Владикавказский математический журнал
2014, Том 16, Выпуск 2, С. 29–37

УДК **517.98**

TRACE CLASS AND LIDSKIĬ TRACE FORMULA ON KAPLANSKY–HILBERT MODULES

U. Gönüllü

In this paper, we introduce and study the concepts of the trace class operators and global eigenvalue of continuous Λ-linear operators in Kaplansky–Hilbert modules. In particular, we give a variant of Lidskiĭ trace formula for cyclically compact operators in Kaplansky–Hilbert modules.

Mathematics Subject Classification (2000): 47B60, 46A19, 46B99, 46L08, 47B07.

Key words: Kaplansky–Hilbert module, cyclically compact operator, global eigenvalue, trace class, Lidskiĭ trace formula.

1. Introduction

Kaplansky–Hilbert module or AW^*-module arose naturally in Kaplansky's study of AW^*-algebras of type I [2]. I. Kaplansky proved some deep and elegant results for such structures, and therefore they have many properties of Hilbert spaces. In [7] A. G. Kusraev established functional representations of Kaplansky–Hilbert modules and AW^*-algebras of type I by spaces of continuous vector-functions and strongly continuous operator-functions, respectively. The functional representations are the main technical tool used in this paper. Cyclically compact sets and operators in lattice-normed spaces were introduced by A. G. Kusraev in [5] and [6], respectively. In [8] (see also [9]) a general form of cyclically compact operators in Kaplansky–Hilbert modules, which, like the Schmidt representation of compact operators in Hilbert spaces, as well as a variant of the Fredholm alternative for cyclically compact operators, was also given. Recently, cyclically compact sets and operators in Banach–Kantorovich spaces over a ring of measurable functions were investigated in [1, 3, 4].

In this paper, we introduce and study the concepts of the trace class operators and global eigenvalue and multiplicity of a global eigenvalue, and give a variant of Lidskiĭ trace formula for cyclically compact operators in Kaplansky–Hilbert modules. We refer to [9] for the whole standard terminology and detailed information.

2. Preliminaries

A C^*-*module* over the Stone algebra Λ is a Λ-module X equipped with a Λ-*valued inner product* $\langle \cdot \mid \cdot \rangle : X \times X \to \Lambda$ satisfying the following conditions:

(1) $\langle x \mid x \rangle \geqslant 0$; $\langle x \mid x \rangle = 0 \Leftrightarrow x = 0$;

(2) $\langle x \mid y \rangle = \langle y \mid x \rangle^*$;

(3) $\langle ax + by \mid z \rangle = a \langle x \mid z \rangle + b \langle y \mid z \rangle$;

(4) X is complete with respect to the norm $|||x||| := \|\langle x \mid x\rangle\|^{\frac{1}{2}}$
for all x, y, z in X and a, b in Λ. As well as its scalar-valued norm $|||\cdot|||$, a C^*-module X has a vector norm, given by $|x| := \sqrt{\langle x \mid x\rangle}$. It is not difficult to deduce $|||x||| = \||x|\|$ and the Cauchy–Bunyakovskiĭ–Schwarz inequality $|\langle x \mid y\rangle| \leqslant |x||y|$.

A *Kaplansky–Hilbert module* or an *AW^*-module* over Λ is a unitary C^*-module over Λ that enjoys the following two properties:

(1) let x be an arbitrary element in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$ with $e_\xi x = 0$ for all $\xi \in \Xi$; then $x = 0$;

(2) let $(x_\xi)_{\xi\in\Xi}$ be a norm-bounded family in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$; then there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$ where $\mathfrak{P}(\Lambda)$ denotes complete Boolean algebra of all projections p of Λ (i. e., $p^2 = p$ and $p^* = p$). We say that X is *faithful* if for every $a \in \Lambda$ the condition $ax = 0$ for all $x \in X$ implies that $a = 0$.

Throughout this paper the letters X and Y denote faithful Kaplansky–Hilbert modules over Λ. Moreover, Q and H will denote an extremally disconnected compact space and a Hilbert space, respectively.

Let $B_\Lambda(X,Y)$ denote the set of all continuous Λ-linear operators from X into Y. In case $X = Y$, $B_\Lambda(X) := B_\Lambda(X,X)$ is an AW^*-algebra of type I with center isomorphic to Λ [2, Theorem 7]. Every continuous Λ-linear operator is dominated and *bo*-continuous [9, Theorem 7.5.7.(1)]. Furthermore, for every continuous Λ-linear operator T,

$$|T|\mathbf{1} = \sup\left\{|Tx| : x \in X,\ |x| \leqslant \mathbf{1}\right\} = \sup\left\{|Tx| : x \in X,\ |x| = \mathbf{1}\right\},$$

holds, and $|T| \in \mathrm{Orth}(\Lambda)$ [9, Theorem 5.1.8.], whence we can identify $|T|\mathbf{1}$ and $|T|$ since $\mathrm{Orth}(\Lambda) = \Lambda$.

Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}_{\mathbb{N}}(B)$ the set of sequences $\nu : \mathbb{N} \to B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}_{\mathbb{N}}(B)$, the symbol $\nu_1 \ll \nu_2$ abbreviates the following assertion: if $m, n \in \mathbb{N}$ and $\nu_1(m) \wedge \nu_2(n) \neq 0_B$ then $m < n$. Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \to K$, and a partition $\nu \in \mathrm{Prt}_{\mathbb{N}}(B)$, put $s_\nu := \mathrm{mix}\,(\nu(n)s(n))n \in \mathbb{N}$. A *cyclic subsequence* of $s : \mathbb{N} \to K$ is any sequence of the form $(s_{\nu_k})_{k\in\mathbb{N}}$, where $(\nu_k)_{k\in\mathbb{N}} \subset \mathrm{Prt}_{\mathbb{N}}(B)$ and $\nu_k \ll \nu_{k+1}$ for all $k \in \mathbb{N}$. A subset $C \subset X$ is said to be *cyclically compact* if C is mix-complete and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set. An operator $T \in B_\Lambda(X,Y)$ is called *cyclically compact* if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact in Y. The set of all cyclically compact operators is denoted by $\mathscr{K}(X,Y)$.

Let $x \in X$, $y \in Y$. Define the operator $\theta_{x,y} : X \to Y$ by the formula

$$\theta_{x,y}(z) := \langle z \mid x\rangle\, y, \quad z \in X,$$

and note that $\theta_{x,y} \in \mathscr{K}(X,Y)$.

The techniques employed in [1] yield the following theorem: $U = S_{\widetilde{u}}$ is a cyclically compact opeartor on $C_\#(Q,H)$ if and only if there is a comeager set Q_0 in Q such that $u(q)$ is a compact operator on H for all $q \in Q_0$.

3. The Trace Class

In this section, we study the trace class operators on Kaplansky–Hilbert modules and investigate the dualities of the trace class.

From now onward, it will be assumed that $(e_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, and $(\mu_k)_{k\in\mathbb{N}}$ verify the representation of a cyclically compact operator T as in [9, Theorem 8.5.6]

3.1. Definition. Let $1 \leqslant p < \infty$. The symbol $\mathscr{S}_p(X, Y)$ denotes the set of all cyclically compact operators T such that $(\mu_k^p)_{k\in\mathbb{N}}$ is o-summable in Λ. Put $v_p(T) := \big(o\text{-}\sum_{k\in\mathbb{N}} \mu_k^p\big)^{\frac{1}{p}}$.

$\mathscr{S}_1(X, Y)$ and $\mathscr{S}_2(X, Y)$ are called the *trace class* and the *Hilbert–Schmidt class*, respectively.

3.2. Proposition. *Let $T \in \mathscr{K}(X, Y)$. Then T is in $\mathscr{S}_1(X, Y)$ if and only if there exist families $(x_i)_{i\in I}$ in X and $(y_i)_{i\in I}$ in Y such that $\big(|\!|x_i|\!||\!|y_i|\!|\big)_{i\in I}$ is o-summable and $T = bo\text{-}\sum_{i\in I} \theta_{x_i,y_i}$. In particular, if $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are projection orthonormal families and $(\alpha_i)_{i\in I}$ is a family with positive elements, then $v_1(T) = o\text{-}\sum_{i\in I} \alpha_i|\!|x_i|\!||\!|y_i|\!|$.*

$\lhd$ If T is in $\mathscr{S}_1(X, Y)$, then the result follows from $x_n := \mu_n e_n$ and $y_n := f_n$.

For the converse, assume that the families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the stated conditions. The inequality

$$\sum_{n=1}^{k} \mu_n = \sum_{n=1}^{k} \langle Te_n \mid f_n\rangle = \sum_{n=1}^{k} \left(o\text{-}\sum_{i\in I} \langle e_n \mid x_i\rangle \langle y_i \mid f_n\rangle \right)$$
$$\leqslant o\text{-}\sum_{i\in I} \left(\left(\sum_{n=1}^{k} |\langle e_n \mid x_i\rangle|^2 \right)^{1/2} \left(\sum_{n=1}^{k} |\langle y_n \mid f_i\rangle|^2 \right)^{1/2} \right) \leqslant o\text{-}\sum_{i\in I} |\!|x_i|\!||\!|y_i|\!|$$

holds for each $k \in \mathbb{N}$, and the proof is finished. $\rhd$

3.3. Corollary. *Let $T \in \mathscr{S}_1(X, Y)$ and $\lambda \in \Lambda$. Then $v_1(\lambda T) = |\lambda| v_1(T)$ and $|\!|T|\!| \leqslant v_1(T)$ and*

$$v_1(T) = \inf \left\{ o\text{-}\sum_{i\in I} |\!|x_i|\!||\!|y_i|\!| : \ (x_i)_{i\in I} \subset X,\ (y_i)_{i\in I} \subset Y \right\}$$

where $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy condition (ii) of Proposition 3.2.

3.4. Lemma. *Let $T \in \mathscr{S}_1(X)$. Then the net $(|\langle Te \mid e\rangle|)_{e\in\mathscr{E}}$ is o-summable in Λ for all projection bases $\mathscr{E}$, and the sum $o\text{-}\sum_{e\in\mathscr{E}} \langle Te \mid e\rangle$ is the same for all projection bases $\mathscr{E}$ of X.*

$\lhd$ It is enough to observe that there exist a positive cyclically compact operator R_1 and a cyclically compact operator R_2 in $\mathscr{S}_2(X)$ such that $T = R_1 R_2$ and $\langle Te \mid e\rangle = \langle R_2 e \mid R_1 e\rangle$ hold for every $e \in \mathscr{E}$, namely,

$$R_1 := bo\text{-}\sum_{k=1}^{\infty} \mu_k^{1/2} \theta_{f_k,f_k}, \quad R_2 := bo\text{-}\sum_{k=1}^{\infty} \mu_k^{1/2} \theta_{e_k,f_k}. \ \rhd$$

The *trace* of $T \in \mathscr{S}_1(X)$ is defined by $\mathrm{tr}(T) := o\text{-}\sum_{e\in\mathscr{E}} \langle Te \mid e\rangle$ where $\mathscr{E}$ is a projection bases of X. Observe that $v_1(T) = \mathrm{tr}(T)$ is satisfied for every positive operator T in $\mathscr{S}_1(X)$ and $\mathrm{tr}(T) = o\text{-}\sum_{i\in I} \langle y_i \mid x_i\rangle$ where $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the condition (ii) of Proposition 3.2, and so tr is a Λ-linear operator.

3.5. Lemma. *The following statements hold:*

(i) $\mathrm{tr} : (\mathscr{S}_1(X), v_1(\cdot)) \to \Lambda$ *is a dominated and bo-continuous Λ-linear operator. In particular,* $|\mathrm{tr}(T)| \leqslant v_1(T)$ *and* $|\!|\mathrm{tr}|\!| = \mathbf{1}$*;*

(ii) $\mathrm{tr}(T^*) = \mathrm{tr}(T)^*$ $(T \in \mathscr{S}_1(X))$;

(iii) $\mathrm{tr}(TL) = \mathrm{tr}(LT)$ *whenever* $TL, LT \in \mathscr{S}_1(X)$ $(T \in \mathscr{K}(X)$ *and* $L \in B_\Lambda(X))$*;*

(iv) If $T \in \mathscr{S}_1(Y, X)$ and $L \in B_\Lambda(X, Y)$, then $TL \in \mathscr{S}_1(X)$, $LT \in \mathscr{S}_1(Y)$ and $|\mathrm{tr}(TL)| \leqslant v_1(T)|\!|L|\!|$.

◁ (i) Using the representation of T, we deduce $|\mathrm{tr}(T)| \leqslant v_1(T)$. Thus, tr is *bo*-continuous and subdominated, and hence it is dominated, by virtue of [9, Theorem 4.1.11.(1)].

(ii) Follows immediately from the definition of tr.

(iii) Use the representation of T to obtain $\mathrm{tr}(LT) = \mathrm{tr}(TL)$.

(iv) If $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the condition (ii) of Proposition 3.2 for T, then $(L^*x_i)_{i\in I}$ and $(y_i)_{i\in I}$ also satisfy the same conditions for TL. Therefore, we have $TL \in \mathscr{S}_1(X)$ and the inequality

$$|\mathrm{tr}(TL)| = \Big| o\text{-}\sum_{i\in I} \langle y_i \mid L^*x_i\rangle \Big| = \Big| o\text{-}\sum_{i\in I} \langle Ly_i \mid x_i\rangle \Big| \leqslant o\text{-}\sum_{i\in I} |Ly_i||x_i| \leqslant |L|\, o\text{-}\sum_{i\in I} |y_i||x_i|,$$

and so the desired inequality follows from Corollary 3.2. ▷

Let $(\mathscr{X}, |\cdot|, \Lambda)$ be a Banach–Kantorovich space. Denote by $\mathscr{X}^*$ the set of all Λ-linear operator $\eta : \mathscr{X} \to \Lambda$ such that $(\exists\, c \in \Lambda)\ |\eta(x)| \leqslant c|x|\ (\forall\, x \in \mathscr{X})$, and note that $\mathscr{X}^*$ consists of all $|||\cdot|||$-continuous Λ-linear operators $\eta : \mathscr{X} \to \Lambda$.

3.6. Theorem. *If $\varphi : \mathscr{S}_1(Y,X) \to \mathscr{K}(X,Y)^*$ is defined by $\varphi(T)(A) = \mathrm{tr}(TA)$ for all $A \in \mathscr{K}(X,Y)$ and $T \in \mathscr{S}_1(Y,X)$, then φ satisfies the following properties:*

(i) φ is a bijective Λ-linear operator from $\mathscr{S}_1(Y,X)$ to $\mathscr{K}(X,Y)^*$;

(ii) $v_1(T) = |\varphi(T)|$ $(T \in \mathscr{S}_1(Y,X))$.

◁ By Lemma 3.5 (i) and (iv), φ is a well-defined dominated Λ-linear operator, and $|\varphi(T)| \leqslant v_1(T)$ holds for all $T \in \mathscr{S}_1(Y,X)$. Let $\phi \in \mathscr{K}(X,Y)^*$. Since $\mathscr{S}_2(X,Y)$ is a Kaplansky–Hilbert module, $\phi_{|\mathscr{S}_2(X,Y)}$ is in $\mathscr{S}_2(X,Y)^*$ and there exists a unique $S \in \mathscr{S}_2(X,Y)$ such that $\phi_{|\mathscr{S}_2(X,Y)} = \langle\cdot, S\rangle$. Thus, $\phi_{|\mathscr{S}_2(X,Y)}(A) = \mathrm{tr}(S^*A)$ $(A \in \mathscr{S}_2(X,Y))$. Assume that $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$, and $(\lambda_k)_{k\in\mathbb{N}}$ satisfy representation of S^* as [9, Theorem 8.5.6]. Define $P_m := \sum_{k=1}^m \theta_{y_k,x_k}$ $(m \in \mathbb{N})$, and note that $|P_m| \leqslant \mathbf{1}$. Thus, the following inequality

$$|\phi| = |\phi|\mathbf{1} \geqslant |\phi||P_m| \geqslant |\phi(P_m)| = |\mathrm{tr}(S^*P_m)| = \sum_{k=1}^m \lambda_k.$$

implies that $S^* \in \mathscr{S}_1(Y,X)$. From $\varphi(S^*)$ is *bo*-continuous $\varphi(S^*)(A) = \phi(A)$ is satisfied for all $A \in \mathscr{K}(X,Y)$. Thus, φ is onto and $|\varphi(S^*)| \geqslant v_1(S^*)$ holds, and the proof is finished. ▷

The proof of the following lemma can be extracted from the proof of [10, Proposition 1.3].

3.7. Lemma. *If the mapping $\sigma : X \times Y \to \Lambda$ satisfies the properties:*

(i) $\sigma(\lambda x_1 + \mu x_2, y) = \lambda\sigma(x_1,y) + \mu\sigma(x_2,y)$ $(x_1, x_2 \in X,\ y \in Y,\ \lambda,\mu \in \Lambda)$;

(ii) $\sigma(x, \lambda y_1 + \mu y_2) = \lambda^*\sigma(x,y_1) + \mu^*\sigma(x,y_2)$ $(x \in X,\ y_1, y_2 \in Y,\ \lambda,\mu \in \Lambda)$;

(iii) There exists some $\lambda \in \Lambda_+$ such that $|\sigma(x,y)| \leqslant \lambda|x||y|$ $(x \in X,\ y \in Y)$

then there exists a unique $A \in B_\Lambda(X,Y)$ such that $|A| \leqslant \lambda$ and $\sigma(x,y) = \langle Ax \mid y\rangle$.

3.8. Theorem. *If $\psi : (B_\Lambda(X,Y), |\cdot|) \to (\mathscr{S}_1(Y,X)^*, |\cdot|_1)$ is defined by $\psi(L)(T) = \mathrm{tr}(TL)$ for all $L \in B_\Lambda(X,Y)$ and $T \in \mathscr{S}_1(Y,X)$, then ψ satisfies the following properties:*

(i) ψ is a bijective Λ-linear operator from $B_\Lambda(X,Y)$ to $\mathscr{S}_1(Y,X)^*$;

(ii) $|L| = |\psi(L)|_1$ $(L \in B_\Lambda(X,Y))$.

◁ By Lemma 3.5 (i) and (iv), ψ is a well-defined dominated Λ-linear operator, and $|\psi(L)|_1 \leqslant |L|$ holds for all $L \in B_\Lambda(X,Y)$. Let $\tau \in \mathscr{S}_1(Y,X)^*$. Define $\sigma : X \times Y \to \Lambda$ by $\sigma(x,y) := \tau(\theta_{y,x})$, and observe that

$$|\sigma(x,y)| = |\tau(\theta_{y,x})| \leqslant |\tau|_1 v_1(\theta_{y,x}) \leqslant |\tau|_1|x||y|.$$

Therefore, there exists $A \in B_\Lambda(X,Y)$ with $\sigma(x,y) = \langle Ax \mid y\rangle$. This implies that $\psi(A)(\theta_{y,x}) = \tau(\theta_{y,x})$ and $|Ax|^2 \leqslant |\tau_1|Ax||x|$. Thus, we have $|A| \leqslant |\tau|_1$ and $\psi(A)(T) = \tau(T)$ $(T \in \mathscr{S}_1(Y,X))$, and the proof is finished. $\rhd$

4. Lidskiĭ trace formula

Our main aim in this section is to prove Lidskiĭ trace formula for cyclically compact operators in a Kaplansky–Hilbert modules.

Set $[\lambda] = \inf\{\pi \in \mathfrak{P}(\Lambda) : \pi\lambda = \lambda\}$, the *support* of λ in Λ.

4.1. Definition. Let T be an operator on X. A scalar $\lambda \in \Lambda$ is said to be an *eigenvalue* if there exists nonzero $x \in X$ such that $Tx = \lambda x$. A nonzero eigenvalue λ is called a *global eigenvalue* if for every nonzero projection $\pi \in \Lambda$ with $\pi \leqslant [\lambda]$ there exists a nonzero $x \in \pi X$ such that $Tx = \lambda x$.

4.2. Proposition. *Let T be a continuous Λ-linear operator on X and λ be a nonzero scalar. Then the following statements are equivalent:*

(1) *The scalar $\lambda \in \Lambda$ is a global eigenvalue of T;*

(2) *There is $x \in X$ such that $Tx = \lambda x$ and $|x| \in \mathfrak{P}(\Lambda)$ with $|x| \geqslant [\lambda]$.*

$\lhd$ (2) $\Rightarrow$ (1) : Obvious.

(1) $\Rightarrow$ (2) : Let λ be a global eigenvalue of T. Consider the set

$$C := \big\{(|x|, x) : |x| \in \mathfrak{P}(\Lambda),\ 0 < |x| \leqslant [\lambda]\,,\ Tx = \lambda x\big\}\,.$$

The definition of global eigenvalue and [2, Lemma 4.] yield $[\lambda] = \sup\big\{|x| : (\pi, x) \in C\big\}$. From this and the Exhaustion Principle, there exists an antichain $(\mu_\alpha)_{\alpha\in A}$ in $\mathfrak{P}(\Lambda)$ such that $\sup_{\alpha\in A}\mu_\alpha = [\lambda]$, and for each $\alpha \in A$ there is $(|x_\alpha|, x_\alpha) \in C$ with $\mu_\alpha \leqslant |x_\alpha|$. Hence, we get $x := bo\text{-}\sum_{\alpha\in A}\mu_\alpha x_\alpha$ with $|x| = [\lambda]$ and $Tx = \lambda x$, whence the proof. $\rhd$

Let T be in $B_\Lambda(X,Y)$. For an eigenvalue λ of T define

$$N_\lambda := \bigcup_{n\in\mathbb{N}} \ker(T - \lambda I)^n.$$

The following lemma gives a relation between N_λ and $\ker(T-\lambda I)^n$ $(n \in \mathbb{N})$

4.3. Lemma. *Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. If π is a nonzero projection with $\pi \leqslant [\lambda]$, then there exist a nonzero projection μ with $\mu \leqslant \pi$ and $n \in \mathbb{N}$ such that $\mu N_\lambda = \mu \ker(T - \lambda I)^n$.*

$\lhd$ Assume by way of contradiction that the assertion is false. Then a sequence $(x_n)_{n\in\mathbb{N}}$ can be constructed such that $x_n \in \pi\big((\ker(T-\lambda I)^n)^\perp \cap \ker(T-\lambda I)^{n+1}\big)$ and $\pi = |x_n|$. Therefore, it follows from

$$(T-\lambda I)^n\left((T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m\right) = 0 \quad (m < n)$$

that $(T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m \in \ker(T-\lambda I)^n$, and so

$$\begin{aligned}|Tx_n - Tx_m|^2 &= |\lambda x_n + ((T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m)|^2\\ &\geqslant |\lambda x_n|^2 + |(T-\lambda I)x_n - \lambda x_m - (T-\lambda I)x_m|^2\\ &\geqslant |\lambda|^2|x_n| = \pi|\lambda|^2 \neq 0\end{aligned}$$

which contradicts cyclically compactness of T. This proves the lemma. $\rhd$

Let T be a cyclically compact operator on X. For a global eigenvalue λ of T and for each $N \in \mathbb{N}$ define

$$\rho_N(\lambda) := \sup\left\{\pi \in \mathfrak{P}(\Lambda) : \pi N_\lambda = \pi \ker(T-\lambda I)^N,\ \pi \leqslant [\lambda]\right\}.$$

Using the lemma above, we immediately have the following corollary.

4.4. Corollary. *Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. The following conditions are satisfied:*

(1) $\rho_N(\lambda) \leqslant \rho_{N+1}(\lambda)$;

(2) $\rho_N(\lambda)N_\lambda = \rho_N(\lambda)\ker(T-\lambda I)^N$;

(3) $[\lambda] = \sup\{\rho_N(\lambda) :\ N \in \mathbb{N}\}$.

According to [9, Theorem 7.4.7 (2)], for each $N \in \mathbb{N}$, there exists a partition $(b_\xi)_{\xi\in\Xi}$ of $\rho_N(\lambda)$ such that $b_\xi N_\lambda$ is a strictly $\varkappa(b_\xi)$-homogeneous Kaplansky–Hilbert module over $b_\xi\Lambda$. Since T is cyclically compact, $\varkappa(b_\xi)$ must be a finite number. From [9, Theorem 7.4.7.(1)], we can assume that $\Xi = \mathbb{N}$ and $\varkappa(\tau_{\lambda,N}(n)) = n$ where $\tau_{\lambda,N}(n) := b_n$. So, there is a unique sequence $(\tau_{\lambda,l})_{l\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)^{\mathbb{N}}$ such that $\tau_{\lambda,l} := (\tau_{\lambda,l}(n))_{n\in\mathbb{N}}$ is a partition of $\rho_l(\lambda)$ and $\tau_{\lambda,l}(n)N_\lambda = \tau_{\lambda,l}(n)\ker(T-\lambda I)^l$ is a strictly n-homogeneous Kaplansky–Hilbert module over $\tau_{\lambda,l}(n)\Lambda$. Moreover, $\tau_{\lambda,l}(n) \leqslant \tau_{\lambda,l+1}(n)$ and $\tau_{\lambda,l}(n) \wedge \tau_{\lambda,k}(m) = 0$ are satisfied for all $k, l, m, n \in \mathbb{N}$ with $n \neq m$. So, $(\tau_\lambda(n))_{n\in\mathbb{N}}$ is a partition of $[\lambda]$ where $\tau_\lambda(n) := \sup_{l\in\mathbb{N}}\{\tau_{\lambda,l}(n)\}$.

Now, we define the multiplicity of global eigenvalues of cyclically compact operators on X which is an element of the universally complete vector lattice $(\mathrm{Re}\Lambda)^\infty$, which in turn is the universal completion of $\mathrm{Re}\Lambda$.

4.5. Definition. Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. The *multiplicity* of λ will be denoted by $\overline{\tau}_\lambda$ and is described as follows:

$$\overline{\tau}_\lambda := o\text{-}\sum_{n\in\mathbb{N}} n\tau_\lambda(n) = o\text{-}\sum_{n\in\mathbb{N}} n \sup_{l\in\mathbb{N}}\{\tau_{\lambda,l}(n)\} = \sup_{l,n\in\mathbb{N}}\{n\tau_{\lambda,l}(n)\} \in (\mathrm{Re}\,\Lambda)^\infty.$$

Now, we define the multiplicity of global eigenvalues of cyclically compact operators on X which is an element of the universally complete vector lattice $(\mathrm{Re}\,\Lambda)^\infty$, which in turn is the universal completion of $\mathrm{Re}\,\Lambda$.

4.6. Lemma. *Let $U = S_{\widetilde{u}}$ be in $\mathrm{End}\,(C_\#(Q,H))$ and λ be a global eigenvalue of U. Then there is a meager subset B_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus B_0$.*

$\lhd$ By Proposition 4.2, $U\widetilde{x} = \lambda\widetilde{x}$ is satisfied for some $\widetilde{x} \in C_\#(Q,H)$ with $|\widetilde{x}| = [\lambda]$. Thus, $u(q)x(q) = \lambda(q)x(q)$ holds for all $q \in Q_0 := \mathrm{dom}\,u \cap \mathrm{dom}\,x$. Define $B_0 := Q_0^c \cup (A_\lambda \setminus \{q \in Q : \lambda(q) \neq 0\})$, and note that B_0 is a meager set in Q. The lemma follows. $\rhd$

4.7. Lemma. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q,H)$ and λ be a global eigenvalue of U. Then there is a meager subset A_0 such that for all $q \in A_\lambda \setminus A_0$ the following equality holds:*

$$\ker(U-\lambda I)(q) := (\ker(U-\lambda I))\,(q) = \ker(u(q)-\lambda(q)I).$$

$\lhd$ Clearly, $q \in \mathrm{dom}\,u$ implies $\ker(U-\lambda I)(q) \subset \ker(u(q)-\lambda(q)I)$. As U is a cyclically compact operator, there exists a partition of $[\lambda]$, $(b_k)_{k\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)$ such that $b_n\ker(U-\lambda I)$ is a strictly n-homogeneous Kaplansky–Hilbert module over $b_nC(Q)$. Fix $k \in \mathbb{N}$. Let $\{\widetilde{e}_i : i = 1,\dots,k\}$ be a basis for $b_k\ker(U-\lambda I)$. Then for some meager set A_k the set $\{e_i(q) : i = 1,\dots,k\}$ is a basis of $\ker(U-\lambda I)(q)$ for all $q \in V_k \setminus A_k$, where V_k is the clopen set corresponding to the projection b_k. From the lemma above we obtain a meager subset B_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus B_0$. Define

$$C_k := \big\{q \in V_k \setminus (A_k \cup B_0) : \ker(U - \lambda I)(q) \neq \ker(u(q) - \lambda(q)I)\big\}.$$

Then we can see that C_k is meager, and so $A_0 = \big(A_\lambda \setminus \big(\bigcup_{k\in\mathbb{N}} V_k\big)\big) \cup \big(\bigcup_{k\in\mathbb{N}} A_k \cup C_k\big) \cup B_0$ is meager. Therefore, $\ker(U - \lambda I)(q) = \ker(u(q) - \lambda(q)I)$ holds for all $q \in A_\lambda \setminus A_0$, as desired. $\rhd$

An immediate consequence of the preceding results is the following.

4.8. Corollary. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q, H)$ and λ be a global eigenvalue of U. Then there exists a meager set B_0 such that for all $q \in A_\lambda \setminus B_0$ the following statements hold:*

(1) *$\lambda(q)$ is a nonzero eigenvalue of compact operator $u(q)$;*

(2) *$\big(\ker(U - \lambda I)^k\big)(q) = \ker(u(q) - \lambda(q)I)^k$ $(k \in \mathbb{N})$;*

(3) *$N_\lambda(q) = N_{\lambda(q)}$ where $N_{\lambda(q)}$ is the generalized eigenspace, corresponding to the eigenvalue $\lambda(q)$;*

(4) *$\overline{\tau}_\lambda(q) = m(\lambda(q))$ where $m(\lambda(q))$ is the algebraic multiplicity of $\lambda(q)$.*

Denote by $\mathrm{Sp}^*(u(q))$ the set of all non-zero eigenvalues of $u(q)$, that is $\mathrm{Sp}^*(u(q)) = \mathrm{Sp}(u(q)) \setminus \{0\}$.

4.9. Lemma. *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q, H)$ and let Σ be a finite subset of $C(Q)$ consisting of global eigenvalues of U and the set*

$$A_u \subset \{q \in \mathrm{dom}(u) : \mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\} \neq \varnothing\}$$

be non meager in Q. If λ_q is in $\mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$ for each $q \in A_u$, then there is a global eigenvalue λ of U and a comeager set Q_0 that satisfy the following conditions:

(1) *$[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ where π_N is the projection corresponding to clopen set $U_N := \mathrm{int}\,(\mathrm{cl}\,(A_N))$ with*

$$A_N := \{q \in A_u : (\forall\, \sigma \in \Sigma) |\sigma(q) - \lambda_q| \geqslant 1/N \text{ and } |\lambda_q| \geqslant 1/N\}\,;$$

(2) *$\pi_N|\lambda| \geqslant \frac{1}{N}\pi_N$ and $\pi_N|\sigma - \lambda| \geqslant \frac{1}{2N}\pi_N$ $(N \in \mathbb{N}, \sigma \in \Sigma)$;*

(3) *If q is in $A_N \cap Q_0$, then $|\lambda(q)| \geqslant \frac{1}{N}$ and $|\sigma(q) - \lambda(q)| \geqslant \frac{1}{2N}$ hold for each $\sigma \in \Sigma$;*

(4) *If $\lambda(q) \neq 0$ holds for some $q \in Q_0$, then $\lambda(q) \in \mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$;*

(5) *If $\lambda(q) = 0$ holds for some $q \in Q_0$, then $q \notin A_u$.*

$\lhd$ Without loss of generality we may assume that $u(q)$ is a compact operator on H for each $q \in \mathrm{dom}\,u$. Since $A_u = \bigcup_{N\in\mathbb{N}} A_N$ is not meager, $U_{N_0} \neq \varnothing$ holds for some N_0. Let h_q be an eigenvector of $u(q)$ corresponding to λ_q with $\|h_q\| = 1$ for every $q \in A_u$. For every $N, n \in \mathbb{N}$ and $q \in U_N \cap A_N$ we can find a clopen set $U_{q,n,N} \subset U_N$ such that

$$\|u(w)h_q - \lambda_q h_q\| \leqslant \frac{1}{n} \quad \text{and} \quad |\sigma(w) - \lambda_q| \geqslant \frac{1}{2N} \quad (\sigma \in \Sigma)$$

for all $w \in U_{q,n,N} \cap \mathrm{dom}\,u$. We can establish a global eigenvalue λ_N of U such that $[\lambda_N] = \pi_N$ and

$$|\lambda_N| \geqslant \frac{1}{N}\pi_N \quad \text{and} \quad \pi_N|\sigma - \lambda_N| \geqslant \frac{1}{2N}\,\pi_N \quad (\sigma \in \Sigma).$$

Therefore, if we define $\lambda := \pi_1\lambda_1 + o\text{-}\sum_{N\in\mathbb{N}}(\pi_{N+1} - \pi_N)\lambda_{N+1}$, then $[\lambda] = \bigvee_{N\in\mathbb{N}} \pi_N$ and λ is a global eigenvalue of U. This and Proposition 4.6 complete the proof. $\rhd$

4.10. Theorem. *Let T be a cyclically compact operator on X. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting of global eigenvalues of T or zeros in Λ with the following properties:*

(1) $|\lambda_k| \leqslant |T|$, $[\lambda_k] \geqslant [\lambda_{k+1}]$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim_{k\to\infty} \lambda_k = 0$*;*

(2) *There exists a projection π_∞ in Λ such that $\pi_\infty|\lambda_k|$ is a weak order-unity in $\pi_\infty\Lambda$ for all $k \in \mathbb{N}$;*

(3) *There exists a partition (π_k) of the projection $\pi_\infty^\perp$ such that $\pi_0\lambda_1 = 0$, $\pi_k \leqslant [\lambda_k]$, and $\pi_k\lambda_{k+m} = 0$, $m, k \in \mathbb{N}$;*

(4) *$\pi\lambda_{k+m} \neq \pi\lambda_k$ for every nonzero projection $\pi \leqslant \pi_\infty + \pi_k$ and for all $m, k \in \mathbb{N}$;*

(5) *Every global eigenvalue λ of T is of the form $\lambda = \operatorname{mix}_{k\in\mathbb{N}}(p_k\lambda_k)$, where $(p_k)_{k\in\mathbb{N}}$ is a partition of $[\lambda]$.*

◁ The theorem will be proved in case of $X = C_\#(Q, H)$ and $T = S_{\widetilde{u}}$. General case can be obtained by the functional representations of Kaplansky–Hilbert modules and bounded linear operators on them (see [9, Theorems 7.4.12 and 7.5.12]). Now, by induction and Lemma 4.9, a sequence (λ_n) consisting of global eigenvalues of $S_{\widetilde{u}}$ or zeros, and a decreasing sequence of comeager sets (Q_n), can be established as follows:

(i) if $\lambda_n(q) \neq 0$ holds for some $q \in Q_n$, then $\lambda_n(q) \in \operatorname{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, \dots, n-1\}$;

(ii) if $\lambda_n(q) = 0$ holds for some $q \in Q_n$, then $\operatorname{Sp}^*(u(q)) \setminus \{\lambda_i(q) : i = 1, \dots, n-1\} = \varnothing$;

(iii) $\operatorname{Sp}^*(u(q)) = \{\lambda_n(q) : \lambda_n(q) \neq 0 \ (n \in \mathbb{N})\}$ is satisfied for all $q \in Q_0 := \bigcap Q_n$.

Define $\pi_\infty := \bigwedge_{k\in\mathbb{N}} [\lambda_k]$ and $\pi_0 := [\lambda_1]^\perp$ and $\pi_k := [\lambda_k] \wedge [\lambda_{k+1}]^\perp$ $(k \in \mathbb{N})$. Then this implies (2), (3) and (4). Moreover, since $|\lambda_n(q)| \leqslant \|u(q)\|$ and $\lim_{k\to\infty} \lambda_k(q) = 0$ hold for all $q \in Q_0$, we have $|\lambda_n| \leqslant |U|$ and $o\text{-}\lim \lambda_k = 0$, and so this implies (1). Let λ be an global eigenvalue of U. Then we can assume that the meager set A_0 satisfies the condition of the Lemma . From (iii) we have $(A_\lambda \cap Q_0) \setminus A_0 = \bigcup_{k\in\mathbb{N}} A_k$ where

$$A_k := \{q \in A_\lambda \setminus A_0 : \lambda(q) = \lambda_k(q)\} \quad (k \in \mathbb{N}).$$

Since $A_k \setminus \operatorname{int}(\operatorname{cl}(A_k))$ is nowhere dense, $[\lambda] = \bigvee_{k\in\mathbb{N}} \mu_k$ and $\mu_k\lambda = \mu_k\lambda_k$ where μ_k denotes the projection corresponding clopen set $\operatorname{int}(\operatorname{cl}(A_k))$. Thus, there exists a partition $(p_k)_{k\in\mathbb{N}}$ of $[\lambda]$ such that $\lambda = \operatorname{mix} p_k\lambda_k k \in \mathbb{N}$ holds, and the proof is finished. ▷

Let $(\lambda_k)_{k\in\mathbb{N}}$ be as Theorem 4.10. If $\lambda_k = 0$, take $\overline{\tau}_{\lambda_k} = 0$.

4.11. Definition. The sequence $(\lambda_k(T))_{k\in\mathbb{N}}$, where $\lambda_k(T) := \lambda_k$ is given by the above theorem, is called a *global eigenvalue sequence* of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$ where $\overline{\tau}_k(T) := \overline{\tau}_{\lambda_k}$.

4.12. Theorem (Lidskiĭ trace formula)**.** *Let T be in $\mathscr{S}_1(X)$ and $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. Then the following equality holds*

$$\operatorname{tr}(T) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(T)\lambda_k(T).$$

◁ As in Theorem 4.10, the theorem will be proved in case of $X = C_\#(Q, H)$ and $T = S_{\widetilde{u}}$. Let $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. From Corollary 4.8 and Theorem 4.10, there exists a comeager set Q_0 such that for each $q \in Q_0$ the following statements hold:

(i) $\operatorname{tr}(T)(q) = \operatorname{tr}(u(q))$ and $v_1(T)(q) = v_1(u(q))$;

(ii) $\operatorname{Sp}^*(u(q)) = \{\lambda_n(T)(q) : \lambda_n(T)(q) \neq 0\}$;

(iii) $\lambda_n(T)(q) \neq \lambda_m(T)(q)$ if $\lambda_n(T)(q) \neq 0$ or $\lambda_m(T)(q) \neq 0$ for $n \neq m$;

(iv) if $\lambda_k(T)(q) \neq 0$, then $\overline{\tau}_k(T)(q) = m(\lambda_k(T)(q)) \in \mathbb{N}$ where $m(\lambda_k(T)(q))$ is the algebraic multiplicity of $\lambda_k(T)(q)$.

From (i), (ii), (iii), (iv) and Lidskiĭ trace formula for the compact operator $u(q)$, we see that

$$\operatorname{tr}(T)(q) = \operatorname{tr}(u(q)) = \sum_{k\in\mathbb{N}} \overline{\tau}_k(T)(q)\lambda_k(T)(q)$$

is absolutely convergent on the comeager set Q_0, and so we have

$$\operatorname{tr}(T) = o\text{-}\sum_{k\in\mathbb{N}} \overline{\tau}_k(T)\lambda_k(T). \ \triangleright$$

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Received September 15, 2013.

UĞUR GÖNÜLLÜ
Department of Mathematics and Computer Science
İstanbul Kültür University
Bakırköy, 34156, İstanbul, TURKEY
E-mail: `u.gonullu@iku.edu.tr`

КЛАСС ОПЕРАТОРОВ СО СЛЕДОМ И ФОРМУЛА ЛИДСКОГО В МОДУЛЯХ КАПЛАНСКОГО — ГИЛЬБЕРТА

Гёнюллю У.

Вводятся и изучаются класс операторов со следом и глобальные собственные значения непрерывных гомоморфизмов в модулях Капланского — Гильберта. В частности, устанавливается вариант формулы Лидского о следе для циклически компактных операторов в модулях Капланского — Гильберта.

Ключевые слова: модуль Капланского — Гильберта, циклически компактный оператор, глобальное собственное значение, класс операторов со следом, формула Лидского о следе.