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A STUDY ON A CLASS OF p-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS[1]

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In this paper, we study and introduce the majorization properties of a new class of analytic p-valent functions of complex order defined by the generalized hypergeometric function. Some known consequences of our main result will be given. Moreover, we investigate the coefficient estimates for this class.

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1. Introduction

Let $\mathscr{A}_p$ be the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}, \tag{1.1}$$

which are analytic and p-valent in the unit disc $\mathbb{U}$. Let f and g be analytic in the open unit disc $\mathbb{U}$. We say that f is majorized by g in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

if there exists a function φ, analytic in $\mathbb{U}$ such that

$$|\varphi(z)| \leqslant 1, \quad f(z) = \varphi(z) g(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

For $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that f is subordinate to g if there exists the Schwarz function ω, analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

If $f(z)$ and $g(z)$ belong to $\mathscr{A}_p$, then the Hadamard product $f * g$ is defined by

$$f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad p \in \mathbb{N}.$$

El-Ashwah [2] studied the following p-valent function, which defined by generalized hypergeometric functions

$${}_r\mathscr{G}_s(a_1, b_1; z^p) = z^p + \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^{p+n}}{n!}, \quad p \in \mathbb{N},$$

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where $a_i \in \mathbb{C}, b_q \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, (i=1,\ldots,r, q=1,\ldots,s)$, and $r \leqslant s+1; r,s \in \mathbb{N}_0$, and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n=0, \\ x(x+1)\cdots(x+n-1), & n=\{1,2,3,\ldots\}. \end{cases}$$

Let $\mathscr{L}^{m,b}_{\lambda_1,\lambda_2,p} \in \mathscr{A}_p$ is defined by

$$\mathscr{L}^{m,b}_{\lambda_1,\lambda_2,p} = z^p + \sum_{n=1}^{\infty} \left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m z^{p+n}, \quad p \in \mathbb{N},$$

where $m, b \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\lambda_2 \geqslant \lambda_1 \geqslant 0$.

Corresponding to ${}_r\mathscr{G}_s(a_1, b_1; z^p)$, $\mathscr{L}^{m,b}_{\lambda_1,\lambda_2,p}$ and using the Hadamard product, we define a new generalized differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)$ as follows:

Definition 1.1. Let $f \in \mathscr{A}_p$, then a generalized differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) : \mathscr{A}_p \to \mathscr{A}_p$ is given as

$$\begin{aligned} D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) &= ({}_r\mathscr{G}_s(a_1,b_1;z^p) * \mathscr{L}^{m,b}_{\lambda_1,\lambda_2,p} * f(z)) \\ &= z^p + \sum_{n=1}^{\infty} \left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\ldots(a_r)_n}{(b_1)_n\ldots(b_s)_n} \frac{a_{p+n}z^{p+n}}{n!}. \end{aligned} \tag{1.4}$$

It follows from the above definition that

$$\begin{aligned} &(p+\lambda_2 n+b)\, D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) \\ &= (p+\lambda_2 n - p\lambda_1 + b)\, D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z) + \lambda_1 z\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)'. \end{aligned} \tag{1.5}$$

Remark 1.1. It should be remarked that the linear operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_1)f(z)$ is a generalization of many operators considered earlier. Let us see some of the examples:

For $\lambda_2 = b = 0$, the operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f$ reduces to the operator was given by Selvaraj and Karthikeyan [1].

For $m = 0$, the operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f$ reduces to the operator was given by El-Ashwah [2].

For $m = 0$ and $p = 1$, the operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f$ reduces to the well-known operator introduced by Dziok and Srivastava [3].

For $\lambda_2 = b = 0$ and $p = 1$, we get the operator studied by Selvaraj and Karthikeyan [4].

For $m = 0$, $r = 2$, $s = 1$ and $p = 1$, we obtain the operator which was given by Hohlov [5].

For $r = 1$, $s = 0$, $a_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = b = 0$ and $p = 1$, we get the Sălăgean derivative operator [6].

For $r = 1$, $s = 0$, $a_1 = 1$, $\lambda_2 = b = 0$ and $p = 1$, we get the generalized Sălăgean derivative operator introduced by Al-Oboudi [7].

For $m = 0$, $r = 1$, $s = 0$, $a_1 = \delta + 1$ and $p = 1$, we obtain the operator introduced by Ruscheweyh [8].

For $r = 1$, $s = 0$, $a_1 = \delta + 1$ and $p = 1$, we obtain the operator studied by El-Yagubi and Darus [9].

For $m = 0$, $r = 2$ and $s = 1$, $a_2 = 1$ and $p = 1$, we obtain the operator studied by Carlson and Shaffer [10].

For $r = 1$, $s = 0$, $a_1 = 1$, $\lambda_2 = 0$ and $p = 1$, we get the operator introduced by Cátás [11].

Next, by using the generalized differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)$, we study the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ as follows:

Definition 1.2. Let $f \in \mathscr{A}_p$, then $f \in S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ of p-valent functions of complex order $\gamma \neq 0$ in $\mathbb{U}$, if and only if

$$\left\{1 + \frac{1}{\gamma}\left(\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))^{(j+1)}}{(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))^{(j)}} - p + j\right)\right\} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{U}, \tag{1.6}$$

where $p \in \mathbb{N}$, $m, b, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda_2 \geqslant \lambda_1 \geqslant 0$, $-1 \leqslant B < A \leqslant 1$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$ $(i = 1, \ldots, r,\ q = 1, \ldots, s)$ and $r \leqslant s + 1$; $r, s \in \mathbb{N}_0$.

Clearly, we have the following relationships:

(i) when $m = 0$, $p = 1$, $j = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = 1$, $A = 1$ and $B = -1$, then the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ reduces to the class $S(\gamma)$.

(ii) when $m = 0$, $p = 1$, $j = 1$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = 1$, $A = 1$ and $B = -1$, then the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ reduces to the class $C(\gamma)$.

(iii) when $m = 0$, $p = 1$, $j = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = 1$, $A = 1$, $B = -1$ and $\gamma = 1 - \alpha$, then the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ reduces to the class $S^*(\alpha)$ for $0 < \alpha < 1$.

The classes $S(\gamma)$ and $C(\gamma)$ are said to be classes of starlike and convex of complex order $\gamma \neq 0$ in $\mathbb{U}$, were considered by Nasr and Aouf [12] and $S^*(\alpha)$ denote the class of starlike functions of order α in $\mathbb{U}$.

2. Majorization Problem

A majorization problem for functions f belong to the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ is considered.

Theorem. *Let $f \in \mathscr{A}_p$ and suppose that $g \in S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$. If $(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))^{(j)}$ is majorized by $(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)g(z))^{(j)}$ in $\mathbb{U}$, then*

$$\left|(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))^{(j)}\right| \leqslant \left|(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)g(z))^{(j)}\right| \quad for \quad |z| \leqslant r_0, \tag{2.1}$$

where $r_0 = r_0(p, \gamma, \lambda_1, \lambda_2, b, A, B)$ is the smallest positive root of the equation

$$\begin{gathered} r^3\left|\gamma(A - B) + \left(\frac{p + \lambda_2 n + b}{\lambda_1}\right)B\right| - \left[\frac{p + \lambda_2 n + b}{\lambda_1} + 2|B|\right]r^2 \\ -\left[\left|\gamma(A - B) - \left(\frac{p + \lambda_2 n + b}{\lambda_1}\right)B\right| + 2\right]r + \left(\frac{p + \lambda_2 n + b}{\lambda_1}\right) = 0, \\ -1 \leqslant B < A \leqslant 1; \quad \lambda_2 \geqslant \lambda_1 \geqslant 0; \quad b \in \mathbb{N}_0; \quad P \in \mathbb{N}; \quad \gamma \in \mathbb{C}\backslash\{0\}. \end{gathered} \tag{2.2}$$

$\lhd$ Since $g \in S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ we can get from (1.6), that

$$1 + \frac{1}{\gamma}\left(\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)g(z))^{(j+1)}}{(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)g(z))^{(j)}} - p + j\right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.3}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $j, p \in \mathbb{N}$, $p > j$ and w is analytic in $\mathbb{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}).$$

From (2.3), we get

$$\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z)\big)^{(j+1)}}{\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z)\big)^{(j)}} = \frac{(p-j) + [\gamma(A-B) + (p-j)B]w(z)}{1 + Bw(z)}. \tag{2.4}$$

By noting that

$$\begin{aligned} z\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j+1)} &= \left(\frac{p+\lambda_2 n + b}{\lambda_1}\right)\big(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j)} \\ &+ \left(p - j - \left(\frac{p+\lambda_2 n + b}{\lambda_1}\right)\right)\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j)}, \end{aligned} \tag{2.5}$$

and by virtue of (2.4) and (2.5) we get

$$\begin{aligned} &\left|(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z))^{(j)}\right| \\ &\leqslant \frac{\frac{p+\lambda_2 n+b}{\lambda_1}[1+|B|z|]}{\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - \left|\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)|B|\right||z|}\big|(D^{m+1,n}_{\lambda_1,\lambda_2,p} g(z))^{(j)}\big|. \end{aligned} \tag{2.6}$$

Next, since $(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z))^{(j)}$ is majorized by $(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z))^{(j)}$ in the unit disc $\mathbb{U}$, thus from (1.3) we have

$$\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j)} = \varphi(z)\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z)\big)^{(j)}.$$

Differentiating it with respect to z and multiplying by z we get

$$\begin{aligned} &z\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j+1)} \\ &= z\varphi'(z)\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z)\big)^{(j)} + z\varphi(z)\big(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z)\big)^{(j+1)}. \end{aligned}$$

Now by using (2.5) in the above equation, it yields

$$\begin{aligned} &\big(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z)\big)^{(j)} \\ &= \frac{z\varphi'(z)(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z))^{(j)}}{\frac{p+\lambda_2 n+b}{\lambda_1}} + \varphi(z)(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z))^{(j)}. \end{aligned} \tag{2.7}$$

Thus, by noting that $\varphi \in \Omega$ satisfies the inequality

$$|\varphi'(z)| \leqslant \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z \in \mathbb{U}), \tag{2.8}$$

by using (2.6) and (2.8) in (2.7), we get

$$\begin{aligned} &\left|(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z))^{(j)}\right| \\ &\leqslant \left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{|z|(1+|B||z|)}{\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - \left|\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)|B|\right||z|}\right]\big|(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)g(z))^{(j)}\big|, \end{aligned} \tag{2.9}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leqslant \rho \leqslant 1)$$

leads us to the inequality

$$\begin{aligned} &\left|(D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_1,b_1)f(z))^{(j)}\right| \\ &\leqslant \frac{\phi(\rho)}{(1-r^2)\left[\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - \left|\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)B\right|r\right]} \left|(D^{m+1,b}_{\lambda_1,\lambda_2,p} g(z))^{(j)}\right| \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \phi(\rho) &= -r(1+|B|r)\rho^2 + (1-r^2) \\ &\times \left[\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - \left|\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)B\right|r\right]\rho + r(1+|B|r) \end{aligned} \tag{2.11}$$

takes its maximum value at $\rho = 1$ with $r_0 = r_0(p,\gamma,\lambda_1,\lambda_2,b,A,B)$. Here $r_1(p,\gamma,\lambda_1,\lambda_2,b,A,B)$ is the smallest positive root of the equation (2.2).

Furthermore, if $0 \leqslant \rho \leqslant r_1(p,\gamma,\lambda_1,\lambda_2,b,A,B)$, then the function $\psi(\rho)$ defined by

$$\begin{aligned} \psi(\rho) &= -\sigma(1+|B|\sigma)\rho^2 + (1-\sigma^2)| \\ &\times \left[\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - |\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)B|\sigma\right]\rho + \sigma(1+|B|\sigma) \end{aligned} \tag{2.12}$$

is seen to be an increasing function on the interval $0 \leqslant \rho \leqslant 1$, so that

$$\begin{aligned} \psi(\rho) \leqslant \psi(1) = (1-\sigma^2)\left[\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) - \left|\gamma(A-B) + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)B\right|\sigma\right] \\ 0 \leqslant \rho \leqslant 1 \quad (0 \leqslant \sigma \leqslant r_1(p,\gamma,\lambda_1,\lambda_2,b,A,B)). \end{aligned} \tag{2.13}$$

Hence upon setting $\rho = 1$ in (2.10) we conclude that (2.1) of Theorem 2.1 holds true for $|z| \leqslant r_1(p,\gamma,\lambda_1,\lambda_2,A,B)$ where $r_1(p,\gamma,\lambda_1,\lambda_2,A,B)$ is the smallest positive root of equation (2.2).

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we have the following result:

Corollary 2.1. *Let* $f \in \mathscr{A}_p$ *and suppose that* $g \in S^{m,b,j}_{\lambda_1,\lambda_2,p}(a_1,b_1,\gamma)$. *If* $(D^m_{\lambda_1,p}(a_1,b_1)f(z))^{(j)}$ *is majorized by* $(D^m_{\lambda_1,p}(a_1,b_1)g(z))^{(j)}$ *in* $\mathbb{U}$, *then*

$$\left|\left(D^{m+1}_{\lambda_1,p}(a_1,b_1)f(z)\right)^{(j)}\right| \leqslant \left|\left(D^{m+1}_{\lambda_1,p}(a_1,b_1)g(z)\right)^{(j)}\right| \quad \text{for} \quad |z| \leqslant r_0, \tag{2.14}$$

where

$$r_0 = r_0(p,\gamma,\lambda_1,\lambda_2,b) = \frac{k - \sqrt{k^2 - 4\left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)\left|2\gamma - \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)\right|}}{2\left|2\gamma - \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)\right|},$$

$$k = 2 + \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right) + \left|2\gamma - \left(\frac{p+\lambda_2 n+b}{\lambda_1}\right)\right|,$$

$$p \in \mathbb{N}; \quad \gamma, \lambda_1 \in \mathbb{C}\backslash\{0\}, \quad b \in \mathbb{N}_0, \quad \lambda_2 \geqslant 0$$

and $S^{m,b,j}_{\lambda_1,\lambda_2,p}(a_1,b_1,\gamma)$ *be a special case of* $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1,b_1,A,B,\gamma]$ *when* $A = 1$ *and* $B = -1$.

Setting $p = 1$, $m = \lambda_2 = b = 0$, $\lambda_1 = 1$, $j = 0$, $r = 2$, $s = 1$, $a_1 = b_1$ and $a_2 = 1$ in Corollary 2.1, we get the following corollary:

Corollary 2.2. *Let $f \in \mathscr{A}_p$ and suppose that $g \in S(\gamma)$. If $f(z)$ is majorised by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| < r_3),$$

where

$$r_0 = r_0(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|},$$

which is a known result obtained by Altintas et al. [13].

For $\gamma = 1$, the Corollary 2.2 reduces to the following result:

Corollary 2.3. *Let $f(z) \in \mathscr{A}_p$ and suppose that $g \in S^* = S^*(0)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant 2 - \sqrt{3}),$$

which is a known result obtained by MacGregor [14].

3. Coefficient Estimates

The coefficient estimate for the class $S^{m,b,j}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ is obtained, when $j = 0$.

Definition 3.1. Let $S^{m,b}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$ denote the subclass of p-valent functions which satisfy the condition

$$1 + \frac{1}{\gamma}\left(\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z)} - p\right) \prec \frac{1 + Az}{1 + Bz}, \tag{3.1}$$

where $p \in \mathbb{N}$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda_2 \geqslant \lambda_1 \geqslant 0$, $m, b \in \mathbb{N}_0$, $-1 \leqslant B < A \leqslant 1$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, $(i = 1, \ldots, r, q = 1, \ldots, s)$, and $r \leqslant s + 1$; $r, s \in \mathbb{N}_0$.

Theorem 3.1. *Let $f \in \mathscr{A}_p$. If it satisfies the condition:*

$$\frac{\sum_{n=1}^{\infty}\left[n + |\gamma(A - B) - nB|\right]\left[\frac{p + (\lambda_1 + \lambda_2)n + b}{p + \lambda_2 n + b}\right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n\, n!}|a_{p+n}|}{|\gamma|(A - B)} \leqslant 1, \tag{3.2}$$

then $f \in S^{m,b}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$.

$\lhd$ Let $f \in S^{m,b}_{\lambda_1,\lambda_2,p}[a_1, b_1, A, B, \gamma]$, then we can write (3.1) as follows:

$$1 + \frac{1}{\gamma}\left(\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z)} - p\right) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

which gives

$$\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z)} - p = \left[\gamma(A - B) - B\left(\frac{z(D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z))'}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_1, b_1)f(z)} - p\right)\right] w(z). \tag{3.3}$$

From (3.3), we obtain

$$\frac{pz^p + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}(p+n)a_{p+n}z^{p+n}}{z^p + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{p+n}} - p$$

$$= \left\{\gamma(A-B) - B\left[\frac{pz^p + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}(p+n)a_{p+n}z^{p+n}}{z^p + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{p+n}} - p\right]\right\} w(z)$$

which yields

$$\frac{\sum_{n=1}^{\infty} n\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}}{1 + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}}$$

$$= \left\{\gamma(A-B) - B\left[\frac{n\sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}}{1 + \sum_{n=1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}}\right]\right\} w(z).$$

Since $|w(z)| \leqslant 1$,

$$\left|\sum_{n=1}^{\infty} n\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}\right|$$

$$\leqslant \left|\gamma(A-B) - \sum_{n=1}^{\infty}[Bn-\gamma(A-B)]\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}a_{p+n}z^{n}\right|.$$

Letting $|z| \to 1^-$ through real values, we have

$$\sum_{n=1}^{\infty}[n + |\gamma(A-B) - Bn|]\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}|a_{p+n}| \leqslant |\gamma|(A-B),$$

therefore,

$$\frac{\sum_{n=1}^{\infty}[n + |\gamma(A-B) - Bn|]\left[\frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2 n+b}\right]^m \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n\, n!}|a_{p+n}|}{|\gamma|(A-B)} \leqslant 1.$$

REMARK 3.1. Other works related to different classes of p-valent functions can be found in [15, 16].

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ИССЛЕДОВАНИЕ ОДНОГО КЛАССА p-ВАЛЕНТНЫХ ФУНКЦИЙ, ПОРОЖДЕННОГО ОБОБЩЕННОЙ ГИПЕРГЕОМЕТРИЧЕСКОЙ ФУНКЦИЕЙ

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Изучаются свойства мажорации для нового класса аналитических p-валентных функций комплексного порядка, порожденного гипергеометрической функцией. Приводятся некоторые известные следствия полученных результатов. Даны также оценки коэффициентов для этого класса.

Ключевые слова: мажорация, p-валентная функция, гипергеометрическая функция.