УДК **517.98**

REVERSIBLE AJW-ALGEBRAS

Sh. A. Ayupov, F. N. Arzikulov

The main result states that every special AJW-algebra can be decomposed into the direct sum of totally irreversible and reversible subalgebras. In turn, every reversible special AJW-algebra decomposes into a direct sum of two subalgebras, one of which has purely real enveloping real von Neumann algebra, and the second one contains an ideal, whose complexification is a C*-algebra and the annihilator of this complexification in the enveloping C^*-algebra of this subalgebra is equal to zero.

Mathematics Subject Classification (2000): 17C65, 46L57.

AJW-algebra, reversible AJW-algebra, AW*-algebra, Enveloping C*-algebra.

1. Introduction

This article is devoted to abstract Jordan operator algebras, which are analogues of abstract W*-algebras (AW*-algebras) of Kaplansky. These Jordan operator algebras can be characterized as a JB-algebra satisfying the following conditions

(1) in the partially ordered set of all projections any subset of pairwise orthogonal projections has the least upper bound in this JB-algebra;

(2) every maximal associative subalgebra of this JB-algebra is generated by it's projections (i. e. coincides with the least closed subalgebra containing all projections of the given subalgebra).

In the articles [3, 4] the second author introduced analogues of annihilators for Jordan algebras and gave algebraic conditions equivalent to (1) and (2). Currently, these JB-algebras are called AJW-algebras or Baer JB-algebras in the literature. Further, in [5] a classification of these algebras has been obtained. It should be noted that many of facts of the theory of JBW-algebras and their proofs hold for AJW-algebras. For example, similar to a JBW-algebra an AJW-algebra is the direct sum of special and purely exceptional Jordan algebras [5].

It is known from the theory of JBW-algebras that every special JBW-algebra can be decomposed into the direct sum of totally irreversible and reversible subalgebras. In turn, every reversible special JBW-algebra decomposes into a direct sum of a subalgebra, which is the hermitian part of a von Neumann algebra and a subalgebra, enveloping real von Neumann algebra of which is purely real [6, 7]. In this paper we prove a similar result for AJW-algebras, the proof of which requires a different approach. Namely, we prove that for every special AJW-algebra A there exist central projections $e, f, g \in A$, $e+f+g = 1$ such that (1) eA is reversible and there exists a norm-closed two sided ideal I of $C^*(eA)$ such that $eA = {}^\perp({}^\perp(I_{sa})_+)_+$; (2) fA is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$; (3) gA is a totally nonreversible AJW-algebra.

2. Preliminary Notes

We fix the following terminology and notations.

Let $\mathscr{A}$ be a real Banach $*$-algebra. $\mathscr{A}$ is called a real C*-algebra, if $\mathscr{A}_c = \mathscr{A} + i\mathscr{A} = \{a + ib : a, b \in \mathscr{A}\}$, can be normed to become a (complex) C*-algebra, and keeps the original norm on $\mathscr{A}$ [11].

Let A be a JB-algebra, $P(A)$ be a set of all projections of A. Further we will use the following standard notations: $\{aba\} = U_a b := 2a(ab) - a^2 b$, $\{abc\} = a(bc) + (ac)b - (ab)c$ and $\{aAb\} = \{\{acb\} : c \in A\}$, where $a, b, c \in A$. A JB-algebra A is called an AJW-algebra, if the following conditions hold:

(1) in the partially ordered set $P(A)$ of projections any subset of pairwise orthogonal projections has the least upper bound in A;

(2) every maximal associative subalgebra A_o of the algebra A is generated by it's projections (i. e. coincides with the least closed subalgebra containing $A_o \cap P(A)$).

Let

$$(S)^\perp = \{a \in A : (\forall x \in S) \, U_a x = 0\},$$
$$^\perp(S) = \{x \in A : (\forall a \in S) \, U_a x = 0\},$$
$$^\perp(S)_+ = {}^\perp(S) \cap A_+.$$

Then for a JB-algebra A the following conditions are equivalent:

(1) A is an AJW-algebra;

(2) for every subset $S \subset A_+$ there exists a projection $e \in A$ such that $(S)^\perp = U_e(A)$;

(3) for every subset $S \subset A$ there exists a projection $e \in A$ such that $^\perp(S)_+ = U_e(A_+)$ [3].

Let A be a real or complex $*$-algebra, and let S be a nonempty subset of A. Then the set $R(S) = \{x \in A : sx = 0 \text{ for all } s \in S\}$ is called the right annihilator of S and the set $L(S) = \{x \in A : xs = 0 \text{ for all } s \in S\}$ is called the left annihilator of S. A $*$-algebra A is called a Baer $*$-algebra, if the right annihilator of any nonempty set $S \subseteq A$ is generated by a projection, i. e. $R(S) = gA$ for some projection $g \in A$ $(g^2 = g = g^*)$. If $S = \{a\}$ then the projection $1 - g$ such that $R(S) = gA$ is called the right projection and denoted by $r(a)$. Similarly one can define the left projection $l(a)$. A (real) C*-algebra A, which is a Baer (real) $*$-algebra, is called an (real) AW*-algebra [1, 8]. Real AW*-algebras were introduced and investigated in [1, 2]. In these papers it was shown that for a real AW*-algebra $\mathscr{A}$ the C*-algebra $M = \mathscr{A} + i\mathscr{A}$ is not necessarily a complex AW*-algebra.

Let A be an AJW-algebra. By [5, Theorem 2.3] we have the equality $A = A_I \oplus A_{II} \oplus A_{III}$, where A_I is an AJW-algebra of type I, A_{II} is an AJW-algebra of type II and A_{III} is an AJW-algebra of type III [5]. By [5, Theorem 3.7] A_I, in its turn, is a direct sum of the following form

$$A_I = A_\infty \oplus A_1 \oplus A_2 \oplus \ldots,$$

where A_n for every n either is $\{0\}$ or an AJW-algebra of type I$_n$, A_∞ is a direct sum of AJW-algebras of type I$_\alpha$ with α infinite. If $A = A_1 \oplus A_2 \oplus \ldots$ then A is called an AJW-algebra of type I$_{fin}$ and denoted by $A_{I_{fin}}$ and if $A = A_\infty$ then A is called an AJW-algebra of type I$_\infty$ and denoted by A_{I_∞}. We say that A is properly infinite if A has no nonzero central modular projection. The fact that an AJW-algebra A_{II} of type II is a JC-algebra can be proved similar to JBW-algebras [9]. Therefore, it is isomorphic to some AJW-algebra defined in [14] (i. e. to some AJW-algebra of self-adjoint operators), and by virtue of [14] $A_{II} = A_{II_1} \oplus A_{II_\infty}$, where A_{II_1} is a modular AJW-algebra of type II and A_{II_∞} is an AJW-algebra of type II, which is properly infinite. So, we have the decomposition

$$A = A_{I_{fin}} \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}.$$

It is easy to verify that the part $A_{I_{fin}} \oplus A_{II_1}$ is modular, and $A_{I_\infty} \oplus A_{II_\infty} \oplus A_{III}$ is properly infinite (i. e. properly nonmodular).

3. Reversibility of AJW-algebras

Let A be a special AJW-algebra on a complex Hilbert space H. By $R^*(A)$ we denote the uniformly closed real $*$-algebra in $B(H)$, generated by A, and by $C^*(A)$ the C*-algebra, generated by A. Thus the set of elements of kind

$$\sum_{i=1}^{n} \prod_{j=1}^{m_i} a_{ij} \quad (a_{ij} \in A)$$

is uniformly dense in $R^*(A)$. Let $iR^*(A)$ be the set of elements of kind ia, $a \in R^*(A)$. Then $C^*(A) = R^*(A) + iR^*(A)$ [7, 13].

Lemma 3.1. *The set $R^*(A) \cap iR^*(A)$ is a uniformly closed two sided ideal in $C^*(A)$.*

◁ If $a, b \in R^*(A)$ and $c = id \in R^*(A) \cap iR^*(A)$, then $(a+ib)c = ac+ibid = ac-bd \in R^*(A)$. Similarly $(a+ib)c \in iR^*(A)$, i. e. $(a+ib)c \in R^*(A) \cap iR^*(A)$. Since $R^*(A) \cap iR^*(A)$ is uniformly closed and the set of elements of kind $a + ib$, $a, b \in A$ is uniformly dense in $C^*(A)$, we have $R^*(A) \cap iR^*(A)$ is a left ideal in $C^*(A)$. By the symmetry $R^*(A) \cap iR^*(A)$ is a right ideal. ▷

Let R be a $*$-algebra, R_{sa} be the set of all self-adjoint elements of R, i. e. $R_{sa} = \{a \in R : a^* = a\}$.

DEFINITION 3.2. A JC-algebra A is said to be reversible if $a_1 a_2 \ldots a_n + a_n a_{n-1} \ldots a_1 \in A$ for all $a_1, a_2, \ldots, a_n \in A$.

Similar to JW-algebras we have the following criterion.

Lemma 3.3. *An AJW-algebra A is reversible if and only if $A = R^*(A)_{sa}$.*

◁ It is clear that, if $A = R^*(A)_{sa}$, then A is reversible since

$$\left(\prod_{i=1}^{n} a_i + \prod_{i=n}^{1} a_i \right)^* = \prod_{i=n}^{1} a_i + \prod_{i=1}^{n} a_i \in R^*(A)_{sa} = A,$$

for all $a_1, a_2, \ldots, a_n \in A$. Conversely, let A be a reversible AJW-algebra. The inclusion $A \subset R^*(A)_{sa}$ is evident. If $a = \sum_{i=1}^{n} \prod_{j=1}^{m_i} a_{ij} \in R^*(A)_{sa}$, then

$$a = \frac{1}{2}(a + a^*) = \frac{1}{2} \sum_{i=1}^{n} \left(\prod_{j=1}^{m_i} a_{ij} + \prod_{j=m_i}^{1} a_{ij} \right) \in A.$$

Hence the converse inclusion holds, i. e. $R^*(A)_{sa} = A$. ▷

Lemma 3.4. *Let A be an AJW-algebra and let I be a norm-closed ideal of A. Then there exists a central projection g such that $^\perp(^\perp(I_{sa})_+)_+ = gA_+$.*

◁ Since A is an AJW-algebra there exists a projection g in A such that

$$^\perp(I_{sa})_+ = U_{(1-g)}(A_+), \quad ^\perp(^\perp(I_{sa})_+)_+ = U_g(A_+),$$

where $^\perp(S)_+ = \{x \in A_+ : (\forall a \in S)\ U_a x = 0\}$ for $S \subseteq A$.

Let (u_λ) be an approximate identity of the JB-subalgebra I and a be an arbitrary positive element in I. Then there exists a maximal associative subalgebra A_o of A containing a. Let

v_μ be an approximate identity of A_o. Then $(v_\mu) \subseteq (u_\lambda)$ and $\|av_\mu - a\| \to 0$. Let $b \in A_+$ and $U_{v_\mu}b = 0$ for every μ. Then $U_a U_{v_\mu} b = U_{av_\mu} b = 0$ and $U_c U_{av_\mu} b = 0$, where c is an element in A such that $b = c^2$. Hence $U_c U_{av_\mu} c^2 = 0$, $(U_c(av_\mu))^2 = 0$, $U_c(av_\mu) = 0$ and $U_c U_c(av_\mu) = U_b(av_\mu) = 0$ for every μ. We have

$$\|U_b(av_\mu) - U_b a\| = \|U_b(av_\mu) - a)\| \to 0$$

because $\|av_\mu - a\| \to 0$ and the operator U_b is norm-continuous. Hence $U_b a = 0$. We may assume that $a = d^2$ for some element $d \in A$. Then

$$U_d U_b a = U_d U_b d^2 = (U_d b)^2 = 0, \quad U_d b = 0.$$

Thus $U_d U_d b = U_{d^2} b = U_a b = 0$. Therefore, if $b \in^\perp ((u_\lambda))_+$ then $b \in^\perp (I_{sa})_+$. Hence $^\perp((u_\lambda))_+ \subseteq^\perp (I_{sa})_+$. It is clear that $^\perp(I_{sa})_+ \subseteq^\perp ((u_\lambda))_+$ and

$$^\perp(I_{sa})_+ =^\perp ((u_\lambda))_+.$$

This implies that $^\perp((u_\lambda))_+ = U_{(1-g)}(A_+)$ and

$$\sup_\lambda u_\lambda = g.$$

Let us prove that $U_g(A)$ is an ideal of A. Indeed, let x be an arbitrary element in A. Then $U_x u_\lambda \in I_{sa}$, i. e. $U_x u_\lambda \in U_g(A)$. By [9, Proposition 3.3.6] and the proof of [9, Lemma 4.1.5] we have U_x is a normal operator in A. Hence

$$\sup_\lambda U_x u_\lambda = U_x(\sup_\lambda u_\lambda) = U_x g.$$

At the same time

$$\sup_\lambda U_x u_\lambda \in U_g(A).$$

Hence $U_x g \in U_g(A)$. By [9, 2.8.10] we have

$$4(xg)^2 = 2gU_x g + U_x g^2 + U_g x^2 = 2gU_x g + U_x g + U_g x^2.$$

Therefore $(xg)^2 \in U_g(A)$ and $xg \in U_g(A)$.

Now, let y be an arbitrary element in $U_g A$. Then $y = U_g y$ and

$$xy = (U_g x + \{gx(1-g)\} + U_{1-g}x)U_g y = U_g x U_g y + \{gx(1-g)\}U_g y \in U_g A$$

since $\{gx(1-g)\} \in U_g A$. Hence $U_g A$ is a norm-closed ideal of A. Therefore $\{gA(1-g)\} = \{0\}$ and

$$A = U_g A \oplus U_{1-g} A.$$

This implies that g is a central projection in A and $^\perp(^\perp(I_{sa})_+)_+ = gA_+$. ▷

Lemma 3.5. *Let A be a reversible AJW-algebra on a Hilbert space H. Then there exist two central projections e, f in A and a norm-closed two sided ideal I of $C^*(A)$ such that $e + f = 1$, $eA =^\perp(^\perp(I_{sa})_+)_+$ and $R^*(fA) \cap iR^*(fA) = \{0\}$.*

◁ Let $I = R^*(A) \cap iR^*(A)$. Since A is reversible by Proposition 3.3 we have $I_{sa} \subseteq A$. By [7, 3.1] I is a two sided ideal of $C^*(A)$. Hence I_{sa} is an ideal of the AJW-algebra A. By Proposition 3.4 we have there exists a central projection g such that $^\perp(^\perp(I_{sa})_+)_+ = gA_+$. It is clear that g is a central projection also in $C^*(A)$.

By the definitions of I and g we have

$$R^*((1-g)A) \cap iR^*((1-g)A) = \{0\}. \quad ▷$$

Lemma 3.6. *Let A be an AJW-algebra and let J be the set of elements $a \in A$ such that $bac + c^*ab^* \in A$ for all $b, c \in C^*(A)$. Then J is a norm-closed ideal in A. Moreover J is a reversible AJW-algebra.*

$\lhd$ Let $a, b \in J$, $s, t \in C^*(A)$. Then

$$s(a + b)t + t^*(a + b)s^* = (sat + t^*as^*) + (sbt + t^*bs^*) \in A,$$

i. e. J is a linear subspace of A. Now, if $a \in J$, $b \in A$, $s, t \in C^*(A)$, then

$$s(ab + ba)t + t^*(ab + ba)s^* = (sa(bt) + (bt)^*as^*) + ((sb)at + t^*a(sb)^*) \in A,$$

i. e. J is a norm-closed ideal of A.

Let $a_1 \in J$, $a_2, \ldots, a_n \in A$ and $a = \prod_{i=2}^{n} a_i$. Then $a_1 a + a^* a_1 \in A$ by the definition of J. Let us show that $a_1 a + a^* a_1 \in J$; then, in particular, in the case of $a_2, \ldots, a_n \in J$ this will imply that J is reversible. For all b, $c \in C^*(A)$ we have

$$b(a_1 a + a^* a_1)c + c^*(a_1 a + a^* a_1)b^* = (ba_1(ac) + (ac)^* a_1 b^*) + ((ba^*)a_1 c + c^* a_1 (ba^*)^*) \in A,$$

i. e. $a_1 a + a^* a_1 \in J$. $\rhd$

DEFINITION 3.7. An AJW-algebra A is said to be *totally nonreversible*, if the ideal J in Lemma 3.6 is equal to $\{0\}$, i. e. $J = \{0\}$.

Theorem 3.8. *Let A be a special AJW-algebra. Then there exist central projections $e, f, g \in A$, $e + f + g = 1$ such that*

(1) $J = (e + f)A$, J is the ideal from Lemma 3.6;

(2) eA is reversible and there exists a norm-closed two sided ideal I of $C^*(eA)$ such that $eA = {}^\perp({}^\perp(I_{sa})_+)_+$;

(3) fA is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$;

(4) gA is a totally nonreversible AJW-algebra and

$$gA = \sum_{\omega \in \Omega} C(Q_\omega, \mathbf{R} \oplus H_\omega),$$

where Ω is a set of indices, $\{Q_\omega\}_{\omega \in \Omega}$ is an appropriate family of extremal compacts and $\{H_\omega\}_{\omega \in \Omega}$ is a family of Hilbert spaces.

$\lhd$ We have

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}$$

and the subalgebra (without the part A_2)

$$A_1 \oplus A_3 \oplus A_4 \oplus \cdots \oplus A_{I_\infty} \oplus A_{II_1} \oplus A_{II_\infty} \oplus A_{III}$$

is reversible. The last statement can be proven similar to [9, Theorem 5.3.10]. By [10] the subalgebra A_2 can be represented as follows

$$A_2 = \sum_{i \in \Xi} C(X_i, \mathbf{R} \oplus H_i),$$

where Ξ is a set of indices, $\{X_i\}_{i \in \Xi}$ is a family of extremal compacts and $\{H_i\}_{i \in \Xi}$ is a family of Hilbert spaces. Hence by [9, Theorem 6.2.5] there exist central projections h, g such that $A = hA \oplus gA$, hA is reversible and gA is totally nonreversible. For all $a, b_1, \ldots, b_n, c_1, \ldots, c_m$ in hA we have

$$b_1 \ldots b_n a c_1 \ldots c_m + c_m c_{m-1} \ldots c_1 a b_n b_{n-1} \ldots b_1 \in hA$$

since hA is reversible. Similarly for all $b, c \in R^*(hA)$, $a \in hA$ we have

$$bac + c^*ab^* \in hA.$$

Hence $hA = J$.

By Proposition 3.5 there exist two central projections e, f in hA and a norm-closed two sided ideal I of $C^*(hA)$ such that $e + f = h$, $eA = {}^\perp({}^\perp(I_{sa})_+)_+$, fA is a reversible AJW-algebra and $R^*(fA) \cap iR^*(fA) = \{0\}$. This completes the proof. $\triangleright$

Let A be a special AJW-algebra. Despite the fact that for the real AW*-algebra $R^*(A)$ the C*-algebra $\mathscr{M} = R^*(A) + iR^*(A)$ is not necessarily a complex AW*-algebra we consider, that

CONJECTURE. Under the conditions of Theorem 3.8 the following equality is valid

$$eA = I_{sa}.$$

References

1. *Albeverio S., Ayupov Sh. A. and Abduvaitov A. H.* On Real AW*-algebras. Methods of Functional Analysis and Topology.—2005.—Vol. 11, № 2.—P. 99–112.
2. *Albeverio S., Ayupov Sh. A. and Abduvaitov A. H.* On the coincidence of types of a real AW*-algebra and its comlexification // News of Russian Acad. of Sci. Math. ser.—2004.—Vol. 68.—P. 3–12.—[Russian].
3. *Arzikulov F. N.* On abstract JW-algebras // Sib. Math. J.—1998.—Vol. 39.—P. 20–27.
4. *Arzikulov F. N.* On an analog of the Peirce decomposition // Sib. Math. J.—1999.—Vol. 40.—P. 413–419.
5. *Arzikulov F. N.* AJW-algebras of type I and their classification // Sib. Adv. Math.—1998.—Vol. 8.—P. 30–48.
6. *Ayupov Sh. A.* Traces on JW-algebras and enveloping W^*-algebras // Math. Z.—1987.—Vol. 194.—P. 15–23.
7. *Ayupov Sh.* Classification and representation of ordered Jordan algebras Fan.—Tashkent, 1986.—[Russian].
8. *Chilin V. I.* Partial ordered Baer involutive algebras // Sovrem. Problemy Matematiki, Itogi nauki i tehkniki. VINITI.—1985.—Vol. 27.—P. 99–128.—[Russian].
9. *Hanche-Olcen H., Størmer E.* Jordan Operator Algebras.—Boston etc: Pitman Publ. Inc., 1984.
10. *Kusraev A. G.* On the structure of AJW-algebras of type I_2 // Sib. Math. J.—1999.—Vol. 40.—P. 905–917.
11. *Li B.* Real operator algebras. Institute of Mathematics.—China: Acad. Sinica Beijing, 2002.
12. *Sakai S.* C^*-algebras and W^*-algebras.—Springer, 1971.
13. *Størmer E.* Jordan algebras of type I // Acta. Math.—1966.—Vol. 115.—P. 165–184.
14. *Topping D. M.* Jordan algebras of self-adjoint operators // Mem. Am. Math. Soc.—1965.—Vol. 53.

AYUPOV SHAVKAT ABDULLAYEVICH
National University of Uzbekistan,
Director of the Institute of Math.
Do'rmon yo'li st., Tashkent, 1000125, Uzbekistan
Email: `sh_ayupov@mail.ru`

ARZIKULOV FARHODJON NEMATJONOVICH
Andizhan State University,
Department of Mathematics docent,
University street, Andizhan, 710020, Uzbekistan
Email: `arzikulovfn@rambler.ru`

ОБРАТИМЫЕ AJW-АЛГЕБРЫ

Аюпов Ш. А., Арзикулов Ф. Н.

Основной результат статьи гласит, что каждая специальная AJW-алгебра раскладывается в прямую сумму тотально необратимой и обратимой подалгебр. В свою очередь, каждая обратимая AJW-алгебра раскладывается в прямую сумму подалгебры, которая содержит идеал такой, что аннулятор комплексификации этого идеала в обертывающей C^*-алгебре этой подалгебры равен нулю и подалгебры, обертывающая вещественная алгебра фон Неймана которой является чисто вещественной.

Ключевые слова: AJW-алгебра, обратимая AJW-алгебра, AW^*-алгебра, обертывающая *-алгебра.