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ORDER BORNOLOGICAL LOCALLY CONVEX LATTICE CONES

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In this paper, we introduce the concepts of *us*-lattice cones and order bornological locally convex lattice cones. In the special case of locally convex solid Riesz spaces, these concepts reduce to the known concepts of seminormed Riesz spaces and order bornological Riesz spaces, respectively. We define solid sets in locally convex cones and present some characterizations for order bornological locally convex lattice cones.

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1. Introduction

The theory of locally convex cones as developed in [5] and [11], uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. Examples of locally convex cones contain classes of functions that take infinite values and families of convex subsets of vector spaces. These type of structures are not vector space and also may not even be embedded into a larger vector spaces in order to apply technics from topological vector spaces. These structures are studied in the general theory of locally convex cones. The class of bornological locally convex spaces is an important class of locally convex spaces which are introduced by Mackey in 1946. Every bounded linear operator on a bornological space is continuous. These structures have an advantage which they can be written as an inductive limit of seminormed spaces. Therefore every complete Hausdorff bornological locally convex space is the inductive limit of Banach spaces. We establish these results for locally convex cones in [3]. Also, We investigated the bornological convergence for cones in [2]. In the case of locally convex lattice cones, we want to study the order bornological locally convex lattice cones. The investigating of these structure is interesting, since these structures are the order inductive limit of *us*-lattice cones which are the extensions of seminormed Riesz spaces. We note that in the case of vector lattices the concept of separated *us*-lattice cones reduces to the concept of normed Riesz spaces and the concept of symmetric complete separated *us*-lattice cones reduces to the concept of Banach lattices, which have many applications in Economics. This research can be useful for researchers in mathematical economic theory. For recent researches see [2–4, 6, 9].

A *cone* is a set $\mathscr{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathscr{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathscr{P}$ and $\alpha, \beta \geqslant 0$.

Let $\mathscr{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathscr{P}^2 = \mathscr{P} \times \mathscr{P}$ is called *a convex quasiuniform structure* on $\mathscr{P}$, if the following properties hold:

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(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ $(\Delta = \{(a,a) : a \in \mathscr{P}\})$;
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathscr{P}^2$, by $U \circ V$ we mean the set of all $(a,b) \in \mathscr{P}^2$ such that there is some $c \in \mathscr{P}$ with $(a,c) \in U$ and $(c,b) \in V$.

Let $\mathscr{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathscr{P}$. We shall say $(\mathscr{P}, \mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathscr{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0, a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathscr{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathscr{P} : (b,a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in \mathscr{P} : (a,b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathscr{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathscr{W}$ be convex quasiuniform structures on $\mathscr{P}$. We say that $\mathfrak{U}$ is finer than $\mathscr{W}$ if $\mathscr{W} \subseteq \mathfrak{U}$.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. We set $\mathscr{V} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \left\{(a,b) \in \overline{\mathbb{R}}^2 : \ a \leqslant b + \varepsilon\right\}.$$

Then $\tilde{\mathscr{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathscr{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathscr{P}$ and $\mathscr{Q}$, a mapping $T : \mathscr{P} \to \mathscr{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathscr{P}$ and $\alpha \geqslant 0$. If both $(\mathscr{P}, \mathfrak{U})$ and $(\mathscr{Q}, \mathscr{W})$ are locally convex cones, the operator T is called (*uniformly*) continuous if for every $W \in \mathscr{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathscr{P}$ is a linear operator $\mu : \mathscr{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathscr{P}^*$ of a locally convex cone $(\mathscr{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathscr{P}$. The polar of the neighborhood $U \in \mathfrak{U}$ is defined as folows:

$$U^\circ = \{\mu \in \mathscr{P}^* : \ \mu(a) \leqslant \mu(b) + 1, \ \forall (a,b) \in U\}.$$

Let $\mathfrak{U}$ be a convex quasiuniform structure on $\mathscr{P}$. The subset $\mathscr{B}$ of $\mathfrak{U}$ is called a *base* for $\mathfrak{U}$, whenever for every $U \in \mathfrak{U}$ there are $n \in \mathbb{N}$, $U_1, \ldots, U_n \in \mathscr{B}$ and $\lambda_1, \ldots, \lambda_n > 0$ such that $\lambda_1 U_1 \cap \cdots \cap \lambda_n U_n \subseteq U$.

Suppose that $(\mathscr{P}, \mathfrak{U})$ is a locally convex cone. We shall say that $F \subseteq \mathscr{P}^2$ is *u-bounded* (*uniformly-bounded*) if it is absorbed by each $U \in \mathfrak{U}$. A subset A of $\mathscr{P}$ is called *bounded above* (*below*) whenever $A \times \{0\}$ (res. $\{0\} \times A$) is u-bounded (see [3]).

2. Solid sets and us-lattice cones

Locally convex lattice cones as a generalization of locally solid Riesz spaces has been introduced by Walter Roth in [10]. Here, we use the definition of this structure which have been presented in the terms of convex quasiuniform structures. We define solid sets in locally convex lattice cones and use them for our aim.

Definition 1. Let $\mathscr{P}$ be a cone and $\leqslant$ be a reflexive, transitive and antisymmetric order on $\mathscr{P}$ ($\mathscr{P}$ is an ordered cone). We shall say that $\mathscr{P}$ is a $\vee$ (or $\wedge$)-*lattice cone* whenever

(1) $a, b \in \mathscr{P}$ implies that $a \vee b \in \mathscr{P}$ (or $a \wedge b \in \mathscr{P}$);

(2) for $a, b, c \in \mathscr{P}$, $(a+c) \vee (b+c) = a \vee b + c$(or $(a+c) \wedge (b+c) = a \wedge b + c$).

The cone $\mathscr{P}$ is called a *lattice cone* if it is a $\vee$ and $\wedge$-lattice cone.

Let $\mathscr{P}$ and $\mathscr{Q}$ be $\vee$ (or $\wedge$)-lattice cones. The linear operator $T : \mathscr{P} \to \mathscr{Q}$ is called $\vee$ (or $\wedge$)-*lattice homomorphism* whenever $T(a \vee b) = T(a) \vee T(b)$ (or $T(a \wedge b) = T(a) \wedge T(b)$) for $a, b \in \mathscr{P}$.

Let E be a Riesz space. A subset A of E is called solid whenever $|b| \leqslant |a|$ and $a \in A$ imply that $b \in A$ (see [1]). We note that for $a \in E$, $|a| = a \vee (-a)$. Now, we present a definition of solid sets in lattice cones.

Definition 2. Let $\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice cone. We shall say that a subset B of $\mathscr{P}^2$, is $\vee$ (or $\wedge$)-*solid*, whenever

(1) $a \leqslant b$ implies that $(a, b) \in B$;

(2) $(a, b) \in \alpha B$ and $(c, b) \in \beta B$ imply that $(a \vee c, b) \in (\alpha+\beta)B$ (or $(a, b) \in \alpha B$ and $(a, c) \in \beta B$ imply that $(a, b \wedge c) \in (\alpha+\beta)B$).

If $\mathscr{P}$ is a lattice cone, the subset B is called *solid* whenever it is $\vee$-solid and $\wedge$-solid.

The $\vee$ (or $\wedge$)-solid hull of a subset B of $\mathscr{P}^2$ is the smallest (with respect to the set inclusion) $\vee$ (or $\wedge$)-solid subset of $\mathscr{P}^2$, which contains B, we denote it by $sh_{\vee}(B)$ (or $sh_{\wedge}(B)$). Also we denote the solid hull of B by $sh(B)$.

If E is a Riesz space and $A \subseteq E$ is solid (in the sense of the Riesz spaces) and convex, then $\tilde{A} = \{(c, b) \in E^2 : \exists\, a \in A,\ c \leqslant b + a\}$ is solid in the sense of lattice cones. Indeed, if $a \leqslant b$ for $a, b \in E$, then $(a, b) \in \tilde{A}$, since $0 \in A$ and $a \leqslant b + 0$. Now, let $(a, b) \in \gamma\tilde{A}$ and $(c, b) \in \lambda\tilde{A}$ for $\gamma, \lambda > 0$. Then we have $a \leqslant b + \gamma t$ and $c \leqslant b + \lambda t'$ for some $t, t' \in A$. Now, we have $t \vee 0, t' \vee 0 \in A$, since A is solid. Then $a \leqslant b + \gamma(t \vee 0) + \lambda(t' \vee 0)$ and $c \leqslant b + \gamma(t \vee 0) + \lambda(t' \vee 0)$. This shows that $a \vee c \leqslant b + \gamma(t \vee 0) + \lambda(t' \vee 0)$. Since A is convex, we conclude that $\gamma(t \vee 0) + \lambda(t' \vee 0) \in (\gamma+\lambda)A$. Therefore $(a \vee c, b) \in (\gamma+\lambda)\tilde{A}$. Similarly, we can prove that $\tilde{A}$ is $\wedge$-solid.

Definition 3. Let $\mathscr{P}$ be an ordered cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathscr{P}$. We shall say that $\mathfrak{U}$ is compatible with the order structure of $\mathscr{P}$ whenever $a \leqslant b$ implies that $(a, b) \in U$ for all $U \in \mathfrak{U}$ for $a, b \in \mathscr{P}$.

Definition 4. Let $\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice cone and $\mathfrak{U}$ be a compatible convex quasiuniform structure on $\mathscr{P}$ such that $(\mathscr{P}, \mathfrak{U})$ is a locally convex cone. Then we shall say that $(\mathscr{P}, \mathfrak{U})$ is a *locally convex* $\vee$ (or $\wedge$)-*lattice cone*, whenever $\mathfrak{U}$ has a base of $\vee$ (or $\wedge$)-solid sets. If $\mathfrak{U}$ has a base of $\vee$ (or $\wedge$)-solid sets, then it is called $\vee$ (or $\wedge$)-*solid convex quasiuniform structure*. If $\mathscr{P}$ is a lattice cone, then the convex quasiuniform structure $\mathfrak{U}$ is called *solid* whenever it has a base of solid sets. The locally convex cone $(\mathscr{P}, \mathfrak{U})$ is called *locally convex lattice cone* if $\mathfrak{U}$ has a base of solid sets.

Example 1. Let (E, τ) be a locally convex solid Riesz space. Then τ has a base $\mathscr{V}$ of solid, convex and balanced subsets. For $V \in \mathscr{V}$, we set $\tilde{V} = \big\{(a, b) \in E^2 : \exists\, v \in V,\ a \leqslant b + v\big\}$.

Then $\tilde{\mathscr{V}} = \{\tilde{V} : V \in \mathscr{V}\}$ is a solid convex quasiuniform structure on E. Therefore $(E, \tilde{\mathscr{V}})$ is a locally convex lattice cone.

Let $\mathscr{P}$ be a cone. A subset B of $\mathscr{P}^2$ is called *uniformly convex* whenever it has the properties (U_1) and (U_3). The locally convex cone $(\mathscr{P}, \mathfrak{U})$ is called a *uc-cone* whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$ (see [3]). If $\mathscr{P}$ is a $\vee$ (or $\wedge$)-lattice cone and U is $\vee$ (or $\wedge$)-solid, then $(\mathscr{P}, \mathfrak{U})$ is called $\vee_{us}$ (or $\wedge_{us}$)-lattice cone. In the case that $\mathscr{P}$ is a lattice cone and U is solid, $(\mathscr{P}, \mathfrak{U})$ is called *us-lattice cone.* For example normed Riesz spaces and Banach lattices are us-lattice cones as locally convex cones. Also the locally convex cone $(\overline{\mathbb{R}}, \tilde{\mathscr{V}})$ is a us-lattice cone. We note that every us-lattice cone is a locally convex lattice cone.

Let $\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice cone and $B \subseteq \mathscr{P}^2$. We denote the smallest uniformly convex and $\vee$ (or $\wedge$)-solid subset of $\mathscr{P}^2$, which contains B by $us_\vee(B)$ (or $us_\wedge(B)$), and we call it the *uniformly convex* $\vee$ (or $\wedge$)-*solid* hull of B. If $\mathscr{P}$ be a lattice cone, then we denote the uniformly convex solid hull of B, by $us(B)$.

Proposition 1. *In a locally convex $\vee$ (or $\wedge$)-lattice cone, the $\vee$ (or $\wedge$)-solid hull of a u-bounded set is u-bounded.*

$\lhd$ Let $(\mathscr{P}, \mathfrak{U})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone and B be a u-bounded subset of $\mathscr{P}^2$. Let $\mathfrak{B}$ be a base of $\vee$ (or $\wedge$)-solid sets for $\mathfrak{U}$. For every $U \in \mathfrak{B}$ there is $\lambda > 0$ such that $B \subseteq \lambda U$. This shows that $us_\vee(B) \subseteq us_\vee(\lambda U) = \lambda U$ (or $us_\wedge(B) \subseteq us_\wedge(\lambda U) = \lambda U$), since U is $\vee$ (or $\wedge$)-solid. Therefore $us_\vee(B)$ (or $us_\wedge(B)$) is u-bounded. $\rhd$

Corollary 1. *In a locally convex lattice cone, the solid hull of a u-bounded set is u-bounded.*

Proposition 2. *Let $\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice cone and $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)_{\gamma \in \Gamma}$ be a family of locally convex $\vee$ (or $\wedge$)-lattice cones. Also, let for every $\gamma \in \Gamma$, $g_\gamma : \mathscr{P} \to \mathscr{P}_\gamma$ is a $\vee$ (or $\wedge$)-lattice homomorphism. Then the coarsest convex quasiuniform structure $\mathfrak{U}$ on $\mathscr{P}$, which makes all g_γ continuous, is $\vee$ (or $\wedge$)-solid and $(\mathscr{P}, \mathfrak{U})$ is a locally convex $\vee$ (or $\wedge$)-lattice cone.*

$\lhd$ It is enough to show that for every $\gamma \in \Gamma$ and $\vee$ (or $\wedge$)-solid $U_\gamma \in \mathfrak{U}_\gamma$, $(g_\gamma \times g_\gamma)^{-1}(U_\gamma)$ is $\vee$ (or $\wedge$)-solid in the $\vee$ (or $\wedge$)-lattice cone $\mathscr{P}$. We prove the assertion for the case that $\mathscr{P}$ is a $\vee$-lattice cone. Indeed, let $a \leqslant b$ for $a, b \in \mathscr{P}$. Then $g_\gamma(a) \leqslant g_\gamma(b)$ for each $\gamma \in \Gamma$, since g_γ is a $\vee$-lattice homomorphism for each $\gamma \in \Gamma$. This implies that $(g_\gamma(a), g_\gamma(b)) \in U_\gamma$, since U_γ is $\vee$-solid for each $\gamma \in \Gamma$. Then $(a, b) \in (g_\gamma \times g_\gamma)^{-1}(U_\gamma)$. Now, let $(a, b) \in \alpha(g_\gamma \times g_\gamma)^{-1}(U_\gamma)$ and $(c, b) \in \beta(g_\gamma \times g_\gamma)^{-1}(U_\gamma)$ for $a, b, c \in \mathscr{P}$ and $\gamma \in \Gamma$. Then $(g_\gamma(a), g_\gamma(b)) \in \alpha U_\gamma$ and $(g_\gamma(c), g_\gamma(b)) \in \beta U_\gamma$. Now, since U_γ is $\vee$-solid an g_γ is $\vee$-lattice homomorphism, we conclude that $(g_\gamma(a \vee c), g_\gamma(b)) = (g_\gamma(a) \vee g_\gamma(c), g_\gamma(b)) \in (\alpha + \beta) U_\gamma$. Therefore $(a \vee c, b) \in (\alpha + \beta)(g_\gamma \times g_\gamma)^{-1}(U_\gamma)$. $\rhd$

Under the assumptions of Proposition 2, $(\mathscr{P}, \mathfrak{U})$ is called $\vee$ (or $\wedge$)-*order projective limit* of locally convex $\vee$ (or $\wedge$)-lattice cones $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)_{\gamma \in \Gamma}$ by the $\vee$ (or $\wedge$)-lattice homomorphisms g_γ, $\gamma \in \Gamma$. Similarly, the concept of *order projective limit* can be defined.

Proposition 3. *Every locally convex $\vee$ (or $\wedge$)-lattice cone is the $\vee$ (or $\wedge$)-order projective limit of some $\vee_{us}$ (or $\wedge_{us}$)-lattice cones.*

$\lhd$ Let $(\mathscr{P}, \mathfrak{U})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone. Then $\mathfrak{U}$ has a base $\mathfrak{B}$ of $\vee$ (or $\wedge$)-solid sets. For $B \in \mathfrak{B}$, we set $\mathfrak{U}_B = \{\alpha B : \alpha > 0\}$. Then $\mathfrak{U}_B$ is a $\vee$ (or $\wedge$)-solid convex quasiuniform structure on $\mathscr{P}$ and $(\mathscr{P}, \mathfrak{U}_B)$ is a locally convex $\vee$ (or $\wedge$)-lattice cone for each $B \in \mathfrak{B}$. Now, it is easy to see that $(\mathscr{P}, \mathfrak{U})$ is the $\vee$ (or $\wedge$)-order projective limit of $(\mathscr{P}, \mathfrak{U}_B)_{B \in \mathfrak{B}}$ by the identity mappings. $\rhd$

Corollary 2. *Every locally convex lattice cone is the order projective limit of some us-lattice cones.*

In the special case of locally convex solid Riesz spaces, Proposition 3 yields that every (Hausdorff) locally convex solid Riesz space is the projective limit of (normed) seminormed Riesz spaces.

3. Order bornological locally convex lattice cones

Suppose that $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ are locally convex cones and $T:\mathscr{P}\to\mathscr{Q}$ is a linear operator. We shall say T is *u-bounded* if $(T\times T)(F)$ is u-bounded in $\mathscr{Q}^2$ for every u-bounded subset F of $\mathscr{P}^2$. We shall say $(\mathscr{P},\mathfrak{U})$ is a *bornological cone* if every u-bounded linear operator from $(\mathscr{P},\mathfrak{U})$ into any locally convex cone is continuous (see [3]).

Let $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ be locally convex cones. The linear operator $T:\mathscr{P}\to\mathscr{Q}$ is called *bounded below* whenever T maps bounded below subsets of $\mathscr{P}$ into bounded below subsets of $\mathscr{Q}$. The locally convex cone $(\mathscr{P},\mathfrak{U})$ is called b-*bornological* whenever every bounded below linear operator from $(\mathscr{P},\mathfrak{U})$ into other locally convex cone is continuous (see [3]).

Bornological and b-bornological locally convex cones have been studied in [3]. Firstly, we review the construction of this structure briefly: Let $\mathscr{P}$ be a cone and U be a uniformly convex subset of $\mathscr{P}$. We set $\mathscr{P}_U=\{a\in\mathscr{P}:\exists\,\lambda>0,\ (0,a)\in\lambda U\}$ and $\mathfrak{U}_U=\{\alpha U:\alpha>0\}$. Then $(\mathscr{P}_U,\mathfrak{U}_U)$ is a locally convex cone (a *uc*-cone). In [3], we proved that there is the finest convex quasiuniform structure $\mathfrak{U}_\tau$ (or $\mathfrak{U}_{b\tau}$) on locally convex cone $(\mathscr{P},\mathfrak{U})$ such that $\mathscr{P}^2$ (or $\mathscr{P}$) has the same u-bounded (or bounded below) subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$(or $\mathfrak{U}_{b\tau}$). The locally convex cone $(\mathscr{P},\mathfrak{U}_\tau)$ is the inductive limit of the *uc*-cones $(\mathscr{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B}$ is the collection of all uniformly convex u-bounded subsets of $\mathscr{P}^2$. Also $(\mathscr{P},\mathfrak{U}_{b\tau})$ is the inductive limit of the *uc*-cones $(\mathscr{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B}=\{uch(\{0\}\times B): B \textit{ is bounded below}\}$. If $(\mathscr{P},\mathfrak{U})$ is bornological or b-bornological, then $\mathfrak{U}$ is equivalent to $\mathfrak{U}_\tau$ or $\mathfrak{U}_{b\tau}$, respectively.

Definition 5. We shall say that the locally convex $\vee$ (or $\wedge$)-lattice cone $(\mathscr{P},\mathfrak{U})$ is $\vee$ (or $\wedge$)-order bornological whenever every u-bounded $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P},\mathfrak{U})$ in to any locally convex $\vee$ (or $\wedge$)-lattice cone is continuous.

Obviously, every bornological locally convex $\vee$ (or $\wedge$)-lattice cone is $\vee$ (or $\wedge$)-order bornological. For example, every $\vee_{us}$ (or $\wedge_{us}$)-lattice cone is $\vee$ (or $\wedge$)-order bornological. Also every *us*-lattice cone is order bornological. It has been proved in [3], that every locally convex cone which its convex quasiuniform structure has countable base is bornological. This shows that if $(\mathscr{P},\mathfrak{U})$ is a locally convex lattice cone and $\mathfrak{U}$ has a countable base, then $(\mathscr{P},\mathfrak{U})$ is order bornological.

Example 2. Let X be a topological space, and let $\mathscr{P}$ be the cone of all $\overline{\mathbb{R}}_+$-valued continuous functions on X, where $\overline{\mathbb{R}}_+$ is endowed with the usual, that is the one-point compactification topology. We consider on $\mathscr{P}$ the pointwise order. For each $\varepsilon>0$, we set $\tilde{\varepsilon}=\{(f,g)\in\mathscr{P}^2:\forall\,x\in X,\ f(x)\leqslant g(x)+\varepsilon\}$. Then for each $\varepsilon>0$, $\tilde{\varepsilon}$ is a solid set and $\mathfrak{U}=\{\tilde{\varepsilon}:\varepsilon>0\}$ is a solid convex qusiuniform structure. Then $(\mathscr{P},\mathfrak{U})$ is a locally convex lattice cone. We note that $(\mathscr{P},\mathfrak{U})$ is order bornological locally convex lattice cone, since it is a *us*-lattice cone. The cone $\mathscr{P}$ is not a vector space and it may not be embedded in any vector space.

Theorem 1. *Let $(\mathscr{P}_\gamma,\mathfrak{U}_\gamma)_{\gamma\in\Gamma}$ be a family of locally convex $\vee$ (or $\wedge$)-lattice cones. Also let $\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice cone and for each $\gamma\in\Gamma$, $f_\gamma:\mathscr{P}_\gamma\to\mathscr{P}$ be a $\vee$ (or $\wedge$)-lattice homomorphism such that $\mathscr{P}=span\,(\bigcup_{\gamma\in\Gamma}f_\gamma(\mathscr{P}_\gamma))$. Then $\mathscr{P}$ endowed with the convex quasiuniform structure $\mathfrak{U}$ created by the sets of the form $us_\vee(\bigcup_{\gamma\in\Gamma}(f_\gamma\times f_\gamma)(U_\gamma))$ (or $us_\wedge(\bigcup_{\gamma\in\Gamma}(f_\gamma\times f_\gamma)(U_\gamma))$), where $U_\gamma\in\mathfrak{U}_\gamma$, is a locally convex $\vee$ (or $\wedge$)-lattice cone.*

$\lhd$ We consider the case that $(\mathscr{P}_\gamma,\mathfrak{U}_\gamma)_{\gamma\in\Gamma}$ are locally convex $\vee$-lattice cones. Firstly, we prove that the elements of $\mathscr{P}$ are bounded below with respect to the sets $us_\vee(\bigcup_{\gamma\in\Gamma}f_\gamma(U_\gamma))$.

Let $a \in \mathscr{P}$. Then there are $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $a_{\gamma_i} \in \mathscr{P}_{\gamma_i}$, $i = 1, \ldots, n$, such that $a = \sum_{i=1}^{n} f_{\gamma_i}(a_{\gamma_i})$. There are $\lambda_i > 0$, $i = 1, \ldots n$, such that $(0, a_{\gamma_i}) \in \lambda_i U_{\gamma_i}$. This shows that $(0, a) \in \lambda us(\bigcup_{\gamma\in\Gamma} f_\gamma(U_\gamma))$, where $\lambda = \max_{1\leqslant i\leqslant n} \lambda_i$. Then $(\mathscr{P}, \mathfrak{U})$ is a locally convex cone. Since the sets $us_\vee(\bigcup_{\gamma\in\Gamma} f_\gamma(U_\gamma))$ are $\vee$-solid, we conclude that $(\mathscr{P}, \mathfrak{U})$ is a locally convex $\vee$-lattice cone. A similar argument yields our claim for the case that $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)_{\gamma\in\Gamma}$ are locally convex $\wedge$-lattice cones. $\rhd$

The projective and inductive limits had been investigated for topological vector spaces and locally convex cones in [8] and [7], respectively. Under the assumptions of Theorem 1, $(\mathscr{P}, \mathfrak{U})$ is called the $\vee$ (or $\wedge$)-order inductive limit of the locally convex $\vee$ (or $\wedge$)-lattice cones $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)$, under the $\vee$ (or $\wedge$)-lattice homomorphisms $f_\gamma : \mathscr{P}_\gamma \to \mathscr{P}$. Similarly, the concept of order inductive limit can de defined.

Corollary 3. *An order inductive limit of locally convex lattice cones is a locally convex lattice cone.*

Corollary 4. *An order inductive limit of locally convex solid Riesz spaces is a locally convex solid Riesz space.*

Proposition 4. *Let $(\mathscr{P}, \mathfrak{U})$ be the $\vee$ (or $\wedge$)-order inductive limit of locally convex $\vee$ (or $\wedge$)-lattice cones $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)\gamma \in \Gamma$, under lattice homomorphisms $f_\gamma : \mathscr{P}_\gamma \to \mathscr{P}$, $\gamma \in \Gamma$, and $(\mathscr{Q}, \mathscr{W})$ be a locally convex cone. Then the linear mapping $T : \mathscr{P} \to \mathscr{Q}$ is continuous if and only if for every $\gamma \in \Gamma$, Tof_γ is continuous.*

$\lhd$ The mapping T is continuous if and only if for each $W \in \mathscr{W}$, $(T \times T)^{-1}(W) \in \mathfrak{U}$. By Theorem 1, this holds if and only if for every $\gamma \in \Gamma$

$$(f_\gamma \times f_\gamma)^{-1}((T \times T)^{-1}(W)) = (\mathrm{Tof}_\gamma \times \mathrm{Tof}_\gamma)^{-1}(W) \in \mathfrak{U}_\gamma.$$

In the other words, we require the continuity of each Tof_γ for each $\gamma \in \Gamma$. $\rhd$

Proposition 5. *An $\vee$ (or $\wedge$)-order inductive limit of $\vee$ (or $\wedge$)-order bornological locally convex lattice cones is $\vee$ (or $\wedge$)-order bornological.*

$\lhd$ Let $(\mathscr{P}, \mathfrak{U})$ be the $\vee$ (or $\wedge$)-order inductive limit of $\vee$ (or $\wedge$)-order bornological locally convex lattice cones $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)$ by the $\vee$ (or $\wedge$)-lattice homomorphisms f_γ, $\gamma \in \Gamma$. Also suppose that T be a u-bounded $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P}, \mathfrak{U})$ into another locally convex $\vee$ (or $\wedge$)-lattice cone $(\mathscr{Q}, \mathscr{W})$. Then for every $\gamma \in \Gamma$, Tof_γ is a u-bounded $\vee$ (or $\wedge$)-lattice homomorphism on $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)$. Since $(\mathscr{P}_\gamma, \mathfrak{U}_\gamma)$ is $\vee$ (or $\wedge$)-order bornological, we conclude that Tof_γ is continuous for each $\gamma \in \Gamma$, by Proposition 4. Therefore T is continuous by Proposition 4. $\rhd$

Similarly, one can prove that an order inductive limit of order bornological locally convex lattice cones is order bornological.

Theorem 2. *Let $(\mathscr{P}, \mathfrak{U})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone. Then there is the finest $\vee$ (or $\wedge$)-solid convex quasiuniform structure $\mathfrak{U}^{\vee}_{|\tau|}$ (or $\mathfrak{U}^{\wedge}_{|\tau|}$) on $\mathscr{P}$ under which $\mathscr{P}^2$ has the same u-bounded subsets as under $\mathfrak{U}$. Under the convex quasiuniform structure $\mathfrak{U}^{\vee}_{|\tau|}$ (or $\mathfrak{U}^{\wedge}_{|\tau|}$), $\mathscr{P}$ is a $\vee$ (or $\wedge$)-order bornological cone, the $\vee$ (or $\wedge$)-order inductive limit of a family of $\vee_{us}$ (or $\wedge_{us}$)-sublattice cones of $\mathscr{P}$. The locally convex cone $(\mathscr{P}, \mathfrak{U})$ is $\vee$ (or $\wedge$)-order bornological if and only if $\mathfrak{U}$ and $\mathfrak{U}^{\vee}_{|\tau|}$ (or $\mathfrak{U}^{\wedge}_{|\tau|}$) are equivalent.*

$\lhd$ We prove the theorem for the case that $(\mathscr{P}, \mathfrak{U})$ is a locally convex $\vee$-lattice cone. Let $\mathfrak{B}$ be the collection of all u-bounded $\vee$-solid subsets of $\mathscr{P}^2$. For $B \in \mathfrak{B}$, we set

$$\mathscr{P}_B = \{a \in \mathscr{P} : \exists\, \lambda > 0 \ s.t.\ (0, a) \in \lambda B\} \quad \text{and} \quad \mathfrak{U}_B = \{\alpha B : \alpha > 0\}.$$

We consider on $\mathscr{P}_B$ the order induced by the original order of $\mathscr{P}$. Since B is $\vee$-solid, it is easy to see that $(\mathscr{P}_B, \mathfrak{U}_B)$ is a locally convex $\vee_{us}$-lattice cone. We have $\mathscr{P} = \bigcup_{B\in\mathfrak{B}} \mathscr{P}_B$. Indeed, for $a \in \mathscr{P}$, Let B' be the smallest uniformly convex $\vee$-solid subset of $\mathscr{P}^2$, which contains $\{(0,a)\}$. Then $B' \in \mathfrak{B}$ and $a \in \mathscr{P}_{B'}$. Now let $(\mathscr{P}, \mathfrak{U}^{\vee}_{|\tau|})$ be the $\vee$-order inductive limit of $(\mathscr{P}_B, \mathfrak{U}_B)_{B\in\mathfrak{B}}$, under the inclusion mappings $I_B : \mathscr{P}_B \to \mathscr{P}$. Then $(\mathscr{P}, \mathfrak{U}^{\vee}_{|\tau|})$ is a locally convex $\vee$-lattice cone by Theorem 1. The u-boundedness of $B \in \mathfrak{B}$ shows that $I_B : (\mathscr{P}_B, \mathfrak{U}_B) \to (\mathscr{P}, \mathfrak{U})$ is continuous. Now, we conclude that $\mathfrak{U}^{\vee}_{|\tau|}$ is finer than $\mathfrak{U}$ by the definition of $\vee$-order inductive limit. Then u-boundedness in $\mathfrak{U}^{\vee}_{|\tau|}$ implies u-boundedness in $\mathfrak{U}$. On the other hand, if $F \subseteq \mathscr{P}^2$ is u-bounded with respect to $\mathfrak{U}$, then it is u-bounded in $(\mathscr{P}_{\tilde{F}}, \mathfrak{U}_{\tilde{F}})$, where $\tilde{F} = us_{\vee}(F)$. Now, the continuity of $I_{\tilde{F}}$ yields that F is u-bounded with respect to $\mathfrak{U}^{\vee}_{|\tau|}$. Also $(\mathscr{P}, \mathfrak{U}^{\vee}_{|\tau|})$ is $\vee$-order bornological by Proposition 5. For the final part of theorem, we note that the identity mapping $I : (\mathscr{P}, \mathfrak{U}) \to (\mathscr{P}, \mathfrak{U}^{\vee}_{|\tau|})$ is a u-bounded $\vee$-lattice homomorphism. Now, if $(\mathscr{P}, \mathfrak{U})$ is order bornological, then I is continuous. Therefore $\mathfrak{U}$ is finer than $\mathfrak{U}^{\vee}_{|\tau|}$. On the other hand $\mathfrak{U}_{|\tau|}$ is finer than $\mathfrak{U}$. Then they are equivalent. Similarly we can prove the theorem for the case that $(\mathscr{P}, \mathfrak{U})$ is a locally convex $\wedge$-lattice cone. $\rhd$

Corollary 5. *If $(\mathscr{P}, \mathfrak{U})$ is a $\vee$ (or $\wedge$)-order bornological locally convex $\vee$ (or $\wedge$)-lattice cone, then $\mathfrak{U} = \mathfrak{U}^{\vee}_{|\tau|}$ (or $\mathfrak{U} = \mathfrak{U}^{\wedge}_{|\tau|}$).*

Corollary 6. *If $(\mathscr{P}, \mathfrak{U})$ is a locally convex lattice cone, then there is the finest solid convex quasiuniform structure $\mathfrak{U}_{|\tau|}$ on $\mathscr{P}$, under which $\mathscr{P}^2$ has the same u-bounded subsets as under $\mathfrak{U}$. Under the convex quasiuniform structure $\mathfrak{U}_{|\tau|}$, $\mathscr{P}$ is an order bornological cone, the order inductive limit of a family of us-sublattice cones of $\mathscr{P}$. The locally convex cone $(\mathscr{P}, \mathfrak{U})$ is order bornological if and if $\mathfrak{U}$ and $\mathfrak{U}_{|\tau|}$ are equivalent.*

Proposition 6. *Let $(\mathscr{P}, \mathfrak{U})$ be a locally convex lattice cone. Then $\mathfrak{U}_{\tau}$ is finer than $\mathfrak{U}_{|\tau|}$.*

$\lhd$ Since $\mathscr{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}_{|\tau|}$, and $\mathfrak{U}_{\tau}$ is the finest convex quasiuniform structure that has this property, we conclude that $\mathfrak{U}_{\tau}$ is finer than $\mathfrak{U}_{|\tau|}$. $\rhd$

Proposition 7. *Every bornological locally convex lattice cone is order bornological.*

$\lhd$ Let $(\mathscr{P}, \mathfrak{U})$ be a bornological locally convex lattice cone. Then we have $\mathfrak{U} = \mathfrak{U}_{\tau}$. Now, since $\mathfrak{U} \subseteq \mathfrak{U}_{|\tau|} \subseteq \mathfrak{U}_{\tau}$, we conclude that $\mathfrak{U} = \mathfrak{U}_{|\tau|}$. $\rhd$

In the following theorem we characterize $\vee$ (or $\wedge$)-order bornological locally convex $\vee$ (or $\wedge$)-lattice cones

Theorem 3. *For locally convex $\vee$ (or $\wedge$)-lattice cone $(\mathscr{P}, \mathfrak{U})$ the followings are equivalent:*

(a) *$(\mathscr{P}, \mathfrak{U})$ is $\vee$(or $\wedge$)-order bornological;*

(b) *for every uniformly convex $\vee$ (or $\wedge$)-solid subset V of $\mathscr{P}^2$ that absorbs all u-bounded subsets, there is $U \in \mathfrak{U}$ such that $U \subseteq V$;*

(c) *every u-bounded $\vee$ (or $\wedge$)-lattice homomorphism from $\mathscr{P}$ into any $\vee_{us}$ (or $\wedge_{us}$)-lattice cone is continuous.*

(d) *$(\mathscr{P}, \mathfrak{U})$ is the $\vee$ (or $\wedge$)-order inductive limit of some us-lattice subcones of $(\mathscr{P}, \mathfrak{U})$.*

$\lhd$ The statements (a) and (d) are equivalent by Proposition 5 and Theorem 2.

($a \to b$): Let (a) holds and V be a uniformly convex $\vee$ (or $\wedge$)-solid subset of $\mathscr{P}^2$, that absorbs all u-bounded subsets. We set $\mathscr{V} = \{\alpha V : \alpha > 0\}$. Then $(\mathscr{P}, \mathscr{V})$ is a locally convex $\vee$ (or $\wedge$)-lattice cone. The identity mappings $I : (\mathscr{P}, \mathfrak{U}) \to (\mathscr{P}, \mathscr{V})$ is u-bounded, since V absorbs all u-bounded subsets. On the other hand I is a $\vee$ (or $\wedge$)-lattice homomorphism. Now (a) yields that I is continuous. Therefore there exists $U \in \mathfrak{U}$ such that $U \subseteq V$.

($b \to a$): Let (b) holds and T be a u-bounded $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P}, \mathfrak{U})$ into another locally convex $\vee$ (or $\wedge$)-lattice cone $(\mathscr{Q}, \mathscr{W})$. Let $W \in \mathscr{W}$ be $\vee$ (or $\wedge$)-solid. Then

$(T\times T)^{-1}(W)$ is a uniformly convex $\vee$ (or $\wedge$)-solid subset of $\mathscr{P}^2$, since T is a $\vee$ (or $\wedge$)-lattice homomorphism. Now there is $U\in\mathfrak{U}$ such that $U\subseteq (T\times T)^{-1}(W)$ by (b). Then T is continuous.

$(a\to c)$: The proof is clear.

$(c\to a)$: Suppose that (c) holds and T is a u-bounded $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P},\mathfrak{U})$ into another locally convex $\vee$ (or $\wedge$)-lattice cone $(\mathscr{Q},\mathscr{W})$. For $W\in\mathscr{W}$, we set $\mathscr{W}_W=\{\alpha W:\alpha>0\}$. Clearly for every $W\in\mathscr{W}$, $T:(\mathscr{P},\mathfrak{U})\to(\mathscr{Q},\mathscr{W}_W)$ is a u-bounded $\vee$ (or $\wedge$)-lattice homomorphism and then it is continuous by (c). Now, let $W\in\mathscr{W}$. Then we have $W\in\mathscr{W}_W$. Therefore there is $U\in\mathfrak{U}$ such that $(T\times T)(U)\subseteq W$. $\rhd$

Corollary 7. *As a special case in locally convex solid Riesz spaces, Theorem 3, (c) yields that a locally convex solid Riesz space E is order bornological if and only if every bounded lattice homomorphism from E into a seminormed Riesz space is continuous.*

Definition 6. Let $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ be locally convex $\vee$ (or $\wedge$)-lattice cones. We shall say that T is $|\tau|_\vee$ (or $|\tau|_\wedge$)-continuous whenever $T:(\mathscr{P},\mathfrak{U}^\vee_{|\tau|})\to(\mathscr{Q},\mathscr{W}^\vee_{|\tau|})$ (or $T:(\mathscr{P},\mathfrak{U}^\wedge_{|\tau|})\to(\mathscr{Q},\mathscr{W}^\wedge_{|\tau|})$) is continuous. Similarly, we can define the concept of $|\tau|$-continuity.

Proposition 8. *Let $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ be locally convex $\vee$ (or $\wedge$)-lattice cones. Then the $\vee$ (or $\wedge$)-lattice homomorphism $T:(\mathscr{P},\mathfrak{U})\to(\mathscr{Q},\mathscr{W})$ is u-bounded if and only if T is $|\tau|_\vee$ (or $|\tau|_\wedge$)-continuous.*

$\lhd$ Let $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ be locally convex $\vee$-lattice cones and $T:(\mathscr{P},\mathfrak{U})\to(\mathscr{Q},\mathscr{W})$ be a u-bounded $\vee$-lattice homomorphism. Then $T:(\mathscr{P},\mathfrak{U}^\vee_{|\tau|})\to(\mathscr{Q},\mathscr{W}^\vee_{|\tau|})$ is u-bounded, since $\mathscr{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}^\wedge_{|\tau|}$. Now, since $(\mathscr{P},\mathfrak{U}^\vee_{|\tau|})$ is $\vee$-order bprnological, we conclude that T is $|\tau|_\vee$-continuous. One the other hand if T is $|\tau|_\vee$-continuous, then $T:(\mathscr{P},\mathfrak{U}^\vee_{|\tau|})\to(\mathscr{Q},\mathscr{W}^\vee_{|\tau|})$ is u-bounded. This implies that $T:(\mathscr{P},\mathfrak{U})\to(\mathscr{Q},\mathscr{W})$ is u-bounded. On can prove the assertion for the case that $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ are locally convex $\wedge$-lattice cones. $\rhd$

Corollary 8. *Let $(\mathscr{P},\mathfrak{U})$ and $(\mathscr{Q},\mathscr{W})$ be locally convex lattice cones. Then the lattice homomorphism $T:(\mathscr{P},\mathfrak{U})\to(\mathscr{Q},\mathscr{W})$ is u-bounded if and only if it is $|\tau|$-continuous.*

Corollary 9. *Every continuous $\vee$ (or $\wedge$)-lattice homomorphism is $|\tau|_\vee$ (or $|\tau|_\wedge$)-continuous. Also, every continuous lattice homomorphism is $|\tau|$-continuous.*

Let $(\mathscr{P},\mathfrak{U})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone. We investigate the behavior of the convex quasiuniform structure $\mathfrak{U}^\vee_{|\tau|}$ (or $\mathfrak{U}^\wedge_{|\tau|}$) under $\vee$ (or $\wedge$)-order inductive limit.

Theorem 4. *Let $(\mathscr{P},\mathfrak{U})$ be the $\vee$ (or $\wedge$)-order inductive limit of locally convex lattice cones $(\mathscr{P}_\gamma,\mathfrak{U}_\gamma)_{\gamma\in\Gamma}$ under the $\vee$ (or $\wedge$)-lattice homomorphisms $f_\gamma:\mathscr{P}_\gamma\to\mathscr{P}$, $\gamma\in\Gamma$. Then $(\mathscr{P},\mathfrak{U}_{|\tau|})$ is the $\vee$ (or $\wedge$)-$\vee$ (or $\wedge$)-order inductive limit of locally convex lattice cones $(\mathscr{P}_\gamma,\mathfrak{U}^\vee_{|\tau|_\gamma})_{\gamma\in\Gamma}$ (or $(\mathscr{P}_\gamma,\mathfrak{U}^\wedge_{|\tau|_\gamma})_{\gamma\in\Gamma}$) under $\vee$ (or $\wedge$)-lattice homomorphisms $f_\gamma:\mathscr{P}_\gamma\to\mathscr{P}$, $\gamma\in\Gamma$.*

$\lhd$ We prove the theorem for the case that $(\mathscr{P},\mathfrak{U})$ is the $\vee$-order inductive limit of locally convex lattice cones $(\mathscr{P}_\gamma,\mathfrak{U}_\gamma)_{\gamma\in\Gamma}$. For every $\gamma\in\Gamma$, $f_\gamma:(\mathscr{P}_\gamma,\mathfrak{U}^\vee_{|\tau|_\gamma})\to(\mathscr{P},\mathfrak{U}_{|\tau|})$ is continuous by Proposition 8. Let $(\mathscr{P},\mathscr{W})$ be the $\vee$-order inductive limit of locally convex lattice cones $(\mathscr{P}_\gamma,\mathfrak{U}^\vee_{|\tau|_\gamma})_{\gamma\in\Gamma}$ under $\vee$-lattice homomorphisms $f_\gamma:\mathscr{P}_\gamma\to\mathscr{P}$, $\gamma\in\Gamma$. Then $\mathfrak{U}^\vee_{|\tau|_\gamma}\subseteq\mathscr{W}$, by the definition of $\vee$-order inductive limit. We claim that $\mathscr{P}^2$ has the same u-bounded subsets under $\mathscr{W}$ and $\mathfrak{U}^\vee_{|\tau|_\gamma}$. If B is u-bounded under $\mathfrak{U}^\vee_{|\tau|_\gamma}$, then it is u-bounded under $\mathfrak{U}$. This shows that for $\gamma\in\Gamma$, $(f_\gamma\times f_\gamma)^{-1}(B)$ is u-bounded under $\mathfrak{U}_\gamma$, and then in $\mathfrak{U}^\vee_{|\tau|_\gamma}$. Therefore B is u-bounded under $\mathscr{W}$. Now, this shows that the identity mapping $I:(\mathscr{P},\mathfrak{U}^\vee_{|\tau|})\to(\mathscr{P},\mathscr{W})$ is

u-bounded. Then it is continuous, since $(\mathscr{P}, \mathfrak{U}^{\vee}_{|\tau|})$ is $\vee$-order bornological. Then $W = \mathfrak{U}^{\vee}_{|\tau|}$. A similar argument yields our claim for the other case. ▷

Corollary 10. *The convex quasiuniform structure $\mathfrak{U}_{|\tau|}$ is stable under the order inductive limit.*

In [3], we introduced the weak convex quasiuniform structure on a locally convex cone. Here, we define the absolute weak convex quasiuniform structure on a locally convex $\vee$ (or $\wedge$)-lattice cone.

Let $(\mathscr{P}, \mathfrak{U})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone and let $L_{\vee}$(or $L_{\wedge}$) is the set of all continuous $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P}, \mathfrak{U})$ into $(\overline{\mathbb{R}}, \tilde{\mathscr{V}})$. We denote by $\mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\vee})$ (or $\mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\wedge})$) the coarsest convex quasiuniform structure on $\mathscr{P}$, that makes all $\mu \in L_{\vee}$ (or $\mu \in L_{\wedge}$), continuous. In fact $(\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\vee}))$ (or $(\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\wedge}))$) is the $\vee$ (or $\wedge$)-order projective limit of $(\overline{\mathbb{R}}, \tilde{\mathscr{V}})$ under all $\mu \in L_{\vee}$ (or $\mu \in L_{\wedge}$). This shows that $(\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\vee}))$ (or $(\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\wedge}))$) is a locally convex $\vee$ (or $\wedge$)-lattice cone. If $(\mathscr{P}, \mathfrak{U})$ is a locally convex lattice cone and L is the set of all continuous lattice homomorphism from $(\mathscr{P}, \mathfrak{U})$ into $(\overline{\mathbb{R}}, \tilde{\mathscr{V}})$, then we can define the solid convex quasiuniform structure on $\mathfrak{U}_{|\sigma|}(\mathscr{P}, L)$ in a similar way.

For a locally convex lattice cone $(\mathscr{P}, \mathfrak{U})$, it is easy to see that $\mathfrak{U}_{\sigma}(\mathscr{P}, \mathscr{P}^*)$ is finer than $\mathfrak{U}_{|\sigma|}(\mathscr{P}, L)$. But for some locally convex lattice cones, these convex quasiuniform structures are equivalent.

Example 3. Let $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. We set $U = \{(a, b) \in (\overline{\mathbb{R}}_+)^2 : a \leqslant b\}$ and $\mathfrak{U} = \{U\}$. If we consider usual order on $\overline{\mathbb{R}}_+$, then $\mathfrak{U}$ is a solid convex quasiuniform structure on $\overline{\mathbb{R}}_+$ and $(\overline{\mathbb{R}}_+, \mathfrak{U})$ is a locally convex lattice cone (a *us*-lattice cone). The dual cone of $(\overline{\mathbb{R}}_+, \mathfrak{U})$ consists of all nonnegative reals an functionals $\overline{0}$ and $\overline{+\infty}$ acting as:

$$\overline{0}(a) = \begin{cases} +\infty, & a = +\infty, \\ 0 & else \end{cases} \quad \text{and} \quad \overline{+\infty}(a) = \begin{cases} 0, & a = 0, \\ +\infty & else, \end{cases}$$

respectively. Since all elements of $\overline{\mathbb{R}}^*_+$ are lattice homomorphism, we conclude that $\mathfrak{U}_{|\sigma|}(\overline{\mathbb{R}}_+, L) = \mathfrak{U}_{\sigma}(\overline{\mathbb{R}}_+, \overline{\mathbb{R}}^*_+)$. In fact, we have $L = \mathscr{P}^*$. It is easy to see that $\mathfrak{U}$ is strictly finer than $\mathfrak{U}_{|\sigma|}(\overline{\mathbb{R}}_+, L)$.

Definition 7. Let $(\mathscr{P}, \mathfrak{U})$ and $(\mathscr{Q}, \mathscr{W})$ be locally convex $\vee$ (or $\wedge$)-lattice cones. The linear operator $T : \mathscr{P} \to \mathscr{Q}$ is called $|\sigma|$-continuous, whenever $T : (\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\vee}))) \to (\mathscr{Q}, \mathscr{W}_{|\sigma|}(\mathscr{Q}, L_{\vee}))$ (or $T : (\mathscr{P}, \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\wedge}))) \to (\mathscr{Q}, \mathscr{W}_{|\sigma|}(\mathscr{Q}, L_{\wedge})))$ is continuous.

Proposition 9. *Let $(\mathscr{P}, \mathfrak{U})$ and $(\mathscr{Q}, \mathscr{W})$ be locally convex $\vee$ (or $\wedge$)-lattice cones. Then every continuous $\vee$ (or $\wedge$)-lattice homomorphism from $(\mathscr{P}, \mathfrak{U})$ into $(\mathscr{Q}, \mathscr{W})$ is $|\sigma|$-continuous.*

◁ We prove the assertion for the case that $(\mathscr{P}, \mathfrak{U})$ and $(\mathscr{Q}, \mathscr{W})$ be locally convex $\vee$-lattice cones. We denote the sets of all continuous $\vee$-lattice homomorphisms on $\mathscr{Q}$ and $\mathscr{P}$ by $L'_{\vee}$ and $L_{\vee}$, respectively. Let $T : (\mathscr{P}, \mathfrak{U}) \to (\mathscr{Q}, \mathscr{W})$ be a continuous $\vee$-lattice homomorphism and $W_{|\sigma|} \in \mathscr{W}_{|\sigma|}(\mathscr{Q}, L'_{\vee})$. Then there are $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in L'_{\vee}$ such that

$$\bigcap_{i=1}^{n} \Lambda_i^{-1}(\tilde{1}) \subseteq W_{|\sigma|},$$

where $\Lambda_i = \mu_i \times \mu_i$, for $i = 1, \ldots, n$. We have $\mu_i o T \in L_{\vee}$ for $i = 1, \ldots, n$, since T is a continuous $\vee$-lattice homomorphism. We set $\Gamma_i = \mu_i o T \times \mu_i o T$, for $i = 1, \ldots, n$. Then $U_{|\sigma|} = \bigcap_{i=1}^{n} \Gamma_i^{-1}(\tilde{1}) \in \mathfrak{U}_{|\sigma|}(\mathscr{P}, L_{\vee})$ and we have $(T \times T)(U_{|\sigma|}) \subseteq W_{|\sigma|}$. ▷

Theorem 5. *Let $(\mathscr{P},\mathfrak{U})$ be a $\vee$ (or $\wedge$)-order bornological locally convex lattice cone and $(\mathscr{Q},\mathscr{W})$ be a locally convex $\vee$ (or $\wedge$)-lattice cone which has the same u-bounded subsets under $\mathscr{W}$ and $\mathscr{W}_{|\sigma|}(\mathscr{Q},L_\vee)$) (or $\mathscr{W}_{|\sigma|}(\mathscr{Q},L_\vee)$)). Then every $|\sigma|$-continuous $\vee$ (or $\wedge$)-lattice homomorphism from $\mathscr{P}$ into $\mathscr{Q}$ is continuous.*

$\lhd$ Let T be a $|\sigma|$-continuous $\vee$ (or $\wedge$)-lattice homomorphism from $\mathscr{P}$ into $\mathscr{Q}$. Since every u-bounded subset is weakly u-bounded, we conclude that T is u-bounded by the assumptions. Now, since $(\mathscr{P},\mathfrak{U})$ is order bornological, T is continuous. $\rhd$

Corollary 11. *Let $(\mathscr{P},\mathfrak{U})$ be a $\vee$ (or $\wedge$)-order bornological locally convex lattice cone. Then every linear functional on $(\mathscr{P},\mathfrak{U})$, which is $\vee$ (or $\wedge$)-lattice homomorphism, is continuous if and only if it is $|\sigma|$-continuous.*

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ПОРЯДКОВО БОРНОЛОГИЧЕСКИЕ ЛОКАЛЬНО ВЫПУКЛЫЕ РЕШЕТОЧНЫЕ КОНУСЫ

Айазех Д., Ранджбари А.

В статье вводятся понятия *us*-решеточного конуса и порядково борнологического локально выпуклого решеточного конуса. В специальном случае локально солидного (= нормального) пространства Рисса (= векторной решетки) эти понятия сводятся к хорошо известным понятиям полунормированного пространства Рисса и порядково борнологического пространства Рисса, соответственно. Вводится также понятие солидного множества в локально выпуклом конусе и даются некоторые характеризации порядково борнологических локально выпуклых решеточных конусов.

Ключевые слова: локально выпуклый решеточный конус, порядково борнологический конус.