Владикавказский математический журнал
2015, Том 17, Выпуск 2, С. 32–36

УДК **510.67+512.55**

ARTIN'S THEOREM FOR f-RINGS[1]

A. G. Kusraev

To Vladimir Kojbaev
on occasion of his 60th birthday

The main result states that each positive polynomial p in N variables with coefficients in a unital Archimedean f-ring K is representable as a sum of squares of rational functions over the complete ring of quotients of K provided that p is positive on the real closure of K. This is proved by means of Boolean valued interpretation of Artin's famous theorem which answers Hilbert's 17th problem affirmatively.

Mathematics Subject Classification (2010): 03C25, 12D15, 13B25.

Key words: f-ring, complete ring of quotients, real closure, polynomial, rational function, Artin's theorem, Hilbert 17th problem, Boolean valued representation.

The aim of this note is to prove that each positive polynomial p in N variables with coefficients in a unital Archimedean f-ring K is representable as a sum of squares of rational functions over the complete ring of quotients of K provided that p is positive on the real closure of K. For an ordered field K this is Artin's famous theorem which answers Hilbert's 17th problem affirmatively.

Recall some basic notions of the theory of rings; see J. Lambek [11]. Everywhere below K is a commutative unital ring. The *complete ring of quotients* of a commutative ring K is denoted by $Q(K)$. We call K *rationally complete* if $Q(K) \simeq K$ canonically or, equivalently, every irreducible fraction has domain K. Given a subset A of a commutative ring K, define the *annihilator* of A as $A^* := \{k \in K : kA = \{0\}\}$. The ideals of the form A^* are called *annihilator ideals*. Thus J is an annihilator ideal if and only if $J = A^*$ for some subset A of K, and this is equivalent to saying that $J^{**} := (J^*)^* = J$. A commutative ring K is called *semiprime* if its prime radical is 0, that is if it has no nonzero nilpotent elements. The annihilator ideals in a commutative semiprime ring K form a complete Boolean algebra $\mathbb{A}(K)$, with intersection as infimum and annihilator as complementation. If K is commutative, semiprime, and rationally complete, then every annihilator of K is a direct summand and $\mathbb{P}(K) \simeq \mathbb{A}(K)$ with $\mathbb{P}(K)$ being the Boolean algebra of idempotents of K. The *lateral* (or *orthogonal*) completion of a commutative semiprime ring K is the least subring $K' \subset Q(K)$ such that for all families $(x)_{\xi\in\Xi}$ in K and $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(Q(K))$ with $e_\xi e_\eta = 0$ $(\xi \neq \eta)$ there exists $x \in K'$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$.

The ring K is called *formally real* if $a_1^2 + \dots + a_n^2 \in J$ implies $a_1, \dots, a_n \in J$ for every finite collection $a_1, \dots, a_n \in K$ and every $J \in \mathbb{A}(K)$ or, in terminology of J. Bochnak, M. Coste,

[1]The study was supported by a grant from the Russian Foundation for Basic Research, project 14-01-91339.

and M.-F. Roy [3, Definition 4.1.3], every annihilator ideal in K is real. A semiprime regular ring K is real if and only if $a_1^2 + \dots + a_n^2 = 0$ implies $a_1 = \dots = a_n = 0$ for all $a_1, \dots, a_n \in K$ and $n \in \mathbb{N}$, since every principal annihilator ideal is a direct summand.

Consider commutative unital rings K and L. Say that L *extends* K if K is a subring of L and the mapping $J \mapsto J \cap K$ is one-to-one from $\mathbb{A}(L)$ onto $\mathbb{A}(K)$. Say also that L is *locally algebraic* over K whenever L extends K and, given $x \in L$ and a nonzero $I \in \mathbb{A}(K)$, there exist a nonzero $J \in \mathbb{A}(K)$, natural $n \in \mathbb{N}$, and $a_0, \dots, a_n \in K$ such that $J \subset I$ and $a_0 + a_1 x + \dots + a_n x^n \in J^*$. In the case of of semiprime regular rings L is locally algebraic over K if and only if $\mathbb{P}(K) = \mathbb{P}(L)$ and, given $x \in L$, for every nonzero $d \in \mathbb{P}(K)$ there exist a nonzero $e \in \mathbb{P}(K)$, natural $n \in \mathbb{N}$, and $a_0, \dots, a_n \in K$ such that $e \leqslant d$ and $e(a_0 + a_1 x + \dots + a_n x^n) = 0$.

An *f-ring* is a lattice-ordered ring K such that $y \wedge z = 0$ implies $xy \wedge z = yx \wedge z = 0$ for all $x, y, z \in K_+$. A *band* (or *polar*) in K is each set of the form $A^\perp := \{k \in K : (\forall a \in A)\, |k| \wedge |a| = 0\}$ with $\varnothing \neq A \subset K$. The set of all bands $\mathbb{B}(K)$ in a semiprime Archimedean f-ring K coincides with $\mathbb{A}(K)$ and hence is a complete Boolean algebra, since $A^* = A^\perp$ for every $A \subset K$. In this note we consider only Archimedean f-rings. See more details in [2].

For a unital f-ring K the complete ring of quotients $Q(A)$ can be uniquely made an f-ring with K a sublattice of $Q(A)$. This result is due to F. W. Anderson [1]; see also [8, § 10]. Moreover, the Boolean algebras $\mathbb{P}(Q(K))$ and $\mathbb{A}(K)$ are isomorphic.

A *real closure* of a unital f-ring K is a rationally complete f-ring $\overline{K}$ satisfying the following conditions: 1) $Q(K)$ is a subring and sublattice of $\overline{K}$ with $\overline{K}$ extending $Q(K)$, 2) $\overline{K}$ is locally algebraic over $Q(A)$, and 3) if K' is rationally complete f-ring containing $\overline{K}$ as a subring and sublattice and locally algebraic over $Q(K)$ then $K' = \overline{K}$. Say that K is *real closed* whenever $K = \overline{K}$. See the general concept of real closed rings in [15]. The main result is stated next.

Theorem. *Let K be an Archimedean unital f-ring and let $\overline{K}$ be its real closure, so that the embeddings $K \subset Q(K) \subset \overline{K}$ hold. If a polynomial $p \in K[x_1, \dots, x_N]$ is positive, that is $p(a_1, \dots, a_N) \geqslant 0$ for all $(a_1, \dots, a_N) \in \overline{K}^N$, then the representation $q^2 p = \sum_{j=1}^m k_j p_j^2$ holds for some non-zero-divisors $0 < k_1, \dots, k_m \in Q(K)$ and some polynomials $p_1, \dots, p_m, q \in Q(K)[x_1, \dots, x_N]$ with $eq(a_1, \dots, a_N) = 0$ equivalent to $ep(a_1, \dots, a_N) = 0$ for all $e \in \mathbb{P}(Q(K))$ and $a_1, \dots, a_N \in K$.*

Remark 1. Our proof uses *Boolean valued analysis* which signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models, the *von Neumann universe* $\mathbb{V}$ and a specially-trimmed *Boolean-valued universe* $\mathbb{V}^{(\mathbb{B})}$. Comparative analysis is carried out by means of some interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ which rests on the functors of *canonical embedding* (or *standard name*) $X \mapsto X^\wedge \in \mathbb{V}^{(\mathbb{B})}$ $(X \in \mathbb{V})$, *descent* $\mathscr{X} \mapsto \mathscr{X}\!\downarrow \in \mathbb{V}$ $(\mathscr{X} \in \mathbb{V}^{(\mathbb{B})})$, and *ascent* $Y \mapsto Y\!\uparrow \in \mathbb{V}^{(\mathbb{B})}$ $(Y \subset \mathbb{V}^{(\mathbb{B})})$. Accordingly, our proof is merely an interpretation of Artin's theorem within $\mathbb{V}^{(\mathbb{B})}$, thus demonstrating how does a Boolean valued transfer principle work in real algebra (as presented in [3] and [14]).

Remark 2. In particular, each Archimedean unital f-ring K has a real closure unique up to K-isomorphism. This is a Boolean valued interpretation of Artin–Schreier Theorem for ordered fields; see [14, Theorem 1.3.14] and [13, Theorem 28.7].

Remark 3. For every $0 \neq a \in Q(K)$ there exists a least element $e_a \in \mathbb{P}(Q(K))$ with $e_a a = a$. Moreover, a is a non-zero-divisor of $Q(e_a K)$ and a^{-1} exists in $Q(e_a K)$. Now, given $p, q \in Q(K)[x_1, \dots, x_N]$, we can define $(p/q)(a_1, \dots, a_N) := p(a_1, \dots, a_N) q(a_1, \dots, a_N)^{-1}$ if $q(a_1, \dots, a_N) \neq 0$, while $(p/q)(a_1, \dots, a_N) := 0$, whenever $q(a_1, \dots, a_N) = 0$. Say that p/q is a *rational function* over $Q(K)$ and denote by $Q(K)(x_1, \dots, x_N)$ the set of all rational functions over $Q(K)$. Thus the above representation can be written as $p = \sum_{j=1}^m k_j (p_j^2/q^2)$.

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\mathbb{V}^{(\mathbb{B})}$ is the corresponding Boolean valued model of set theory and $[\![\varphi]\!] \in \mathbb{B}$ is the Boolean truth value of a set theoretic formula φ. All necessary information concerning Boolean values analysis can be found in [9] and [10].

We need the following important result due to E. I. Gordon [6]. Let K be a commutative semiprime ring and $\mathbb{B}$ the Boolean algebra of its annihilator ideals. Then there exist $\mathscr{K}, \mathscr{F} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, \mathscr{K}$ is an integral domain and $\mathscr{F}$ is the quotient field of $\mathscr{K} \,]\!] = \mathbb{1}$, $\mathscr{K}\!\downarrow$ is the lateral completion of K and $\mathscr{F}\!\downarrow$ is the complete ring of quotients of K. In this event $\mathscr{K}$ is called the *Boolean valued representation* of K. Details can be found in [9, Theorem 8.3.5].

Lemma 1. *Let K be a commutative semiprime ring and $\mathscr{K}$ be its Boolean valued representation in $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{A}(K)$. If L is a subring of K and $\mathscr{L} := L\!\uparrow \in \mathbb{V}^{(\mathbb{B})}$ then*

(1) $[\![\, \mathscr{L}$ *is a subring of* $\mathscr{K} \,]\!] = \mathbb{1} \iff K$ *extends* L.

(2) $[\![\, \mathscr{K}$ *is algebraic over* $\mathscr{L} \,]\!] = \mathbb{1} \iff K$ *is locally algebraic over* L.

(3) $[\![\, \mathscr{K}$ *is algebraic closure of* $\mathscr{L} \,]\!] = \mathbb{1} \iff K$ *is algebraic closure of* L.

$\lhd$ This fact can be derived from Gordon's result [6] by straightforward calculation of Boolean truth values; cp. [9, Section 8.3]. $\rhd$

Lemma 2. *Let K, $\mathscr{K}$, L, and $\mathscr{K}$ be the same as in Lemma 1 and, moreover, K is an Archimedean f-ring extending L, while L is a sublattice of K. Then $[\![\mathscr{K}$ and $\mathscr{L}$ are totally ordered integral domains with K extending $\mathscr{L}]\!] = \mathbb{1}$ and K is a real algebraic closure of L if and only if $[\![\mathscr{K}$ is a field, a real algebraic closure of $\mathscr{L} \,]\!] = \mathbb{1}$.*

$\lhd$ The claim can be proved combining the above mentioned result by Anderson and Lemma 1 in the manner similar to that of [9, Theorem 8.5.6] taking into account the fact that an ordered field $\mathscr{K}$ admits a unique real closure up to $\mathscr{K}$-isomorphism; see [3, Theorem 1.3.2]. $\rhd$

Lemma 3. *Let $\mathscr{K}$ be a totally ordered integral domain, $\mathscr{K}'$ be its field of quotients and $\overline{\mathscr{K}}$ is a real closure of $\mathscr{K}'$. If a polynomial $p \in \mathscr{K}[x_1, \dots, x_N]$ is positive, i. e. $p(a_1, \dots, a_N) \geqslant 0$ for all $(a_1, \dots, a_N) \in \overline{\mathscr{K}}^N$, then $q^2 p = k_1 p_1^2 + \dots + k_m p_m^2$ for some $0 < k_1, \dots, k_m \in K'$, $p_1, \dots, p_m, q \in \mathscr{K}'[x_1, \dots, x_N]$ with $q(x_1, \dots, x_N) = 0$ if and only if $p(x_1, \dots, x_N) = 0$.*

$\lhd$ This is an improved version of Artin's theorem; see [3, Theorem 6.1.3], [12, Theorem 1.4.4], [13, Theorem 28.11], and [14, Theorem 2.1.12]. $\rhd$

Lemma 4. *Let K be an Archimedean unital f-algebra, $\mathbb{B} = \mathbb{P}(K)$, and let $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ be its Boolean valued representation. If $[\![\rho \in \mathscr{K}[x_1, \dots, x_N]]\!] = \mathbb{1}$ and $[\![\deg(\rho) \leqslant d^\wedge]\!]$ for some $d \in \mathbb{N}$ then $\rho\!\downarrow \in K'[x_1, \dots, x_N]$ with $K' := \mathscr{K}\!\downarrow$.*

$\lhd$ Assume that $[\![\rho \in \mathscr{K}[x_1, \dots, x_N]\,]\!] = \mathbb{1}$ and fix $u_1, \dots, u_N \in K$. Define $\mathbb{N}_d := \{1, \dots, d\}$ and identify N with $\{0, 1, \dots, N-1\}$. Observe that $(\mathbb{N}_d^\wedge)^{N^\wedge} = (\mathbb{N}_d^N)^\wedge$. There exist two mappings $\alpha, \varkappa : (\mathbb{N}_d^\wedge)^{N^\wedge} \to \mathscr{K}$ such that

$$\rho(u_1, \dots, u_N) = \sum_{\nu \in (\mathbb{N}_d^N)^\wedge} \alpha(\nu) \varkappa(\nu), \quad \varkappa(\nu) = \prod_{j \in N^\wedge} u_j^{\nu(j)}.$$

Let $a := \alpha\!\downarrow$ and $k\!\downarrow$ stand for the modified descents of α and $\varkappa$, respectively, so that $a, k : \mathbb{N}_d^N \to K'$ and $[\![a(\nu) = \alpha(\nu^\wedge)]\!] = \mathbb{1}$, $[\![k(\nu) = \varkappa(\nu^\wedge)]\!] = \mathbb{1}$ for all $\nu \in \mathbb{N}_d^N$, see [9, 5.7.7] and [10, § 1.5]. Define $p \in K'[x_1, \dots, x_N]$ as $p(x_1, \dots, x_N) := \sum_{\nu \in \mathbb{N}_d^N} a_\nu x_1^{\nu(1)} \cdot \ldots \cdot x_N^{\nu(N)}$, $a_\nu := a(\nu)$, and observe that for all $u_1, \dots, u_N \in K'$ we have

$$\left[\!\!\left[\rho(u_1, \dots, u_N) = \sum_{\nu \in \mathbb{N}_d^N} a_\nu \prod_{j \in N} u_j^{\nu(j)} = p(u_1, \dots, u_N) \right]\!\!\right] = \mathbb{1}.$$

It follows that $\rho\!\downarrow = p$. $\rhd$

◁ Proof of the Theorem. Let $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ be the Boolean valued representation of K with $\mathbb{B} = \mathbb{A}(K)$. Then $\mathscr{K}$ is an integral domain within $\mathbb{V}^{(\mathbb{B})}$. By the Boolean valued transfer principle and the maximum principle, within $\mathbb{V}^{(\mathbb{B})}$ there exist the field of quotients $\mathscr{K}'$ of $\mathscr{K}$ and the real closure $\overline{\mathscr{K}}$ of $\mathscr{K}'$. We may assume that $Q(K) = \mathscr{K}'{\downarrow}$ by the above mentioned Gordon's result and $\overline{K} = \overline{\mathscr{K}}{\downarrow}$ by Lemma 2. Take a polynomial $p \in K[x_1, \dots, x_N]$ and assume that $p(a_1, \dots, a_N) \geqslant 0$ for all $(a_1, \dots, a_N) \in \overline{K}^N$. Putting $\pi := p{\uparrow}$, one can prove by direct calculation of Boolean truth values that $\pi \in \mathscr{K}[x_1, \dots, x_N]$ and $\pi(a_1, \dots, a_N) \geqslant 0$ for all $(a_1, \dots, a_N) \in \overline{\mathscr{K}}^N$ within $\mathbb{V}^{(\mathbb{B})}$. By the transfer principle, Lemma 3 holds true within $\mathbb{V}^{(\mathbb{B})}$ and by the maximum principle there exist $m \in \mathbb{N}^{\wedge}$, $\pi_1, \dots, \pi_m, \rho \in \mathscr{K}'[x_1, \dots, x_N]$ such that $\rho^2\pi = \pi_1^2 + \dots + \pi_m^2$ and $\rho(x_1, \dots, x_N) = 0$ if and only if $\pi(x_1, \dots, x_N) = 0$ for all $x_1, \dots, x_N \in \mathscr{K}'$. Moreover, the number of squares $m \leqslant 2^{\wedge N^{\wedge}}$ (see [13, Theorem (Pfister) 29.3]) and the degrees of π_j^2 for every j and ρ^2 bounded by $D = 2^{2^{2^{\alpha}}}$, $\alpha := \deg(\pi)^{4^{N^{\wedge}}}$ (see [12, Theorem 1.4.4]). Observe now that $m \leqslant (2^{\wedge})^{A^{\wedge}} = (2^A)^{\wedge}$ for any finite set A, since $(\mathscr{P}_{\mathrm{fin}}(A))^{\wedge} = \mathscr{P}_{\mathrm{fin}}(A^{\wedge})$, see [9, Proposition 5.1.9]. Thus, we have $(2^N)^{\wedge} = 2^{N^{\wedge}}$ and $\alpha \leqslant \big(\deg(p)^{4^N}\big)^{\wedge}$. (We identify 2 with $2^{\wedge}$ and 4 with $4^{\wedge}$.) It follows the existence of $l \in \mathbb{N}$ with $D \leqslant l^{\wedge}$. Denote $q := \rho{\downarrow}$ and $p := \pi_j{\downarrow}$ $(j := 1, \dots, m)$. Then $q^2 p = \sum_{j=1}^{2^N} k_j p_j^2$ and $\pi_j, q \in Q(K)[x_1, \dots, x_N]$ by Lemma 4. ▷

Let $\mathscr{R}$ be the field of reals within $\mathbb{V}^{(\mathbb{B})}$. Then $\mathbf{R} := \mathscr{R}{\downarrow} \in \mathbb{V}$ (with the descended operations and order; see [9]) is a universally complete vector lattice, i. e. the externalization $\mathbf{R}$ of the internal Boolean valued reals $\mathscr{R}$ is a universally complete vector lattice. This remarkable result discovered by E. I. Gordon [5] tells us that each theorem on the reals (in the framework of Zermelo–Fraenkel set theory) has its counterpart for the corresponding universally complete vector lattices. In particular, $\mathbf{R}$ admits a unique f-ring multiplication for which a given order unit, a positive element $\mathbb{1} \in \mathbf{R}$ with $\{\mathbb{1}\}^{\perp} = \{0\}$, is a ring unit.

Corollary 1. *The vector lattice $\mathbf{R}$ is a real closed f-ring and each positive polynomial in $\mathbf{R}[x_1, \dots, x_N]$ is a sum of squares of rational functions in $\mathbf{R}(x_1, \dots, x_N)$.*

Two important particular cases of $\mathbf{R}$ were independently studied by G. Takeuti, who observed that the vector lattice of cosets of (almost everywhere equal) measurable function and a commutative algebra of (unbounded) self-adjoint operators in Hilbert space can be considered as instances of Boolean valued reals [16, 17].

Corollary 2. *Let (Ω, Σ, μ) be a Maharam measure space and let $L^0 := L^0(\Omega, \Sigma, \mu)$ be the f-ring of all cosets of real measurable functions on Ω. Then any positive polynomial in $L^0[x_1, \dots, x_N]$ is a sum of squares of rational functions in $L^0(x_1, \dots, x_N)$.*

◁ A vector lattice $L^0(\Omega, \Sigma, \mu)$ is a universally complete f-algebra (with identically one function as a ring unit) if and only if the measure space (Ω, Σ, μ) is Maharam (= localizable). ▷

Given a complete Boolean algebra $\mathbb{B}$ of projections in a Hilbert space H, denote by $\mathfrak{S}(\mathbb{B})$ the space of all selfadjoint operators on H whose spectral decompositions are in $\mathbb{B}$; i. e., $A \in \mathfrak{S}(\mathbb{B})$ if and only if $A = \int_{\mathbb{R}} \lambda \, dE_{\lambda}$ with $E_{\lambda} \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$, see [17]. For $A, B \in \mathfrak{S}(\mathbb{B})$ put $A \leqslant B$ if and only if $(Ax, x) \leqslant (Bx, x)$ for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B)$.

Corollary 3. *Let H be a complex Hilbert space and $\mathbb{B}$ a complete Boolean algebra of projections on H. Then any positive polynomial in $\mathfrak{S}(\mathbb{B})[x_1, \dots, x_N]$ is a sum of squares of rational functions in $\mathfrak{S}(\mathbb{B})(x_1, \dots, x_N)$.*

◁ $\mathfrak{S}(\mathbb{B})$ is a universally complete f-algebra (and hence a unital f-ring) [17, Ch. 1, § 3]. ▷

Remark 4. Corollary 2 can be considered (and also proved) as a measurable version of the 'continuous' solution of Hilbert's 17th problem obtained by C. N. Delzell and re-discovered by L. González-Vega and H. Lombardi, see [4] and [14, Theorem 4.3.4].

References

1. *Anderson F. W.* Lattice-ordered rings of quotients // Canad. J. Math.—1965.—Vol. 17.—P. 434–448.
2. *Bigard A., Keimel K., and Wolfenstein S.* Groupes et Anneaux Réticulés.—Berlin etc.: Springer-Verlag, 1977.—xi+334 p.—(Lecture Notes in Math., Vol. 608).
3. *Bochnak J., Coste M., and Roy M.-F.* Real Algebraic Geometry.—Berlin a. o.: Springer, 1998.—x+430 p.
4. *Delzell C. N., González-Vega L., and Lombardi H.* A continuous and rational solution to Hilbert's 17th problem and several cases of the Positivstellensatz // Computational Alg. Geom. / Eds. F. Eyssette and A. Galligo.—Birkhäuser: Boston a. o., 1993.—P. 61–75.—(Progress in Math., Vol. 109).
5. *Gordon E. I.* Real numbers in Boolean-valued models of set theory and K-spaces // Dokl. Akad. Nauk SSSR.—1977.—Vol. 237, № 4.—P. 773–775.
6. *Gordon E. I.* Rationally Complete Semiprime Commutative Rings in Boolean Valued Models of Set Theory.—Gor'kiĭ, 1983.—35 p.—(VINITI, № 3286-83).
7. *Henriksen M. and Isbell J. R.* Lattice-ordered rings and function rings // Pacific J. Math.—1962.—Vol. 12.—P. 533–565.
8. *Knebusch M. and Zhang D.* Convexity, Valuations and Prüfer extensions in real algebra // Documenta Math.—2005.—Vol. 10.—P. 1–109.
9. *Kusraev A. G. and Kutateladze S. S.* Introduction to Boolean Valued Analysis [in Russian].—M.: Nauka, 2005.—526 p.
10. *Kusraev A. G. and Kutateladze S. S.* Boolean Valued Analysis: Selected Topics.—Vladikavkaz: SMI VSC RAS, 2014.—iv+400 p.
11. *Lambek J.* Lectures on rings and modules.—Toronto: Blaisdell Publ. Comp., 1966.—183 p.—(AMS Chelesea publishing. Providence, R. I.).
12. *Lombardi H., Perrucci D., and Roy M.-F.* An elementary recursive bound for effective Positivstellensatz and Hilbert 17th problem.—2014.—arXiv:1404.2338v2 [math.AG].
13. *Prasolov V. V.* Polynomials.—Berlin–Heidelberg: Springer-Verlag, 2010.—301 p.
14. *Prestel A. and Delzell Ch. N.* Positive Polynomials: From Hilbert's 17th Problem to Real Algebra.—Berlin a. o.: Springer, 2001.—viii+267 p.
15. *Schwartz N.* The basic theory of real closed spaces.—Providence (R. I.): Amer. Math. Soc., 1989.—122 p.—(Mem. Amer. Math. Soc. Vol. 77 (397)).
16. *Takeuti G.* Two Applications of Logic to Mathematics.—Princeton: Princeton Univ. Press, 1978.
17. *Takeuti G.* A transfer principle in harmonic analysis // Symbolic Logic.—1979.—Vol. 44, № 3.—P. 417–440.

Received February 16, 2015.

Kusraev Anatoly Georgievich
Southern Mathematical Institute
Vladikavkaz Science Center of the RAS, *Director*
22 Markus street, Vladikavkaz, 362027, Russia
E-mail: `kusraev@smath.ru`

ТЕОРЕМА АРТИНА ДЛЯ f-КОЛЕЦ

Кусраев А. Г.

Основной результат заметки утверждает, что полином p от N переменных с коэфиициентами из унитарного архимедова f-кольца K представляется в виде суммы квадратов рациональных фукнций над полным кольцом частных кольца K, если только p положителен на вещественном замыкании K. Доказательство состоит в булевозначной интерпретации классической теоремы Артина, содержащей положительное решение 17-й проблемы Гильберта.

Ключевые слова: f-кольцо, полное кольцо частных, вещественное замыкание, полином, рациональная функция, теорема Артина, 17-я проблема Гильберта, булевозначное представление.