Владикавказский математический журнал
2014, Том 16, Выпуск 4, С. 41–48

УДК **517.957+517.548**

SOLUTIONS OF THE DIFFERENTIAL INEQUALITY WITH A NULL LAGRANGIAN: HIGHER INTEGRABILITY AND REMOVABILITY OF SINGULARITIES. II[1]

A. A. Egorov

The aim of this paper is to establish a result on removability of singularities for solutions of the differential inequality with a null Lagrangian. Also, we obtain integral estimates for wedge products of closed differential forms and for minors of a Jacobian matrix.

Mathematics Subject Classification (2000): 30C65 (primary), 35F20, 35A15, 35B35, 26B25 (secondary).

Key words: null Lagrangian, removability of singularities, integral estimates, closed differential forms, minors of a Jacobian matrix.

Introduction

In this paper we continue to study the properties of solutions $v\colon V \to \mathbb{R}^m$, $V \subset \mathbb{R}^n$, of the following inequality

$$F(v'(x)) \leqslant KG(v'(x)) + H(x) \quad \text{a.e. } V \tag{1}$$

constructed by means of a continuous function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, a null Lagrangian $G\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, a measurable function $H\colon V \to \mathbb{R}$, and a constant $K \geqslant 1$. Here $v'(x)$ denotes the differential of v at $x \in V$. Using the higher integrability theorem of the previous paper [8], we establish a result on removability of singularities for solutions to (1).

Many investigations have dealt with the problem of removable singularities for quasiconfarmal mappings and mappings with bounded distortion (for example, see [1, 2, 4], [9]–[29] and the bibliography therein). Painlevé's theorem, a classical result in complex function theory, states that sets of zero length are removable for bounded holomorphic functions. More precisely, if E is a closed subset of linear measure zero in a planar domain V and v is a bounded function holomorphic in $V \setminus E$, then v extends to a bounded holomorphic function of V. Observe that the class of planar mappings with 1-bounded distortion coincides with the class of holomrphic functions. The strongest removability conjecture, stated in [14] as the counterpart of Painlevé's theorem for mappings with bounded distortion, suggests that sets of Hausdorff α-measure zero, $\alpha \leqslant n/(K+1) \leqslant n/2$, are removable for bounded mappings with K-bounded distortion in $\mathbb{R}^n$. In the case $n = 2$ this conjecture was verified

© 2014 Egorov A. A.

[1] The study was supported by a grant from the Russian Foundation for Basic Research, project 14-01-00768, the State Maintenance Program for the Leading Scientific Schools of the Russian Federation, grant № NSh-2263.2014.1, and the Integration Grant of the Siberian Division of the Russian Academy of Sciences, 2012, № 56.

by K. Astala [1] for $\alpha < 2/(K+1)$ and K. Astala, A. Clop, J. Mateu, J. Orobitg, I. Uriarte-Tuero [2] for $\alpha = 2/(K+1)$ (also see [3]). The higher integrability results for mappings with bounded distortion are closely related to the removability problems. Caccioppoli-type estimates (one of the key ingredients in proofs of higher integrability results) can be used as the basic tool for proving removability theorems. More precisely, if there exists $\underline{p}(n,K) < n$ such that Caccioppoli-type estimates hold for $p > \underline{p}(n,K)$, then a close set E with the Hausdorff dimension $\dim_H(E) < n - \underline{p}(n,K)$ is removable for bounded mappings with K-bounded distortion in $\mathbb{R}^n$ (for example, see [13, 14, 15, 16]). In such way removability results have been established for the classes of mappings that are close to solutions of linear elliptic partial differential equations and for the classes of quasiregular mappings of several n-dimensional variables (for example, see [5, 6]). Mappings of these classes, as mappings with bounded distortion, can be considered as solutions to (1) with specific functions F, G, and H. Our removability result (Theorem 1.1) contain partially the known results on removability of singularities for mappings of these classes.

In this paper, using the Hodge decomposition theory developed by T. Iwaniec and G. Martin [13, 14, 15], we also obtain integral estimates for wedge products of closed differential forms (Theorem 2.3) and for minors of a Jacobian matrix (Theorem 2.1). These estimates are extensions of integral estimates derived in [15]. They have been used in the proof of the higher integrability theorem in [8].

Some results of this paper have been announced in [7].

This paper is organized as follows. In § 1 we establish a result on removability of singularities for solutions to (1). We derive integral estimates for wedge products of closed differential forms and for minors of a Jacobian matrix in § 2.

We use the notation and terms from [8].

1. Removability of Singularities

Using the higher integrability theorem from [8], we establish the following result on removability of singularities for solutions to (1).

Theorem 1.1 (Removability of singularities). *Let $n, m, k \in \mathbb{N}$ and $t > k$ such that $2 \leqslant k \leqslant \min\{n, m\}$, and let V be a domain in $\mathbb{R}^n$. Suppose that a continuous function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfies*

$$F(\zeta) \geqslant c_F|\zeta|^k, \quad \zeta \in \mathbb{R}^{m\times n}, \tag{2}$$

with some constant $c_F > 0$, a null Lagrangian $G\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is homogeneous of degree k, and a measurable function $H\colon V \to \mathbb{R}$ has $H_+ \in L^t_{\mathrm{loc}}(V)$. Fix $K \geqslant 1$. Let $\underline{p} = \underline{p}(F, G, K)$ be the exponent from [8, *Theorem* 2.1]. *For a closed subset E of V with the Hausdorff dimension $\dim_H(E) < n - \underline{p}$ every bounded solution $v \in W^{1,k}_{\mathrm{loc}}(V \setminus E; \mathbb{R}^m)$ to* (1) *extends to a mapping of the class $W^{1,k}_{\mathrm{loc}}(V; \mathbb{R}^m)$ which is defined over the whole domain V and also satisfies* (1).

We follow the approach, developed in [14] (also see [15]), to prove Theorem 1.1. There are two key components in the proof. Firstly, the assumption on the size of the set E implies that E has zero s-capacity for an appropriate value of s. Secondly, the Cacciappoli-type estimate holds for this particular value of s. The definiton of s-capacity can be find in [9, 10, 15, 22, 23].

$\lhd$ Proof of Theorem 1.1. We have $\underline{p} < n - \dim_H(E)$. Let $s \in (\underline{p}, n - \dim_H(E))$. From [9, Ch. 3, Theorem 5.11] (also see [10, 15]), we obtain that the set E has zero s-capacity.

It is clear that $|E| = 0$. Further, the higher integrability theorem [8, Theorem 2.1] gives the Caccioppoli-type estimate

$$\|\varphi v'\|_{L^s(V;\mathbb{R}^{m\times n})} \leqslant C\big\||v \otimes \varphi'| + |\varphi|(\varepsilon + \varepsilon^{1-k}H_+)\big\|_{L^s(V)} \tag{3}$$

for $\varepsilon > 0$ and $\varphi \in C_0^\infty(V \setminus E)$, where the constant $C = C(F, G, K, s)$ does not depend on the test function φ or the mapping v. Let $\chi \in C_0^\infty(V)$ and $E' := E \cap \operatorname{supp}\chi$. Then E' has zero s-capacity. Therefore there exists a sequence of functions $(\eta_j \in C_0^\infty(V))_{j\in\mathbb{N}}$ such that $0 \leqslant \eta_j \leqslant 1$, $\eta_j = 1$ on some neighbourhood of E', $\lim_{j\to\infty} \eta_j = 0$ almost everywhere in V, and $\lim_{j\to\infty} \int_V |\eta_j'|^s = 0$. Put $\varphi_j := (1 - \eta_j)\chi \in C_0^\infty(V \setminus E)$ and $v_j := \varphi_j v \in W_0^{1,s}(V; \mathbb{R}^m)$. Then the mappings v_j are bounded in $L^\infty(V; \mathbb{R}^m)$ and converge to χv almost everywhere. We have $v_j' = \varphi_j v' + v \otimes \varphi_j'$ and $\varphi_j' = -\chi\eta_j' + (1 - \eta_j)\chi'$. Using (3), we obtain

$$\begin{aligned}
\|v_j'\|_{L^s(V;\mathbb{R}^{m\times n})} &\leqslant \|\varphi_j v'\|_{L^s(V;\mathbb{R}^{m\times n})} + \|v \otimes \varphi_j'\|_{L^s(V;\mathbb{R}^{m\times n})} \\
&\leqslant (1 + C)\Big(\|v \otimes \varphi_j'\|_{L^s(V;\mathbb{R}^{m\times n})} + \|\varphi_j(\varepsilon + \varepsilon^{1-k}H_+)\|_{L^s(V)}\Big) \\
&\leqslant (1 + C)\Big(\|v\|_{L^\infty(V\setminus E;\mathbb{R}^m)}\|\varphi_j'\|_{L^s(V;\mathbb{R}^n)} + \|\varphi_j(\varepsilon + \varepsilon^{1-k}H_+)\|_{L^s(V)}\Big) \\
&\leqslant (1 + C)\Big(\|v\|_{L^\infty(V\setminus E;\mathbb{R}^m)}\|\chi\|_{L^\infty(V)}\|\eta_j'\|_{L^s(V;\mathbb{R}^n)} \\
&\quad + \|v\|_{L^\infty(V\setminus E;\mathbb{R}^m)}\|(1 - \eta_j)\chi'\|_{L^s(V;\mathbb{R}^n)} + \|(1 - \eta_j)|\chi|(\varepsilon + \varepsilon^{1-k}H_+)\|_{L^s(V)}\Big).
\end{aligned}$$

Passing to the limit over j, we get

$$\limsup_{j\to\infty} \|v_j'\|_{L^s(V;\mathbb{R}^{m\times n})} \leqslant (1 + C)\Big(\|v\|_{L^\infty(V\setminus E;\mathbb{R}^m)}\|\chi'\|_{L^s(V;\mathbb{R}^n)} + \|\chi(\varepsilon + \varepsilon^{1-k}H_+)\|_{L^s(V)}\Big). \tag{4}$$

Therefore the sequence $(v_j)_{j\in\mathbb{N}}$ is bounded in $W^{1,s}(V; \mathbb{R}^m)$. Hence there exists its subsequence converged weakly in $W^{1,s}(V; \mathbb{R}^m)$ to a mapping in this Sobolev space. Clearly, this limit coincides with χv almost everywhere in V.

Therefore $\chi v \in W_0^{1,s}(V; \mathbb{R}^m)$ for all test functions $\chi \in C_0^\infty(V)$. This yields $v \in W_{\mathrm{loc}}^{1,s}(V; \mathbb{R}^m)$. Since v is a solution to (1) almost everywhere in V, the higher integrability theorem [8, Theorem 2.1] implies $v \in W_{\mathrm{loc}}^{1,k}(V; \mathbb{R}^m)$. $\rhd$

Remark 1.2. The assumption that v is bounded is of course rather more than we really need in the proof of Theorem 1.1. All that is required is that the sequence $(\|v\otimes\varphi_j'\|_{L^s(V;\mathbb{R}^{m\times n})})_{j\in\mathbb{N}}$ remains bounded as $j \to \infty$. Thus, for instance, Theorem 1.1 can be extended to the case $v \in L^p(V \setminus E; \mathbb{R}^m)$, $p \geqslant ns/(n-s)$, if in addition we require the stronger restriction that the set E has zero r-capacity for $r = sp/(p - s)$. That this requirement is sufficient follows from Hölder's inequality applied to $\|v \otimes \varphi_j'\|_{L^s(V;\mathbb{R}^{m\times n})}$ instead of using the trivial bound $\|v\|_{L^\infty(V\setminus E;\mathbb{R}^m)}\|\varphi_j'\|_{L^s(V;\mathbb{R}^n)}$ used above.

2. Integral Estimates

In the proof of the higher integrability of solutions to (1) ([8, Theorem 2.1]) we have used the following theorem on integral estimates for minors of a Jacobian matrix. This theorem is an extension of T. Iwaniec and G. Martin's result on integral estimates for Jacobians (cf. [15, Theorems 7.8.1 and 13.7.1]).

Theorem 2.1. *Let $n, m, k \in \mathbb{N}$ with $2 \leqslant k \leqslant \min(m, n)$. Then for every distribution $v = (v_1, \dots, v_m) \in \mathscr{D}'(\mathbb{R}^n; \mathbb{R}^m)$ with $v' \in L^p(\mathbb{R}^n; \mathbb{R}^{m\times n})$, $1 \leqslant p < \infty$, and for every $I = (i_1, \dots, i_k) \in \Gamma^k_m$, $J = (j_1, \dots, j_k) \in \Gamma^k_n$ we have the inequality*

$$\left| \int |v'|^{p-k} \frac{\partial v_J}{\partial x_I} \right| \leqslant C \left| 1 - \frac{p}{k} \right| \int |v'|^p \tag{5}$$

with some constant $C = C(k)$ depended only on k.

Remark 2.2. In the case $k = n = m$ Theorem 2.1 coincides with T. Iwaniec and G. Martin's result on integral estimates for Jacobians (see [15, Theorems 7.8.1 and 13.7.1]).

In the proof of Theorem 2.1 we need the following modification of T. Iwaniec and G. Martin's result on integral estimates for wedge products of closed differential forms (cf. [15, Theorem 13.6.1]).

Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$, $l \in \mathbb{N} \cup \{0\}$, be the space of all l-exterior forms on $\mathbb{R}^n$. For $I = (i_1, \dots, i_l) \in \Gamma^l_n$ we denote the l-exterior form $dx_{i_1} \wedge \dots \wedge dx_{i_l}$ by dx_I. We use the convention that $dx_I = 1$ if $l = 0$. For $\omega \in \Lambda^l$ we have $\omega = \sum_{I\in\Gamma^l_n} \gamma_I dx_I$ with some coefficients $\gamma_I \in \mathbb{C}$. We put $|\omega| = \left(\sum_{I\in\Gamma^l_n} |\gamma_I|^2\right)^{1/2}$. For $p \geqslant 1$ we denote by $L^p(\mathbb{R}^n; \Lambda^l)$ the space of differential l-forms on $\mathbb{R}^n$ with coefficients in $L^p(\mathbb{R}^n)$.

Theorem 2.3. *Let $n, k \in \mathbb{N}$ with $2 \leqslant k \leqslant n$. Consider $p_1, \dots, p_k, \varepsilon_1, \dots, \varepsilon_k \in \mathbb{R}$ and $l_1, \dots, l_k \in \mathbb{N}$ such that $1 < p_\varkappa < \infty$, $\frac{1}{p_1} + \ldots + \frac{1}{p_k} = 1$, $-1 \leqslant 2\varepsilon_\varkappa \leqslant \frac{p_\varkappa - 1}{p_\varkappa}$, and $\hat{l} := n - l_1 - \ldots - l_k \geqslant 0$. Let $\hat{I} = (\hat{i}_1, \dots, \hat{i}_{\hat{l}}) \in \Gamma^{\hat{l}}_n$. Suppose that $(\varphi_1, \dots, \varphi_k)$ be a k-tuple of closed differential forms with $\varphi_\varkappa \in L^{(1-\varepsilon_\varkappa)p_\varkappa}(\mathbb{R}^n; \Lambda^{l_\varkappa})$. Then*

$$\int \frac{\varphi_1 \wedge \dots \wedge \varphi_k \wedge dx_{\hat{I}}}{|\varphi_1|^{\varepsilon_1} \dots |\varphi_k|^{\varepsilon_k}} \leqslant C\varepsilon \|\varphi_1\|^{1-\varepsilon_1}_{L^{(1-\varepsilon_1)p_1}(\mathbb{R}^n;\Lambda^{l_1})} \dots \|\varphi_k\|^{1-\varepsilon_k}_{L^{(1-\varepsilon_k)p_k}(\mathbb{R}^n;\Lambda^{l_k})}, \tag{6}$$

where $\varepsilon := \max(|\varepsilon_1|, \dots, |\varepsilon_k|)$ and the constant $C = C(p_1, \dots, p_k)$ depends only on $p_1, \dots, p_k$.

Remark 2.4. In the case $\hat{l} = 0$, i. e. $dx_{\hat{I}} = 1$, Theorem 2.3 coincides with [15, Theorem 13.6.1]. For proving Theorem 2.3 we use the Hodge decomposition technique developed in [13, 14, 15] and follow the proof of Theorem 13.6.1 in [15].

◁ Proof of Theorem 2.3. Observe that $(1 - \varepsilon_\varkappa)p_\varkappa \geqslant \frac{p_\varkappa + 1}{2} > 1$ and $|\varepsilon_\varkappa| < 1/2$, $\varkappa = 1, \dots, k$. We have $\frac{\varphi_\varkappa}{|\varphi_\varkappa|^{\varepsilon_\varkappa}} \in L^{p_\varkappa}(\mathbb{R}^n; \Lambda^{l_\varkappa})$. Denote by $W^{1,p}(\mathbb{R}^n; \Lambda^l)$, $0 \leqslant l \leqslant n$, $p \geqslant 1$, the space of differential l-forms on $\mathbb{R}^n$ with coefficients in $W^{1,p}(\mathbb{R}^n)$. We can consider the following Hodge decomposition in $L^{p_\varkappa}(\mathbb{R}^n; \Lambda^{l_\varkappa})$ ([14, Theorem 6.1], also see [15, § 10.6]):

$$\frac{\varphi_\varkappa}{|\varphi_\varkappa|^{\varepsilon_\varkappa}} = d\alpha_\varkappa + d^*\beta_\varkappa \tag{7}$$

with some $\alpha_\varkappa \in W^{1,p_\varkappa}(\mathbb{R}^n; \Lambda^{l_\varkappa - 1})$ and $\beta_\varkappa \in W^{1,p_\varkappa}(\mathbb{R}^n; \Lambda^{l_\varkappa + 1})$. Here d is the exterior derivative, and d^* is its formal adjoint, the coexterior derivative. The forms $d\alpha_\varkappa$ and $d^*\beta_\varkappa$, $\varkappa = 1, \dots, k$, are uniquely determined and can be expressed by means of the Hodge projection operators

$$E\colon L^p(\mathbb{R}^n; \Lambda^l) \to dW^{1,p}(\mathbb{R}^n; \Lambda^{l-1}) \quad \text{and} \quad E^*\colon L^p(\mathbb{R}^n; \Lambda^l) \to d^*W^{1,p}(\mathbb{R}^n; \Lambda^{l+1})$$

defined by [15, § 10.6, formulas (10.71) and (10.72)] for $1 < p < \infty$ and $1 \leqslant l \leqslant n-1$. Namely we have

$$d\alpha_\varkappa = E\left(\frac{\varphi_\varkappa}{|\varphi_\varkappa|^{\varepsilon_\varkappa}}\right) \quad \text{and} \quad d^*\beta_\varkappa = E^*\left(\frac{\varphi_\varkappa}{|\varphi_\varkappa|^{\varepsilon_\varkappa}}\right). \tag{8}$$

Applying [14, Theorem 6.1], we get the following bound for exact term:

$$\|d\alpha_\varkappa\|_{L^{p_\varkappa}(\mathbb{R}^n;\Lambda^{l_\varkappa})} \leqslant C_1(p_\varkappa)\|\varphi_\varkappa\|^{1-\varepsilon_\varkappa}_{L^{(1-\varepsilon_\varkappa)p_\varkappa}(\mathbb{R}^n;\Lambda^{l_\varkappa})}. \tag{9}$$

By [15, § 10.6, formulas (10.73) and (10.74)] we have

$$\operatorname{Ker} E = \{\varphi \in L^p(\mathbb{R}^n;\Lambda^l) : \ d^*\varphi = 0\}$$

and

$$\operatorname{Ker} E^* = \{\varphi \in L^p(\mathbb{R}^n;\Lambda^l) : \ d\varphi = 0\}$$

for $1 < p < \infty$ and $1 \leqslant l \leqslant n-1$. Then $E^*(\varphi_\varkappa) = 0$. Therefore we can write $d^*\beta_\varkappa$ as a commutator

$$d^*\beta_\varkappa = E^*\left(\frac{\varphi_\varkappa}{|\varphi_\varkappa|^{\varepsilon_\varkappa}}\right) - \frac{E^*(\varphi_\varkappa)}{|E^*(\varphi_\varkappa)|^{\varepsilon_\varkappa}}.$$

Applying [15, Theorem 13.2.1] (also see [13, Theorems 8.1 and 8.2]), we obtain

$$\|d^*\beta_\varkappa\|_{L^{p_\varkappa}(\mathbb{R}^n;\Lambda^{l_\varkappa})} \leqslant C_2(p_\varkappa)|\varepsilon_\varkappa|\|\varphi_\varkappa\|^{1-\varepsilon_\varkappa}_{L^{(1-\varepsilon_\varkappa)p_\varkappa}(\mathbb{R}^n;\Lambda^{l_\varkappa})}. \tag{10}$$

Using (7), we have

$$\begin{aligned}\int \frac{\varphi_1\wedge\cdots\wedge\varphi_k\wedge dx_{\hat I}}{|\varphi_1|^{\varepsilon_1}\ldots|\varphi_k|^{\varepsilon_k}} &= \int (d\alpha_1 + d^*\beta_1)\wedge\cdots\wedge(d\alpha_k + d^*\beta_k)\wedge dx_{\hat I} \\ &= \int d\alpha_1\wedge\cdots\wedge d\alpha_k\wedge dx_{\hat I} + \int \mathscr{B}. \end{aligned}\tag{11}$$

Since $p_1,\ldots,p_k$ represents a Hölder conjugate tuple, by Stokes' formula via an approximation argument we obtain

$$\int d\alpha_1\wedge\cdots\wedge d\alpha_k\wedge dx_{\hat I} = 0. \tag{12}$$

The integrand $\mathscr{B}$ is a sum of wedge products of the type $\psi_1\wedge\cdots\wedge\psi_k\wedge dx_{\hat I}$, where $\psi_\varkappa$ is either $d\alpha_\varkappa$ or $d^*\beta_\varkappa$ and at least one $d^*\beta_\varkappa$ is always present, with at most 2^k-1 terms. Combining Hölder's inequality with (9) and (10), we get

$$\begin{aligned}\int \psi_1\wedge\cdots\wedge\psi_k\wedge dx_{\hat I} &\leqslant C_3(k)\|\psi_1\|_{L^{p_1}(\mathbb{R}^n;\Lambda^{l_1})}\cdots\|\psi_k\|_{L^{p_k}(\mathbb{R}^n;\Lambda^{l_k})} \\ &\leqslant C_4(p_1,\ldots,p_k)\varepsilon\|\varphi_1\|^{1-\varepsilon_1}_{L^{(1-\varepsilon_1)p_1}(\mathbb{R}^n;\Lambda^{l_1})}\cdots\|\varphi_k\|^{1-\varepsilon_k}_{L^{(1-\varepsilon_k)p_k}(\mathbb{R}^n;\Lambda^{l_k})}.\end{aligned}$$

This with (11) and (12) yields (6). $\rhd$

$\lhd$ Proof of Theorem 2.1. Let $p_\varkappa := k$, $\varepsilon_\varkappa := \varepsilon := 1-\frac{p}{k}$, and $l_\varkappa := 1$ for $\varkappa = 1,\ldots,k$. Then $1 < p_\varkappa < \infty$, $\frac{1}{p_1}+\cdots+\frac{1}{p_k} = 1$, $\hat l := n-k = n-l_1-\cdots-l_k \geqslant 0$, $(1-\varepsilon_\varkappa)p_\varkappa = p$, and $\max(|\varepsilon_1|,\ldots,|\varepsilon_k|) = |\varepsilon| = \left|1-\frac{p}{k}\right|$. Let $\varphi_\varkappa := dv_{j_\varkappa} \in L^{(1-\varepsilon_\varkappa)p_\varkappa}(\mathbb{R}^n;\Lambda^{l_\varkappa})$. Let $\hat I = (\hat i_1,\ldots,\hat i_{\hat l}) \in \Gamma^{\hat l}_n$ be the ordered $\hat l$-tuple such that $\{\hat i_1,\ldots,\hat i_{\hat l}\} = \{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}$. We chose the sign $\operatorname{sgn} I$ such that $\operatorname{sgn} I\, dx_I\wedge dx_{\hat I} = dx_1\wedge\cdots\wedge dx_n$.

When p lies outside the interval $\left(\frac{k+1}{2},\frac{3k}{2}\right)$ the estimate is clear as (5) always holds with 1 in place $C(k)\left|1-\frac{p}{k}\right|$. In this case $\left|1-\frac{p}{k}\right| \geqslant \frac{k-1}{2k}$ and inequality (5) holds with $C(k) = \frac{2k}{k-1}$.

Suppose that $k+1 \leqslant 2p \leqslant 3k$. Then $-1 \leqslant 2\varepsilon_\varkappa \leqslant \frac{p_\varkappa - 1}{p_\varkappa}$ and $|\varepsilon| \leqslant 1/2$. Applying Theorem 2.3, we obtain

$$\int \frac{\frac{\partial v_J}{\partial x_I}}{|dv_{j_1}|^\varepsilon \dots |dv_{j_k}|^\varepsilon} = \int \frac{\operatorname{sgn} I\, dv_{j_1} \wedge \dots \wedge dv_{j_k} \wedge dx_{\hat I}}{|dv_{j_1}|^{\varepsilon_1} \dots |dv_{j_k}|^{\varepsilon_k}}$$
$$\leqslant C_1(k)|\varepsilon|\, \|dv_{j_1}\|^{1-\varepsilon}_{L^p(\mathbb{R}^n;\Lambda^1)} \dots \|dv_{j_k}\|^{1-\varepsilon}_{L^p(\mathbb{R}^n;\Lambda^1)} \leqslant C_1(k)\varepsilon \int |v'|^p. \quad (13)$$

Using the elementary inequalities $\left|\frac{\partial v_J}{\partial x_I}\right| \leqslant |dv_{j_1}| \dots |dv_{j_k}|$ and $|a - a^{1-\varepsilon}| \leqslant |\varepsilon|$ for $0 \leqslant a \leqslant 1$ and $-1 < \varepsilon < 1$, we have

$$\left| \frac{\frac{\partial v_J}{\partial x_I}}{|v'|^{\varepsilon k}} - \frac{\frac{\partial v_J}{\partial x_I}}{|dv_{j_1}|^\varepsilon \dots |dv_{j_k}|^\varepsilon} \right|$$
$$= \frac{\left|\frac{\partial v_J}{\partial x_I}\right| |v'|^p}{|dv_{j_1}| \dots |dv_{j_k}|} \left| \frac{|dv_{j_1}| \dots |dv_{j_k}|}{|v'|^k} - \left(\frac{|dv_{j_1}| \dots |dv_{j_k}|}{|v'|^k} \right)^{1-\varepsilon} \right| \leqslant |\varepsilon||v'|^p.$$

Combining this with (13), we obtain

$$\left| \int |v'|^{p-k} \frac{\partial v_J}{\partial x_I} \right| \leqslant \int \left| \frac{\frac{\partial v_J}{\partial x_I}}{|v'|^{\varepsilon k}} - \frac{\frac{\partial v_J}{\partial x_I}}{|dv_{j_1}|^\varepsilon \dots |dv_{j_k}|^\varepsilon} \right| + \int \left| \frac{\frac{\partial v_J}{\partial x_I}}{|dv_{j_1}|^\varepsilon \dots |dv_{j_k}|^\varepsilon} \right|$$
$$\leqslant (C_1(k)+1)|\varepsilon| \int |v'|^p. \ \triangleright$$

Acknowlegement. The author is greteful to A. P. Kopylov, Yu. G. Reshetnyak, A. S. Romanov, and S. K. Vodop'yanov for helpful discussions.

References

1. *Astala K.* Area distortion of quasiconformal mappings // Acta Math.—1994.—Vol. 173, № 1.—P. 37–60.
2. *Astala K., Clop A., Mateu J., Orobitg J., Uriarte-Tuero I.* Distortion of Hausdorff measures and improved Painlevé removability for quasiregular mappings // Duke Math. J.—2008.—Vol. 141, № 3.—P. 539–571.
3. *Astala K., Iwaniec T., Martin G.* Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.—Princeton: Princeton Univ. Press, 2009.—xvi+677 p.—(Princeton Math. Ser. Vol. 48.).
4. *Dairbekov N. S.* Removable singularities of locally quasiconformal maps // Sib. Mat. Zh.—1992. Vol. 33, № 1.—P. 193–195. [Russian]; Engl. transl.: Sib. Math. J.—1992.—Vol. 33, № 1.—P. 159–161.
5. *Dairbekov N. S.* On removable singularities of solutions to first order elliptic systems with irregular coefficients // Sib. Mat. Zh.—1993.—Vol. 34, № 1.—P. 65–69. [Russian]; Engl. transl.: Sib. Math. J.—1993.—Vol. 34, № 1.—P. 55–58.
6. *Dairbekov N. S.* Stability of Classes of Mappings, Beltrami Equations, and Quasiregular Mappings of Several Variables. Doctoral dissertation.—Novosibirsk, 1995.—[Russian].
7. *Egorov A. A.* Solutions of the Differential Inequality with a Null Lagrangian: Regularity and Removability of Singularities.—2010.—URL: http://arxiv.org/abs/1005.3459.
8. *Egorov A. A.* Solutions of the Differential Inequality with a Null Lagrangian: Higher Integrability and Removability of Singularities. I // Vladikavkaz Math. J.—2014.—Vol. 16, № 3.—P. 22–37.
9. *Gol'dshtein V. M., Reshetnyak Yu. G.* Introduction to the Theory of Functions with Generalized Derivatives, and Quasiconformal Mappings.—M.: Nauka, 1983.—285 p.—[Russian].
10. *Gol'dshtein V. M., Reshetnyak Yu. G.* Quasiconformal Mappings and Sobolev Spaces.—Dordrecht etc.: Kluwer Acad. Publ., 1990.—xix+371 p.—(Math. and its Appl. Soviet Ser. Vol. 54.).
11. *Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E.* The Beltrami Equation. A Geometric Approach.—Berlin: Springer, 2012.—xiii+301 p.—(Developments in Math. Vol. 26.).

12. *Iwaniec T.* On L^p-integrability in PDE's and quasiregular mappings for large exponents // Ann. Acad. Sci. Fenn. Ser. AI.—1982.—Vol. 7.—P. 301–322.
13. *Iwaniec T.* p-Harmonic tensors and quasiregular mappings // Ann. Math.—1992.—Vol. 136.—P. 651–685.
14. *Iwaniec T., Martin G.* Quasiregular mappings in even dimensions // Acta Math.—1993.—Vol. 170.—P. 29–81.
15. *Iwaniec T., Martin G.* Geometric Function Theory and Non-Linear Analysis. Oxford Math. Monogr.—Oxford: Oxford Univ. Press, 2001.
16. *Iwaniec T., Migliaccio L., Nania L., Sbordone C.* Integrability and removability results for quasiregular mappings in high dimensions // Math. Scand.—1994.—Vol. 75, № 2.—P. 263–279.
17. *Martio O.* Modern Tools in the Theory of Quasiconformal Maps. Textos de Matematica. Serie B. Vol. 27.—Coimbra: Universidade de Coimbra, Departamento de Matematica, 2000.—43 p.
18. *Martio O., Ryazanov V., Srebro U., Yakubov E.* Moduli in Modern Mapping Theory.—N. Y.: Springer, 2009.—xii+367 p.
19. *Miklyukov V. M.* Removable singularities of quasi-conformal mappings in space // Dokl. Akad. Nauk SSSR.—1969.—Vol. 188.—P. 525–527.—[Russian].
20. *Pesin I. N.* Metric properties of Q-quasiconformal mappings // Mat. Sb. (N. S.).—1956.—Vol. 40 (82), № 3.—P. 281–294.—[Russian].
21. *Poletskii E. A.* On the removal of singularities of quasiconformal mappings // Mat. Sb. (N. S.).—1973.—Vol. 92 (134), № 2 (10).—P. 242–256.—[Russian]; Engl. transl.: Math. USSR Sb.—1973.—Vol. 21, № 2.—P. 240–254.
22. *Reshetnyak Yu. G.* Space Mappings with Bounded Distortion.—Novosibirsk: Nauka, 1982.—288 p.—[Russian].
23. *Reshetnyak Yu. G.* Space Mappings with Bounded Distortion.—Providence (R. I.): Amer. Math. Soc., 1989.—362 p.—(Transl. of Math. Monogr. Vol. 73.).
24. *Rickman S.* Quasiregular Mappings.—Berlin: Springer-Verlag, 1993.—x+213 p.—(Results in Math. and Related Areas (3). Vol. 26.).
25. *Vodop'yanov S. K., Gol'dshtein V. M.* A criterion for the possibility of eliminating sets for the spaces W_p^1, of quasiconformal and quasi-isometric mappings // Dokl. Akad. Nauk SSSR.—1975.—Vol. 220.—P. 769–771.—[Russian]; Engl. transl.: Sov. Math. Dokl.—1975.—Vol. 16.—P. 139–142.
26. *Vodop'yanov S. K., Gol'dshtein V. M.* Criteria for the removability of sets in spaces of L_p^1, quasiconformal and quasi-isometric mappings // Sib. Mat. Zh.—1977.—Vol. 18, № 1.—P. 48–68.—[Russian]; Engl. transl.: Sib. Math. J.—1977.—Vol. 18, № 1.—P. 35–50.
27. *Vodop'yanov S. K., Gol'dshtein V. M., Reshetnyak Yu. G.* On geometric properties of functions with generalized first derivatives // Uspekhi Mat. Nauk.—1979.—Vol. 34, № 1 (205).—P. 17–65.—[Russian]; Engl. transl.: Russ. Math. Surv.—1979.—Vol. 34, № 1.—P. 19–74.
28. *Vuorinen M.* Conformal Geometry and Quasiregular Mappings.—Berlin etc.: Springer-Verlag, 1988.—xix+209 p.—(Lect. Notes Math. Vol. 1319.).
29. *Zorich V. A.* An isolated singularity of mappings with bounded distortion // Mat. Sb. (N. S.).—1970.—Vol. 81 (123), № 4.—P. 634–636.—[Russian]; Engl. transl.: Math. USSR Sb.—1970.—Vol. 10, № 4.—P. 581–583.

Received April 3, 2013.

Egorov Alexander Anatol'evich
Sobolev Institute of Mathematics,
senior researcher
RUSSIA, 630090, Novosibirsk, Koptyug Avenue, 4;
Novosibirsk State University,
associate professor
RUSSIA, 630090, Novosibirsk, Pirogova Str., 2
E-mail: yegorov@math.nsc.ru

РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНОГО НЕРАВЕНСТВА С НУЛЬ-ЛАГРАНЖИАНОМ: ПОВЫШАЮЩАЯСЯ ИНТЕГРИРУЕМОСТЬ И УСТРАНИМОСТЬ ОСОБЕННОСТЕЙ. II

Егоров А. А.

Целью статьи является установление результата о затирании особенностей у решений дифференциального неравенства с нуль-лагранжианом. Также получены интегральные оценки для внешних произведений замкнутых дифференциальных форм и для миноров матрицы Якоби.

Ключевые слова: нуль-лагранжиан, устранимость особенностей, интегральные оценки, внешнее произведение замкнутых дифференциальных форм, миноры матрицы Якоби.