$$

and regard $\mathbb{Z}[q, q^{-1}]$ as a subring of $\mathbb{Q}[[h]]$.

Theorem 2.1 *Let K be a string knot with 0 framing. Then there are unique elements $a_n(K) \in \mathbb{Z}[q, q^{-1}]$ for $n \geq 0$ such that*

$$J_K = \sum_{n \geq 0} a_n(K) \sigma_n.$$

For a knot K with 0-framing and an integer $n \geq 0$, let $a_n(K) \in \mathbb{Z}[q, q^{-1}]$ denote the element determined by Theorem 2.1. If K' is a string knot with 0-framing with the closure equivalent to K, then set $a_n(K') = a_n(K)$.

3 Colored Jones polynomials of links

3.1 Finite dimensional representations of U_h

By a *finite dimensional representation* of U_h we mean as usual a left U_h-module which is free of finite rank as a $\mathbb{Q}[[h]]$-module. For each nonnegative integer n, there is exactly one irreducible $(n+1)$-dimensional representation V_{n+1} of U_h, which corresponds to the unique $(n+1)$-dimensional representation of sl_2. The representation V_{n+1} is defined as follows. As a $\mathbb{Q}[[h]]$-module, V_{n+1} is freely generated by the elements $\mathsf{v}_0, \mathsf{v}_1, \dots, \mathsf{v}_n$. The left action is given by

$$H\mathsf{v}_i = (n - 2i)\mathsf{v}_i, \quad E\mathsf{v}_i = [n + 1 - i]\mathsf{v}_{i-1}, \quad F\mathsf{v}_i = [i + 1]\mathsf{v}_{i+1},$$

for $i = 0, \dots, n$, where we set $\mathsf{v}_{-1} = \mathsf{v}_{n+1} = 0$.

For a finite dimensional representation V of U_h and a $\mathbb{Q}[[h]]$-module endomorphism $g\colon V \to V$, the *(left) quantum trace* $\mathrm{tr}_q^V(f)$ of f is defined by

$$\mathrm{tr}_q^V(f) = \mathrm{tr}(\rho_V(K)f),$$

where $\rho_V\colon U_h \to \operatorname{End} V$ is the action of U_h on V. By abuse of notation, we set for $x \in U_h$,

$$\mathrm{tr}_q^V(x) = \mathrm{tr}_q^V(\rho_V(x)),$$

and call it the (left) quantum trace of x in V. We have

$$\mathrm{tr}_q^V(x) = \mathrm{tr}(\rho_V(Kx)).$$

If $z \in Z(U_h)$, then z acts on each V_{n+1} as a scalar (i.e., an element of $\mathbb{Q}[[h]]$). We have for any $v \in V_{n+1}$

$$zv = \frac{\mathrm{tr}_q^{V_{n+1}}(z)}{[n+1]} v.$$

Here we have $\mathrm{tr}_q^{V_{n+1}}(1) = [n+1]$. We have

$$\frac{\mathrm{tr}_q^{V_{n+1}}(z)}{[n+1]} = s_n(\varphi(z)),$$

where $\varphi\colon U_h \to \mathbb{Q}[H][[h]]$ is the h-adically continuous $\mathbb{Q}[[h]]$-linear map defined on the topological basis of U_h by

$$\varphi(F^i H^j E^k) = \delta_{i,0}\delta_{k,0}H^j$$

for $i,j,k \geq 0$, and $s_n\colon \mathbb{Q}[H][[h]] \to \mathbb{Q}[[h]]$ denote the $\mathbb{Q}[[h]]$-algebra homomorphism defined by $s_n(g(H)) = g(n)$. The restriction of φ onto the h-adic closure $(U_h)_0$ of the $\mathbb{Q}[[h]]$-subalgebra spanned by $F^i H^j E^i$, $i,j \geq 0$, is a $\mathbb{Q}[[h]]$-algebra homomorphism known as the Harish-Chandra homomorphism. It is well known that φ maps the center $Z(U_h) \subset (U_h)_0$ bijectively onto the $\mathbb{Q}[[h]]$-subalgebra of $\mathbb{Q}[H][[h]]$ topologically generated by $(H+1)^2$.

3.2 Colored Jones polynomial of links

Let $L = (L_1, \ldots, L_l)$ be an ordered oriented framed link in S^3 consisting of l components $L_1, \ldots, L_l$. For nonnegative integers $n_1, \ldots, n_l$, we can define the *colored Jones polynomial* $J_L(V_{n_1+1}, \ldots, V_{n_l+1})$ of L associated with the "colors" $(n_1+1, \ldots, n_l+1)$ by

$$J_L(V_{n_1+1}, \ldots, V_{n_l+1}) = (\tilde{\mathrm{tr}}^{V_{n_1+1}} \otimes \cdots \otimes \tilde{\mathrm{tr}}^{V_{n_l+1}})(J_L). \tag{4}$$

Here $\tilde{\mathrm{tr}}^{V_{n_i+1}}\colon U_h/I \to \mathbb{Q}[[h]]$ is defined by

$$\tilde{\mathrm{tr}}^{V_{n_i+1}}([x]_I) = \mathrm{tr}(\rho_{V_{n_i+1}}(x)).$$

(Usual definition of colored Jones polynomial involves braiding operators on finite dimensional representations. Our definition here is equivalent to the usual one.)

We choose an l-component string link $T = (T_1, \ldots, T_l)$ such that the closure of T is ambient isotopic to L. Then we have

$$J_L(V_{n_1+1}, \ldots, V_{n_l+1}) = (\mathrm{tr}_q^{V_{n_1}+1} \otimes \cdots \otimes \mathrm{tr}_q^{V_{n_l}+1})(J_T).$$

3.3 The case of knots of framing 0

Let K be a string knot with 0 framing. Since $J_K \in Z(U_h)$, J_K acts on each representation V_{n+1}, $n \geq 0$, as a scalar, which we will denote by $J_K(V_{n+1})$. It is well known that $J_K(V_{n+1}) \in \mathbb{Z}[q, q^{-1}]$. We have

$$J_{\mathrm{cl}(K)}(V_{n+1}) = \mathrm{tr}_q^{V_{n+1}}(J_K) = [n+1]J_K(V_{n+1})$$

and

$$J_K(V_{n+1}) = s_n(\varphi(J_K)).$$

Theorem 3.1 *Let K be a string knot with 0 framing and let $n \geq 0$ be an integer. Then we have*

$$J_K(V_{n+1}) = \sum_{i=0}^{n} a_i(K) \prod_{n+1-i \leq j \leq n+1+i,\ j \neq n+1} (v^j - v^{-j}). \tag{5}$$

Note that this sum may be regarded as the infinite sum $\sum_{i=0}^{\infty}$ since the terms for $i > n$ vanishes.

Theorem 3.1 provides a new proof for Rozansky's integral version [15] of the Melvin-Morton expansion [11] of the colored Jones polynomials of knots. (We do *not* mean here that Theorem 3.1 implies the Melvin-Morton conjecture, proved in [1], involving the Alexander polynomial.) It follows from (5) that

$$J_K(V_{n+1}) = \sum_{i=0}^{\infty} a_i(K) \prod_{j=1}^{i} (\alpha^2 - (v^j - v^{-j})^2),$$

where $\alpha = v^{n+1} - v^{-n-1}$. The right hand side may be regarded as an element of the completion ring

$$\varprojlim{}_n \mathbb{Z}[q, q^{-1}, \alpha^2] / (\prod_{j=1}^{n} (\alpha^2 - q^j - q^{-j} + 2)),$$

with α^2 being regarded as an indeterminate. There is a natural injective homomorphism from this ring to the formal power series ring $\mathbb{Z}[[q-1, \alpha^2]]$.

By expanding in powers of α^2, we have

$$J_K(V_{n+1}) = \sum_{k=0}^{\infty} \alpha^{2k} (\sum_{i=k}^{\infty} (-1)^{k-i} \tau_{i,i-k} a_i(K)),$$

where

$$\tau_{i,k} = \sum_{1 \leq p_1 < \cdots < p_k \leq i} \prod_{r=1}^{k} (v^{p_r} - v^{-p_r})^2.$$

It is not difficult to see that for each $k \geq 0$, the coefficient

$$\sum_{i=k}^{\infty} (-1)^{k-i} \tau_{i,i-k} a_i(K)$$

of α^{2k} defines an element of the completion ring

$$\widehat{\mathbb{Z}[q]} = \varprojlim_i \mathbb{Z}[q]/((q-1)(q^2-1)\cdots(q^i-1)).$$

In particular, the constant term

$$\sum_{i=0}^{\infty}(-1)^i(\prod_{p=1}^{i}(v^p - v^{-p})^2)a_i(K)$$

specializes to the Kashaev invariants [4, 13] of K by substituting roots of unity for q.

3.4 Examples

Let 3_1^+ (resp. 3_1^-) denote the trefoil knot with positive (resp. negative) signature, and let 4_1 denote the figure eight knot. Then we have for each $n \geq 0$

$$a_n(\text{unknot}) = \delta_{n,0}, \tag{6}$$

$$a_n(3_1^+) = (-1)^n q^{\frac{1}{2}n(n+3)}, \tag{7}$$

$$a_n(3_1^-) = (-1)^n q^{-\frac{1}{2}n(n+3)}, \tag{8}$$

$$a_n(4_1) = 1. \tag{9}$$

A formula for 4_1 in [10] follows from (9) and (5).

3.5 The algebra $\mathcal{R}$ and the basis P'_n

If $m, n \geq 0$, then we have a direct sum decomposition of left U_h-modules

$$V_{m+1} \otimes V_{n+1} \cong \bigoplus_{|m-n| \leq i \leq m+n,\ i \equiv m+n \mod 2} V_{i+1}.$$

The Grothendieck ring $\mathcal{R}_{\mathbb{Z}}$ of finite dimensional representations of U_h is freely spanned over $\mathbb{Z}$ by $V_1 = 1, V_2, V_3, \ldots$, and is isomorphic to $\mathbb{Z}[V_2]$. For a commutative ring with unit k, set

$$\mathcal{R}_k = \mathcal{R}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k = k[V_2].$$

For $n \geq 0$, set

$$P_n = \prod_{i=0}^{n-1}(V_2 - v^{2i+1} - v^{-2i-1}) \in \mathcal{R}_{\mathbb{Z}[v,v^{-1}]}.$$

We will also use the following normalizations

$$P'_n = (v - v^{-1})^{-n}([n]!)^{-1} P_n \in \mathcal{R}_{\mathbb{Q}(v)},$$

and

$$P''_n = (v - v^{-1})^{-2n}([2n+1]!)^{-1} P_n \in \mathcal{R}_{\mathbb{Q}(v)}.$$

Let $L = (L_1, \ldots, L_l)$ be a framed link of l components in S^3. Extend (4) multilinearly to define

$$J_L(x_1, \ldots, x_l) \in \mathbb{Q}(v^{1/2})$$

for $x_1, \ldots, x_l \in \mathcal{R}_{\mathbb{Q}(v)}$. If L is algebraically split (i.e., the linking numbers are all 0) and with all framings 0, then we have

$$J_L(x_1, \ldots, x_l) \in \mathbb{Q}(v).$$

Theorem 3.2 *If K is a string knot with 0 framing and if $n \geq 0$, then we have*

$$a_n(K) = J_{\mathrm{cl}(K)}(P''_n).$$

In particular we have

$$J_{\mathrm{cl}(K)}(P''_n) \in \mathbb{Z}[q, q^{-1}].$$

Let $\mathcal{R}$ denote the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{R}_{\mathbb{Q}(v)}$ generated by the elements P'_n for $n \geq 1$. As an $\mathbb{Z}[v, v^{-1}]$-module, $\mathcal{R}$ is freely generated by the P'_n, $n \geq 0$. For each $m \geq 0$, let $\mathcal{R}_m$ denote the $\mathbb{Z}[v, v^{-1}]$-submodule of $\mathcal{R}$ spanned by $P'_m, P'_{m+1}, \ldots$, which turns out to be an ideal in $\mathcal{R}$. The following theorem is a generalization of Theorem 3.2 to algebraically split framed links.

Theorem 3.3 *Let $L = (L_1, \ldots, L_l)$ be an algebraically split framed link of l components in S^3 with all framings 0. If $x_1, \ldots, x_l \in \mathcal{R}$, then we have*

$$J_L(x_1, \ldots, x_l) \in \mathbb{Z}[v, v^{-1}]$$

If one of the x_i is contained in $\mathcal{R}_m$, $m \geq 0$, then we have

$$J_L(x_1, \ldots, x_l) \in \frac{(v - v^{-1})^m [2m+1]!}{[m]!} \mathbb{Z}[v, v^{-1}]$$

It follows from the first half of Theorem 3.3 that if $L = (L_1, \ldots, L_l)$ is an algebraically split framed link with 0-framings in S^3, then the $\mathbb{Q}(v)$-multilinear map $J_L\colon \mathcal{R}_{\mathbb{Q}(v)} \times \cdots \times \mathcal{R}_{\mathbb{Q}(v)} \to \mathbb{Q}(v)$ restricts to the $\mathbb{Z}[v, v^{-1}]$-multilinear map

$$J_L\colon \mathcal{R} \times \cdots \times \mathcal{R} \to \mathbb{Z}[v, v^{-1}], \tag{10}$$

which induce the $\mathbb{Z}[v,v^{-1}]$-linear map

$$J_L\colon \mathcal{R}\otimes_{\mathbb{Z}[v,v^{-1}]}\cdots\otimes_{\mathbb{Z}[v,v^{-1}]}\mathcal{R}\to\mathbb{Z}[v,v^{-1}]. \tag{11}$$

Set

$$\hat{\mathcal{R}}=\varprojlim{}_m\mathcal{R}/\mathcal{R}_m,$$

which is a commutative $\mathbb{Z}[v,v^{-1}]$-algebra. $\hat{\mathcal{R}}$ consists of the infinite sums $\sum_{m\geq 0} b_m P'_m$, where $b_m\in\mathbb{Z}[v,v^{-1}]$ for $m\geq 0$. It follows from the second half of Theorem 3.3 that J_L in (10) induces a $\mathbb{Z}[v,v^{-1}]$-linear map

$$J_L\colon \hat{\mathcal{R}}^{\hat{\otimes} l}\to\widehat{\mathbb{Z}[v]}, \tag{12}$$

where $\hat{\mathcal{R}}^{\hat{\otimes} l}$ denote the completion of the l-fold tensor product $\hat{\mathcal{R}}\otimes_{\mathbb{Z}[v,v^{-1}]}\cdots\otimes_{\mathbb{Z}[v,v^{-1}]}\hat{\mathcal{R}}$ with respect to the natural tensor product topology induced by the completion topology of $\hat{\mathcal{R}}$, and $\widehat{\mathbb{Z}[v]}=\widehat{\mathbb{Z}[q]}\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Z}[v,v^{-1}]$. (Here recall that we set $q=v^2$.)

4 A universal sl_2 invariant of integral homology spheres

In this section we define an invariant $I(M)\in\widehat{\mathbb{Z}[q]}$ of integral homology spheres M, which we call the "universal sl_2-invariant" of M, since $I(M)$ is "universal" over the sl_2 WRT invariants at various roots of unity.

Remark Recall that Le [9] defined an invariant of closed 3-manifolds M with values in a "functional space" such that the sl_2 WRT invariants of M recovers from it via certain ring homomorphisms. However, his "functional space" is rather large, and since it involves complex functions, it does not give any information on the value of the WRT invariant at each root of unity.

Remark The use of the word "universal" here is with respect to the roots of unity, but the use in Lawrence's sl_2 universal link invariant is with respect to finite dimensional representations. We can unify these two invariants into a "universal sl_2 invariant" $I(M,L)$ of links L in integral homology spheres M. This generalization is an easy modification of the definition of $I(M)$, and the details will appear in [2].

4.1 The definition of the invariant $I(M)$

Set

$$\omega = \sum_{i\geq 0} v^{\frac{1}{2}i(i+3)} P'_i \in \hat{\mathcal{R}},$$

which is invertible in the algebra $\hat{\mathcal{R}}$ with the inverse

$$\omega^{-1} = \sum_{i\geq 0} (-1)^i v^{-\frac{1}{2}i(i+3)} P'_i.$$

Let M be an integral homology 3-sphere. It is well known that there is an algebraically split framed link $L = (L_1, \dots, L_l)$ $(l \geq 0)$ in S^3 with all framings ± 1 such that the surgery $(S^3)_L$ on S^3 along L is orientation-preserving homeomorphic to M. Set

$$I(L) = J_{L_0}(\omega^{-f_1}, \dots, \omega^{-f_l}) \tag{13}$$

where L_0 denotes the framed link obtained from L by changing all the framings into 0, and, for $i = 1, \dots, l$, $f_i = \pm 1$ denotes the framing of the component L_i. We have

$$I(L) \in \widehat{\mathbb{Z}[q]}.$$

Theorem 4.1 *There is a well-defined invariant $I(M)$ of integral homology spheres M with values in $\widehat{\mathbb{Z}[q]}$ such that if L is a algebraically split framed link in S^3 with all framings ± 1, then we have $I((S^3)_L) = I(L)$.*

Theorem 4.1 follows from Theorems 4.2 and 4.3 below.

Theorem 4.2 (Conjectured by Hoste [3]) *Let L and L' be two algebraically split framed links in S^3 with all the framings ± 1. Then L and L' define orientation-preserving homeomorphic results of surgeries $(S^3)_L$ and $(S^3)_{L'}$ if and only if L and L' are related by a finite sequence of* Hoste moves, *i.e., the usual Fenn-Rourke moves (surgery on unknotted component of framing ± 1)* through *algebraically split framed links with ± 1 framings.*

Theorem 4.3 *The invariant $I(L)$ of algebraically split framed links L in S^3 with ± 1 framings is invariant under Hoste moves.*

4.2 Specializations to the WRT invariants at roots of unity

For each root of unity ζ, there is a well-defined ring homomorphism

$$(-)|_{q=\zeta}\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta], \quad f(q) \mapsto f(q)|_{q=\zeta} = f(\zeta).$$

For an integral homology sphere M and a root of unity ζ, let $\tau_\zeta(M)$ be the WRT invariant of M at ζ normalized so that $\tau_\zeta(S^3) = 1$. (For definition of $\tau_\zeta(M)$, see [5], but $\tau_r(M)$ for $r \geq 3$ defined there corresponds to $\tau_{\exp(2\pi i/r)}$. For the other primitive rth roots of unity ζ, $\tau_\zeta(M)$ is obtained from $\tau_r(M)$ by the automorphism of $\mathbb{Z}[\zeta]$ which maps $\exp(2\pi i/r)$ to ζ. For $\zeta = \pm 1$, set $\tau_{\pm 1}(M) = 1$.)

Theorem 4.4 *For an integral homology sphere M and a root of unity ζ, we have*

$$I(M)|_{q=\zeta} = \tau_\zeta(M). \tag{14}$$

The proof of Theorem 4.1 does not involve the existence proofs of the variations of the WRT invariant in the literature. Hence the Theorems 4.1 provides a new definition of the WRT invariant of integral homology spheres via (14).

4.3 Consequences

In the rest of this paper, we list some consequences to Theorems 4.1 and 4.4.

The following was first proved by H. Murakami [12] for the case ζ is a root of unity of odd prime order, and conjectured by Lawrence [8] in the general case.

Corollary 4.5 *For an integral homology sphere M and a root ζ of unity, we have*

$$\tau_\zeta(M) \in \mathbb{Z}[\zeta].$$

By Lawrence's conjecture [7] proved by Rozansky [16], the Ohtsuki series [14] of an integral homology sphere M can be characterized as the formal power series $\tau(M) \in \mathbb{Z}[[q-1]]$ such that for each root of unity ζ of odd prime power order we have

$$\tau(M)|_{q=\zeta} = \tau_\zeta(M), \tag{15}$$

where both sides are regarded as the elements of $\mathbb{Z}_p[\zeta]$ with p the odd prime such that the order of q is a power of p. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Theorem 4.4 provides a new proof of the existence of $\tau(M)$, and moreover the following version of Lawrence's p-adic convergence conjecture for $p = 2$.

Theorem 4.6 *If ζ is a primitive 2^mth root of unity ($m \geq 1$), then we have*

$$\tau(M)|_{q=\zeta} = \tau_\zeta(M) \in \mathbb{Z}_2[\zeta]. \tag{16}$$

Let

$$\iota_1\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[[q-1]]$$

be the homomorphism induced by $\mathrm{id}_{\mathbb{Z}[q]}$.

Theorem 4.7 *If M is an integral homology sphere, then we have*

$$\iota_1(I(M)) = \tau(M). \tag{17}$$

Since ι_1 is injective, $I(M)$ is as strong as $\tau(M)$. The injectivity of ι_1 is also independently proved by Vogel. We also have the following.

Theorem 4.8 *Let M and M' be two integral homology spheres. Then the following conditions are equivalent.*

(1) $I(M) = I(M')$,

(2) $\tau(M) = \tau(M')$,

(3) $\tau_\zeta(M) = \tau_\zeta(M')$ *for all roots of unity* ζ,

(4) $\tau_\zeta(M) = \tau_\zeta(M')$ *for infinitely many roots of unity ζ of prime power order.*

Remark For each root of unity ζ, there is a natural homomorphism

$$\iota_\zeta\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]].$$

For an integral homology sphere M, we may think of $\iota_\zeta(M) \in \mathbb{Z}[\zeta][[q-\zeta]]$ as an "expansion of the WRT invariants of M at $q = \zeta$". This is a generalization of the Ohtsuki series to the expansion at ζ.

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