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ERROR INEQUALITIES FOR SOME NEW QUADRATURE FORMULAS WITH WEIGHT INVOLVING n KNOTS AND THE L_p-NORM OF THE m-th DERIVATIVE ON TIME SCALES[1]

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In this paper we generalize the Ostrowski inequality on time scales for n points and the L_p norm of m-th derivative, where $m, n \in \mathbb{N}$ and $p \in [1, +\infty]$.

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1. Introduction

In 1938, Ostrowski proved the following interesting integral inequality which has received considerable attention from many researchers [1, 5, 6, 13–16].

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) and its derivative $f' : (a, b) \to \mathbb{R}$ is bounded in (a, b), that is $\|f'\|_\infty := \sup_{x\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a, b]$, we have the inequality

$$\left|\int\limits_a^b f(t)\,dt - f(x)(b-a)\right| \leqslant \left(\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2\right)\|f'\|_\infty. \tag{1.1}$$

In [9], the following results was obtained: If $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is an absolutely continuous function and $\gamma_n \leqslant f^{(n)}(x) \leqslant \Gamma_n$ for all $x \in [a, b]$ for some constants γ_n and Γ_n, then

$$\left|\frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{b+a}{2}\Big) + f(b)\Big] - \int\limits_a^b f(t)\,dt\right| \leqslant C_n(\Gamma_n - \gamma_n)(b-a)^{n+1}, \tag{1.2}$$

where the constants $C_1 = \frac{5}{72}$, $C_2 = \frac{1}{162}$ and $C_3 = \frac{1}{1152}$ are sharp in the sense that they cannot be replaced by smaller ones.

Very recently, V. N. Huy et al. [5, 6] have strengthened (1.1) and (1.2) by enlarging the number of knots. More precisely, they proved that

$$\left|\int\limits_a^b f(x)\,dx - \frac{b-a}{n}\sum_{i=1}^n f(a + x_i(b-a))\right| \leqslant A_{a,b,m}(S - s), \tag{1.3}$$

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$$\left|\int_a^b f(x)\,dx - \frac{b-a}{n}\sum_{i=1}^{n} f(a + x_i(b-a))\right| \leqslant B_{a,b,m}\|f^{m+1}\|_p, \tag{1.4}$$

where $s = \inf_{x\in[a,b]} f^{(m)}(x)$ and $S = \sup_{x\in[a,b]} f^{(m)}(x)$, and

$$\sum_{k=1}^{n} x_k^i = \frac{1}{i+1} \quad (\forall\, i = 1, 2, \ldots, m). \tag{1.5}$$

Note that, (1.5) have the solutions only for $n \in [1, 9]$, $n \in \mathbb{N}$ (see [17–19]).

On time scales,the Ostrowski type inequalities have been generalized in various ways. For example, [2, 3, 10]. It proved in [2] the following result on time scales: Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable, $M = \sup_{a<x<b} |f^{\Delta}(x)|$. Then

$$\left|\int_a^b f^{\sigma}(t)\Delta(t) - f(x)(b-a)\right| \leqslant M(h_2(x, a) + h_2(x, b)),$$

where $h_k(\cdot,\cdot)$ is defined in section 2.

In this paper, making use of the above theorem and some simple estimations, we obtain propose a new way of treating a class of quadrature formulas with weight involving n points and the L_p norm of m-th derivative on time scales where $m, n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$.

2. Preliminaries on time scales

A *time scale* is a nonempty closed subset of $\mathbb{R}$ and is denoted by $\mathbb{T}$. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad (\forall\, t \in \mathbb{T}),$$

with $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. We now introduce the set $\mathbb{T}^k$ which is derived from the time scales $\mathbb{T}$, as follows. If $\mathbb{T}$ has a left-scattered maximum m then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. The delta graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t \quad (\forall\, t \in \mathbb{T}).$$

If $f : \mathbb{T} \to \mathbb{R}$ is a function then we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)) \quad (\forall\, t \in \mathbb{T}).$$

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^k$ if there exists a number $f^{\Delta}(t)$ such that for all $\epsilon > 0$ there is a neighborhood U of t (i. e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s| \quad (\forall\, s \in U).$$

We call $f^{\Delta}(t)$ the *delta derivative* of f at t.

For delta differentiable function f and g, the next formula holds:

$$(fg)^{\Delta}(t) = f^{\Delta}g^{\sigma}(t) + f(t)g^{\Delta}(t) = f^{\Delta}g(t) + f^{\sigma}(t)g^{\Delta}(t).$$

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-*continuous* if it is continuous at right-dense points, and its left-side limits exist at left-dense points.

A function $F:\mathbb{T}\to\mathbb{R}$ is called a Δ-*antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Then the Δ-integral of f is defined by $\int_a^b f(t)\Delta t=F(b)-F(a)$.

It is known that every rd-continuous function f has an antiderivative.

The functions $h_k:\mathbb{T}^2\to\mathbb{R}$ are defined recursively as follows:

$$h_0(t,s)=1,\quad h_{k+1}(t,s)=\int_s^t h_k(\gamma,s)\Delta\gamma\quad(\forall\, s,t\in\mathbb{T}).$$

Proposition 2.1. *If $a,b\in\mathbb{T}$, then the assertions hold:*
1. *If $a\leqslant x\leqslant b$ then $0\leqslant h_k(x,a)\leqslant h_k(b,a)$;*
2. *For $a\leqslant b$ we have $0\leqslant h_{k+1}(b,a)\leqslant(b-a)h_k(b,a)$.*

Now, we introduce a useful result, which is well-known in the literature as Taylor's formula with the integral remainder.

Lemma 2.2 [1]. *Assume $f\in C_{rd}^r(\mathbb{T})$ and $x_0\in\mathbb{T}$. Then for all $x\in(a,b)$ we have*

$$f(x)=T_{r-1}(f,x_0,x)+R_{r-1}(f,x_0,x)$$

where $T_{r-1}(f,x_0,\cdot)$ is Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f,x_0,x)=\sum_{k=0}^{r-1}h_k(x,x_0)f^{\Delta^k}(x_0)$$

and the remainder can be given by

$$R_{r-1}(f,x_0,x)=\int_{x_0}^{x}h_{r-1}(x,\sigma(t))f^{\Delta^r}(t)\,\Delta t.$$

We have the Montgomery identity which is stated in the following lemma.

Lemma 2.3 [8]. *Let $a,b,s,t\in\mathbb{T}$, $a<b$ and $f:\mathbb{T}\to\mathbb{R}$ be differentiable. Then*

$$f(t)=\frac{1}{b-a}\int_a^b f^{\sigma}(s)\,\Delta s+\frac{1}{b-a}\int_a^b p(t,s)f^{\Delta}(s)\,\Delta s,$$

where

$$p(t,s)=\begin{cases}s-a, & \text{for } a\leqslant s<t,\\ s-b, & \text{for } t\leqslant s\leqslant b.\end{cases}$$

3. Main results

Let $1\leqslant m,n$ and $1\leqslant p\leqslant\infty$, $0\leqslant\alpha_i\leqslant 1$ satisfies $\sum_{i=1}^n\alpha_i=1$. For each $i=1,\dots,n$, let $a\leqslant x_i\leqslant b$ and we consider the following condition

$$H_i(x_1,x_2,\dots,x_n)=h_{i+1}(b,a)\quad(\forall\, i=1,2,\dots,m-1),\tag{3.1}$$

where $H_i(x_1, x_2, \ldots, x_n) = (b-a)\sum_{k=1}^{n} \alpha_k h_i(x_k, a)$, and

$$\int_a^b h_m(b, \sigma(t))\,\Delta t - (b-a)\sum_{i=1}^{n}\int_a^{x_i} \alpha_i h_{m-1}(x_i, \sigma(t))\,\Delta t = 0. \tag{3.2}$$

We point out the fact that in the continuous case $\mathbb{T} = \mathbb{R}$ and $\alpha_1 = \ldots = \alpha_n = \frac{1}{n}$, conditions (3.1) and (3.2) become

$$\sum_{k=1}^{n} y_k^i = \frac{1}{i+1} \quad (\forall\, i = 1, 2, \ldots, m),$$

where $x_i = a + y_i(b-a)$. Before stating our main result, let us introduce the following notations.

$$I(f) = \int_a^b f(x)\,\Delta x, \quad Q\,(f, n, m, x_1, \ldots, x_n) = (b-a)\sum_{i=1}^{n} \alpha_i f\,(x_i). \tag{3.3}$$

Note that, for the case $\alpha_1 = \ldots = \alpha_n = \frac{1}{n}$ then

$$Q\,(f, n, m, x_1, \ldots, x_n) = \frac{1}{n}(b-a)\sum_{i=1}^{n} f\,(a + y_i(b-a))$$

are also known in [5, 6]. Now, we slightly improve [5, 6] with weights α_k on time scales:

Theorem 3.1. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}^m(\mathbb{T})$. Then, under conditions* (3.1) *and* (3.2), *we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \leqslant 2(b-a)^2(T-s)h_{m-1}(b, a),$$

where $s = \inf_{x\in[a,b]} f^{\Delta^m}(x)$, $T = (f^{\Delta^{m-1}}(b) - f^{\Delta^{m-1}}(a))/(b-a)$.

$\lhd$ Let us first define

$$F(x) = \int_a^x f(x)\,\Delta x.$$

Then $I(f) = F(b) - F(a)$. Applying Lemma 2.2 to the function $F(x)$ with $x = b$ and $x_0 = a$, we get

$$F(b) = F(a) + \sum_{k=1}^{m} h_k(b, a)F^{\Delta^k}(a) + \int_a^b h_m(b, \sigma(t))F^{\Delta^{m+1}}(t)\,\Delta t$$

which yields that

$$I(f) = \sum_{k=0}^{m-1} h_{k+1}(b, a)f^{\Delta^k}(a) + \int_a^b h_m(b, \sigma(t))f^{\Delta^m}(t)\,\Delta t. \tag{3.4}$$

For each $1 \leqslant i \leqslant n$, applying Lemma 2.2 again to the function $f(x)$ with $x = x_i$ and $x_0 = a$, we get

$$f(x_i) = \sum_{k=0}^{m-1} h_k(x_i, a)f^{\Delta^k}(a) + \int_a^{x_i} h_{m-1}(x_i, \sigma(t))f^{\Delta^m}(t)\,\Delta t.$$

By applying to $i=1,\dots,n$ and then summing up, we deduce that

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i f(x_i) &= \sum_{i=1}^{n}\sum_{k=0}^{m-1}\alpha_i h_k(x_i,a)f^{\Delta^k}(a)+\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\\ &= \sum_{k=0}^{m-1}\sum_{i=1}^{n}\alpha_i h_k(x_i,a)f^{\Delta^k}(a)+\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\\ &= \sum_{k=0}^{m-1}\frac{1}{b-a}H_k(x_1,x_2,\dots,x_n)f^{\Delta^k}(a)+\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t.\end{aligned}$$

Thus,

$$\begin{aligned}&Q(f,n,m,x_1,\dots,x_n)\\ &= \sum_{k=0}^{m-1}H_k(x_1,x_2,\dots,x_n)f^{\Delta^k}(a)+(b-a)\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t.\end{aligned}$$

Then it follows from condition (3.1) that

$$\begin{aligned}&Q(f,n,m,x_1,\dots,x_n)\\ &= \sum_{k=0}^{m-1}k_{k+1}(b,a)f^{\Delta^k}(a)+(b-a)\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t.\end{aligned}\tag{3.5}$$

By (3.4), (3.5), we obtain that

$$\begin{aligned}&\Big|I(f)-Q(f,n,m,x_1,\dots,x_n)\Big|\\ &= \Big|\int_a^b h_m(b,\sigma(t))f^{\Delta^m}(t)\,\Delta t-(b-a)\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\Big|.\end{aligned}$$

Then, by using condition (3.2), we have

$$\begin{aligned}&\Big|I(f)-Q(f,n,m,x_1,\dots,x_n)\Big|\\ &= \Big|\int_a^b h_m(b,\sigma(t))[f^{\Delta^m}(t)-s]\,\Delta t-(b-a)\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))[f^{\Delta^m}(t)-s]\,\Delta t\Big|.\end{aligned}\tag{3.6}$$

We estimate the first term of (3.6) as follows

$$\begin{aligned}\Big|\int_a^b h_m(b,\sigma(t))[f^{\Delta^m}(t)-s]\,\Delta t\Big| &\leqslant h_m(b,a)\int_a^b[f^{\Delta^m}(t)-s]\,\Delta t\\ &= h_m(b,a)\big(f^{\Delta^{m-1}}(b)-f^{\Delta^{m-1}}(a)-s(b-a)\big)\\ &= (b-a)h_m(b,a)(T-s)\leqslant(b-a)^2h_{m-1}(b,a)(T-s).\end{aligned}\tag{3.7}$$

For the second one, we first have

$$\left|\int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))\Big[f^{\Delta^m}(t)-s\Big]\,\Delta t\right| \leqslant h_{m-1}(x_i,a)\int_a^b \Big[f^{\Delta^m}(t)-s\Big]\,\Delta t$$
$$= h_{m-1}(x_i,a)\big(f^{\Delta^{m-1}}(b)-f^{\Delta^{m-1}}(a)-s(b-a)\big) \leqslant (b-a)h_{m-1}(x_i,a)(T-s).$$

Hence, summing up the above inequalities with $i=1,2,\dots,n$, using the Proposition 2.6, it implies that

$$\begin{aligned}(b-a)\sum_{i=1}^{n}&\left|\int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))\Big[f^{\Delta^m}(t)-s\Big]\,\Delta t\right| \\ \leqslant (b-a)^2(T-s)&\sum_{i=1}^{n}\alpha_i h_{m-1}(x_i,a) = (b-a)(T-s)\,h_m(b,a) \\ &\leqslant (b-a)^2(T-s)\,h_{m-1}(b,a).\end{aligned} \tag{3.8}$$

Combining relations (3.6), (3.7) and (3.8), we conclude that

$$\Big|I(f)-Q(f,n,m,x_1,\dots,x_n)\Big| \leqslant 2(b-a)^2(T-s)\,h_{m-1}(b,a)$$

and the proof of Theorem 3.1 is now completed. ▷

With the similar arguments as those used in the proof of Theorem 3.1, we also obtain the following theorem.

Theorem 3.2. *Let $a,b\in\mathbb{T}$ and $f\in C^m_{rd}(\mathbb{T})$. Then, under conditions (3.1) and (3.2), we have*

$$|I(f)-Q(f,n,m,x_1,\dots,x_n)| \leqslant 2(b-a)^2(S-T)h_{m-1}(b,a),$$

where $S=\sup_{x\in[a,b]} f^{\Delta^m}(x)$, $T=(f^{\Delta^{m-1}}(b)-f^{\Delta^{m-1}}(a))/(b-a)$.

Since $s=\inf_{x\in[a,b]} f^{\Delta^m}(x)\leqslant T=\frac{1}{b-a}\int_a^b f^{\Delta^m}(x)\Delta x\leqslant \sup_{x\in[a,b]} f^{\Delta^m}(x)=S$, we have the following corollary.

Corollary 3.3. *Let $a,b\in\mathbb{T}$ and $f\in C^m_{rd}(\mathbb{T})$. Then, under conditions (3.1) and (3.2), we have*

$$|I(f)-Q(f,n,m,x_1,\dots,x_n)| \leqslant 2(b-a)^2(S-s)h_{m-1}(b,a),$$

where $S=\sup_{x\in[a,b]} f^{\Delta^m}(x)$, $s=\inf_{x\in[a,b]} f^{\Delta^m}(x)$.

Now, we will give a new quadrature formulas with weight involving n points and L_p norm m-th derivative on time scales.

Theorem 3.4. *Let $1\leqslant p\leqslant\infty$, $a,b\in\mathbb{T}$ and let $f\in C^m_{rd}(\mathbb{T})$. Then, under conditions (3.1), we have*

$$|I(f)-Q(f,n,m,x_1,\dots,x_n)| \leqslant 2h_{m-1}(b,a)\,(b-a)^{(q+1)/q}\,\|f^{\Delta^m}\|_p,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

◁ We have known that

$$\begin{aligned}&\Big|I(f)-Q(f,n,m,x_1,\dots,x_n)\Big| \\ &= \left|\int_a^b h_m(b,\sigma(t))f^{\Delta^m}(t)\,\Delta t-(b-a)\sum_{i=1}^{n}\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\right|.\end{aligned} \tag{3.9}$$

The first term of (3.9) can be estimated by using the Hölder inequality as follows

$$\begin{aligned}\Big|\int_a^b h_m(b,\sigma(t))f^{\Delta^m}(x)\,\Delta x\Big| &\leqslant \Big(\int_a^b \Big[h_m(b,\sigma(t))\Big]^q\,\Delta t\Big)^{1/q}\Big(\int_a^b \Big[f^{\Delta^m}(t)\Big]^p\,\Delta t\Big)^{1/p}\\ &\leqslant h_m(b,a)\Big(\int_a^b \Delta t\Big)^{1/q}\Big(\int_a^b \Big[f^{\Delta^m}(t)\Big]^p\,\Delta t\Big)^{1/p} \leqslant h_m(b,a)\,(b-a)^{1/q}\,\|f^{\Delta^m}\|_p\\ &\leqslant h_{m-1}(b,a)\,(b-a)^{(q+1)/q}\,\|f^{\Delta^m}\|_p.\end{aligned}\tag{3.10}$$

Similarly, we deduce since $x_i \in (0,1)$ and the Hölder inequality that

$$\begin{aligned}\Big|\int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\Big| &\leqslant \alpha_i\Big(\int_a^{x_i} [h_{m-1}(x_i,\sigma(t))]^q\,\Delta t\Big)^{1/q}\Big(\int_a^{x_i} |f^{\Delta^m}(t)|^p\,\Delta t\Big)^{1/p}\\ &\leqslant h_{m-1}(x_i,a)\Big(\int_a^b \Delta t\Big)^{1/q}\Big(\int_a^b |f^{\Delta^m}(t)|^p\,\Delta t\Big)^{1/p} = h_{m-1}(x_i,a)\,(b-a)^{1/q}\,\|f^{\Delta^m}\|_p.\end{aligned}$$

Now, applying the above inequalities with $i = 1, 2, \ldots, n$, we get

$$\begin{aligned}(b-a)\sum_{i=1}^n\Big|\int_a^{x_i}\alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\Big|\\ = (b-a)^{(q+1)/q}\,\|f^{\Delta^m}\|_p.\sum_{i=1}^n \alpha_i h_{m-1}(x_i,a)\\ = h_m(b,a)\,(b-a)^{1/q}\,\|f^{\Delta^m}\|_p \leqslant h_{m-1}(b,a)\,(b-a)^{(q+1)/q}\,\|f^{\Delta^m}\|_p.\end{aligned}\tag{3.11}$$

Relations (3.9), (3.10) and (3.11) imply that

$$\Big|I(f) - Q(f,n,m,x_1,\ldots,x_n)\Big| \leqslant 2h_{m-1}(b,a)\,(b-a)^{(q+1)/q}\,\|f^{\Delta^m}\|_p$$

and thus Theorem 3.4 is completely proved. ▷

Next, we define the Chebyshev functional on a time scale by

$$T_\Delta(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,\Delta x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\Delta x\int_a^b g(x)\,\Delta x.$$

Then

$$T_\Delta(f,f) = \frac{1}{b-a}\int_a^b f^2(x)\Delta x - \frac{1}{(b-a)^2}\Big(\int_a^b f(x)\,\Delta x\Big)^2.$$

We also define $\sigma_\Delta(f) = (b-a)\,T_\Delta(f,f)$. Then, it should be noticed that in [15], N. Ujević obtained the following result for the case $\mathbb{T} = \mathbb{R}$: Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, whose derivative $f' \in L_2(a,b)$. Then it holds that

$$\Big|\frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{b+a}{2}\Big) + f(b)\Big] - \int_a^b f(t)\,\Delta t\Big| \leqslant \frac{(b-a)^{3/2}}{6}\sqrt{\sigma(f')}.$$

In this article, base on the result of N. Ujević we will give a new quadrature formulas with weight involving n points and m-th derivative on time scales by using Chebyshev functional.

Theorem 3.5. *Let $a, b \in \mathbb{T}$ and let $f \in C_{rd}^m(\mathbb{T})$ be such that $f^{\Delta^m} \in L^2(a,b)$. Then, under conditions* (3.1) *and* (3.2), *we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \leqslant 2h_{m-1}(b,a)\sqrt{(b-a)^3\sigma_\Delta(f^{\Delta^m})}.$$

$\lhd$ We have known that

$$\Big|I(f) - Q(f, n, m, x_1, \ldots, x_n)\Big| = \left|\int_a^b h_m(b,\sigma(t))f^{\Delta^m}(t)\,\Delta t - (b-a)\sum_{i=1}^n \int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))f^{\Delta^m}(t)\,\Delta t\right|.$$

Then, by using condition (3.2), we have

$$\Big|I(f) - Q(f, n, m, x_1, \ldots, x_n)\Big| = \left|\int_a^b h_m(b,\sigma(t))\Big[f^{\Delta^m}(t) - T\Big]\,\Delta t - (b-a)\sum_{i=1}^n \int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))[f^{\Delta^m}(t) - T]\,\Delta t\right|, \tag{3.12}$$

where $T = (f^{\Delta^{m-1}}(b) - f^{\Delta^{m-1}}(a))/(b-a)$. The first term of (3.12) can be estimated by using the Hölder inequality as follows

$$\left|\int_a^b h_m(b,\sigma(t))\Big[f^{\Delta^m}(t) - T\Big]\,\Delta t\right| \leqslant \left(\int_a^b \Big[h_m(b,\sigma(t))\Big]^2\,\Delta t\right)^{1/2}\left(\int_a^b \Big[f^{\Delta^m}(t) - T\Big]^2\,\Delta t\right)^{1/2}$$
$$\leqslant h_m(b,a)\sqrt{b-a}\left(\int_a^b \Big[f^{\Delta^m}(t) - T\Big]^2\,\Delta t\right)^{1/2}.$$

Combining this with the fact that

$$\int_a^b \Big[f^{\Delta^m}(t) - T\Big]^2\Delta t = \int_a^b \Big[f^{\Delta^m}(t)\Big]^2\Delta t - 2T\int_a^b f^{\Delta^m}(t)\,\Delta t + (b-a)T^2$$
$$= \int_a^b \Big[f^{\Delta^m}(t)\Big]^2\,\Delta t - \frac{1}{b-a}\left[\int_a^b f^{\Delta^m}(t)\,\Delta t\right]^2 = \sigma_\Delta(f^{\Delta^m})$$

we obtain that

$$\left|\int_a^b h_m(b,\sigma(t))[f^{\Delta^m}(t) - T]\Delta t\right| \leqslant h_m(b,a)\sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})}$$
$$\leqslant (b-a)h_{m-1}(b,a)\sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})}. \tag{3.13}$$

Similarly, we deduce since $x_i \in (a,b)$ and the Hölder inequality that

$$\Big|\int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))\Big[f^{\Delta^m}(t) - T\Big]\,\Delta t\Big|$$

$$\leqslant \alpha_i\Big(\int_a^{x_i} [h_{m-1}(x_i,\sigma(t))]^2\,\Delta t\Big)^{1/2}\Big(\int_a^{x_i}[f^{\Delta^m}(t) - T]^2\,\Delta t\Big)^{1/2}$$

$$\leqslant h_{m-1}(x_i,a)\Big(\int_a^b \Delta t\Big)^{1/2}\Big(\int_a^b \Big[f^{\Delta^m}(t) - T\Big]^2\,\Delta t\Big)^{1/2} = h_{m-1}(x_i,a)\sqrt{b-a}\sqrt{\sigma_\Delta(f^{\Delta^m})}.$$

Now, applying the above inequalities with $i = 1,2,\dots,n$, we get

$$\begin{aligned}(b-a)\sum_{i=1}^n \Big|\int_a^{x_i} \alpha_i h_{m-1}(x_i,\sigma(t))\Big[f^{\Delta^m}(t) - T\Big]\,\Delta t\Big| \\ = \sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})}\sum_{i=1}^n \alpha_i h_{m-1}(x_i,a) = h_m(b,a)\sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})} \\ \leqslant (b-a)h_{m-1}(b,a)\sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})}.\end{aligned} \tag{3.14}$$

Relations (3.12), (3.13) and (3.14) imply that

$$\Big|I(f) - Q(f,n,m,x_1,\dots,x_n)\Big| \leqslant 2(b-a)h_{m-1}(b,a)\sqrt{(b-a)\sigma_\Delta(f^{\Delta^m})}$$

and thus Theorem 3.5 is completely proved. ▷

Base on the inequality in (1.1), by using some simple estimations, we obtain some new quadrature formulas involving n knots on time scales: For $0 \leqslant x_i \leqslant 1$, $a + x_i(b-a) \in \mathbb{T}$, we put

$$Q(f,x_1,x_2,\dots,x_n) = \frac{1}{n}\sum_{i=1}^n f(a + x_i(b-a)).$$

The next result of this paper can be described as follows.

Theorem 3.6. *Let $a,b \in \mathbb{T}$, $a < b$, $f : \mathbb{T} \to \mathbb{R}$ be differentiable, and assume that f^Δ is rd-continuous such that $f^\Delta \in L^2(\mathbb{T})$. Then for $0 \leqslant x_i \leqslant 1$ with $\sum_{i=1}^n x_i = \frac{n}{2}$ we have the following estimate*

$$\Big|Q(f,x_1,x_2,\dots,x_n) - \frac{1}{b-a}\int_a^b f^\sigma(x)\,\Delta x + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b \frac{1}{2}\mu(s)\,\Delta s\Big| \leqslant \sqrt{(b-a)\sigma_\Delta(f^\Delta)}.$$

◁ Put $t_k = a + x_k(b-a)$, then it follows from Lemma 2.3 that

$$f(a + x_k(b-a)) - \frac{1}{b-a}\int_a^b f^\sigma(x)\,\Delta x = \frac{1}{b-a}\int_a^b p(t_k,s)f^\Delta(s)\,\Delta s$$

$$= \frac{1}{b-a}\int_a^b p(t_k,s)\Big[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\Big]\,\Delta s + \frac{1}{b-a}\int_a^b p(t_k,s)\frac{f(b)-f(a)}{b-a}\,\Delta s.$$

Since

$$\int_a^b p(t,s)\,\Delta s = \int_a^t (s-a)\Delta s + \int_t^b (s-b)\Delta s$$

$$= \int_a^b \left(s + \frac{1}{2}\,\mu(s)\right)\Delta s - \int_a^b \frac{1}{2}\,\mu(s)\Delta s - a\int_a^t \Delta s - b\int_t^b \Delta s$$

$$= \frac{b^2-a^2}{2} - \int_a^b \frac{1}{2}\,\mu(s)\Delta s - a(t-a) - b(b-t) = \left(t - \frac{a+b}{2}\right)(b-a) - \int_a^b \frac{1}{2}\,\mu(s)\Delta s,$$

we deduce that

$$f(a+x_k(b-a)) - \frac{1}{b-a}\int_a^b f^\sigma(x)\,\Delta x = \frac{1}{b-a}\int_a^b p(t_k,s)\left[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\right]\Delta s$$

$$+\frac{f(b)-f(a)}{(b-a)^2}\left[(b-a)^2\left(x_k - \frac{1}{2}\right) - \int_a^b \frac{1}{2}\,\mu(s)\,\Delta s\right].$$

Hence,

$$f(a+x_k(b-a)) - \frac{1}{b-a}\int_a^b f^\sigma(x)\Delta x + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b \frac{1}{2}\,\mu(s)\,\Delta s$$

$$= \frac{1}{b-a}\int_a^b p(t_k,s)\left[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\right]\Delta s + [f(b)-f(a)]\left(x_k - \frac{1}{2}\right).$$

By applying to $k = 1, \ldots, n$ and then summing up, since $\sum_{k=1}^n x_k = \frac{n}{2}$, we obtain that

$$Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a}\int_a^b f^\sigma(x)\,\Delta x + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b \frac{1}{2}\,\mu(s)\,\Delta s$$

$$= \frac{1}{n(b-a)}\sum_{k=1}^n \int_a^b p(t_k,s)\left[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\right]\Delta s.$$

We first observe that

$$\int_a^b \left[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\right]^2 \Delta s = \int_a^b \left[f^\Delta(s)\right]^2 \Delta s - \frac{1}{b-a}\left[\int_a^b f^\Delta(s)\,\Delta s\right]^2 = \sigma_\Delta(f^\Delta),$$

which yields

$$\left|Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a}\int_a^b f^\sigma(x)\,\Delta x + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b \frac{1}{2}\mu(s)\,\Delta s\right|$$

$$\leqslant \frac{1}{n(b-a)}\sum_{k=1}^n \left[\left(\int_a^b [p(t_k,s)]^2\Delta s\right)^{\frac{1}{2}}\left(\int_a^b \left[f^\Delta(s) - \frac{f(b)-f(a)}{b-a}\right]^2\Delta s\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{n(b-a)} \sqrt{\sigma_\Delta(f^\Delta)} \sum_{k=1}^{n} \left(\int_a^b [p(t_k,s)]^2 \, \Delta s \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{n(b-a)} \sqrt{\sigma_\Delta(f^\Delta)} \sum_{k=1}^{n} \left(\int_a^b (b-a)^2 \Delta s \right)^{\frac{1}{2}} = \sqrt{(b-a)\sigma_\Delta(f^\Delta)}.$$

The proof of Theorem 3.6 is now completed. $\triangleright$

Theorem 3.7. *Let $a,b \in \mathbb{T}$ and $f \in C^1_{rd}(\mathbb{T})$. We also let $\gamma = \inf_{x\in\mathbb{T}} f^\Delta(x)$ and $T = \frac{f(b)-f(a)}{b-a}$. Then for $0 \leqslant x_i \leqslant 1$ with $\sum_{i=1}^n x_i = \frac{n}{2}$ we have the following estimate*

$$\left| Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a} \int_a^b f^\sigma(x)\Delta x + \frac{\gamma}{b-a} \int_a^b \frac{1}{2}\,\mu(s)\Delta s \right| \leqslant (b-a)(T-\gamma).$$

$\triangleleft$ Put $t_k = a + x_k(b-a)$, then it follows from Lemma 2.3 that

$$f(a + x_k(b-a)) - \frac{1}{b-a} \int_a^b f^\sigma(x)\,\Delta x = \frac{1}{b-a} \int_a^b p(t_k,s) f^\Delta(s)\Delta s$$

$$= \frac{1}{b-a} \int_a^b p(t_k,s)\Big[f^\Delta(s) - \gamma\Big]\,\Delta s + \frac{1}{b-a} \int_a^b p(t_k,s)\gamma\,\Delta s.$$

Combining this with the fact that

$$\int_a^b p(t,s)\,\Delta s = \left(t - \frac{a+b}{2}\right)(b-a) - \int_a^b \frac{1}{2}\,\mu(s)\,\Delta s$$

we get

$$f(a + x_k(b-a)) - \frac{1}{b-a} \int_a^b f^\sigma(x)\,\Delta x$$

$$= \frac{1}{b-a} \int_a^b p(t_k,s)[f^\Delta(s) - \gamma]\,\Delta s + \frac{\gamma}{b-a}\left[(b-a)^2\left(x_k - \frac{1}{2}\right) + \int_a^b \frac{1}{2}\,\mu(s)\,\Delta s\right].$$

Hence,

$$f(a + x_k(b-a)) - \frac{1}{b-a} \int_a^b f^\sigma(x)\,\Delta x + \frac{\gamma}{b-a} \int_a^b \frac{1}{2}\,\mu(s)\,\Delta s$$

$$= \frac{1}{b-a} \int_a^b p(t_k,s)\big[f^\Delta(s) - \gamma\big]\Delta s + [f(b) - f(a)]\left(x_k - \frac{1}{2}\right)$$

and then by $\sum_{k=1}^n x_k = \frac{n}{2}$ that

$$Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a} \int_a^b f^\sigma(x)\,\Delta x + \frac{\gamma}{b-a} \int_a^b \frac{1}{2}\,\mu(s)\,\Delta s$$

$$= \frac{1}{n(b-a)} \sum_{k=1}^{n} \int_{a}^{b} p(t_k, s)\left[f^{\Delta}(s) - \gamma\right] \Delta s + [f(b) - f(a)] \sum_{k=1}^{n} (b-a)\left(x_k - \frac{1}{2}\right)$$

$$= \frac{1}{n(b-a)} \sum_{k=1}^{n} \int_{a}^{b} p(t_k, s)\left[f^{\Delta}(s) - \gamma\right] \Delta s.$$

Hence,

$$\left| Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a} \int_{a}^{b} f^{\sigma}(x)\, \Delta x + \frac{\gamma}{b-a} \int_{a}^{b} \frac{1}{2} \mu(s)\, \Delta s \right|$$

$$\leqslant \frac{1}{n(b-a)} \sum_{k=1}^{n} \int_{a}^{b} (b-a)\left[f^{\Delta}(s) - \gamma\right] \Delta s = (b-a)(T - \gamma).$$

The proof of Theorem 3.7 is now completed. ▷

With the similar arguments as thosed used in the proof of Theorem 3.7 we also conclude the following result.

Theorem 3.8. *Let $a, b \in \mathbb{T}$ and $f \in C^1_{rd}(\mathbb{T})$. We also let $\Gamma = \sup_{x \in \mathbb{T}} f^{\Delta}(x)$ and $T = \frac{f(b)-f(a)}{b-a}$. Then for $0 \leqslant x_i \leqslant 1$ with $\sum_{i=1}^{n} x_i = \frac{n}{2}$ we have*

$$\left| Q(f, x_1, x_2, \ldots, x_n) - \frac{1}{b-a} \int_{a}^{b} f^{\sigma}(x)\, \Delta x + \frac{\Gamma}{b-a} \int_{a}^{b} \frac{1}{2} \mu(s)\, \Delta s \right| \leqslant (b-a)(\Gamma - T).$$

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НЕРАВЕНСТВА ДЛЯ НЕКОТОРЫХ НОВЫХ КВАДРАТУРНЫХ ФОРМУЛ С ВЕСОМ

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В настоящей работе обобщены неравенства Островского на шкале времени для n точек и L_p-норм m-й производной, где $m, n \in \mathbb{N}$ и $p \in [1, +\infty]$.

Ключевые слова: неравенства ошибок, n узлы, шкала времени.