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BLOW-UP RESULT FOR A CLASS OF WAVE p-LAPLACE EQUATION WITH NONLINEAR DISSIPATION IN $\mathbb{R}^N$

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Abstract. The Laplace equations has been studied in several stages and has gradually developed over the past decades. Beginning with the well-known standard equation $\Delta u = 0$, where it has been well studied in all aspects, many results have been found and improved in an excellent manner. Passing to p-Laplace equation $\Delta_p u = 0$ with a constant parameter, whether in stationary or evolutionary systems, where it experienced unprecedented development and was studied in almost exhaustively. In this article, we consider initial value problem for nonlinear wave equation containing the p-Laplacian operator. We prove that a class of solutions with negative initial energy blows up in finite time if $p \geqslant r \geqslant m$, by using contradiction argument. Additional difficulties due to the constant exponents in $\mathbb{R}^n$ are treated in order to obtain the main conclusion. We used a contradiction argument to obtain a condition on initial data such that the solution extinct at finite time. In the absence of the density function, our system reduces to the nonlinear damped wave equation, it has been extensively studied by many mathematicians in bounded domain.

Key words: blow-up, finite time, nonlinear damping, p-Laplace equation, weighted spaces.

Mathematical Subject Classification (2010): 35L05, 35L15, 35L70, 35B05, 35B40.

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1. Introduction

The study of Laplace equations has been studied in several stages and has gradually developed over the past decades. Beginning with the well-known standard equation $\Delta u = 0$, where it has been well studied in all aspects, many results have been found and improved in an excellent manner. Passing to p-Laplace equation $\Delta_p u = 0$ with a constant parameter, whether in stationary or evolutionary systems, where it experienced unprecedented development and was studied in almost exhaustively.

In this article, we consider IBVP for nonlinear wave equation containing the p-Laplacian operator, $\Delta_p u = \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u)$,

$$\big(\partial_{tt} - \varrho(x)\Delta_p\big)u + \mu|\partial_t u|^{m-2}\,\partial_t u = b|u|^{r-2}u, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u^0(x), \quad \partial_t u(x,0) = u^1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\varrho(x) > 0$, $\forall\, x \in \mathbb{R}^n$, $\mu \neq 0$, $(\varrho(x))^{-1} = \nu(x)$. The density function is $\nu\colon \mathbb{R}^n \to \mathbb{R}^*_+$, $\nu(x) \in C^{0,\theta}(\mathbb{R}^n)$ with $\theta \in (0,1)$ and $\nu \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

In the absence of the density function (that is, if $\varrho(x) \equiv 1$), equation (1.1) reduces to the nonlinear damped wave equation, it has been extensively studied by many mathematicians in bounded domain.

In the famous work [1], Georgiev and Todorova extended Levine's result to the case of nonlinear damping of the form $|\partial_t u|^{m-1}\partial_t u$. More precisely, in [1], V. Georgiev and G. Todorova and by combining the Galerkin approximation with the contraction mapping theorem, showed that problem

$$\partial_{tt}u(x,t) - \Delta_x u(x,t) + a\,|\partial_t u|^{m-2}\,\partial_t u = b\,|u|^{p-2}\,u, \tag{1.3}$$

in a bounded domain with initial and boundary conditions of Dirichlet type has a unique solution in the interval $[0;T)$ provided that T is small enough. Also, they proved that the obtained solutions continue to exist globally in time if $m \geqslant p$ and the initial data are small enough. Whereas for $p > m$ the unique solution of problem (1.3) blows up in finite time provided that the initial data are large enough (i. e., the initial energy is sufficiently negative). This result was generalized to an abstract setup by Levine and Serrin [2], Levine et al. [3] and Vitillaro [4].

Among the most recent work concerning the p-Laplace equation, we can review Lazer et al. [5], where the authors tried to demonstrate the existence of periodic solutions for models of nonlinear supported bending beams and periodic flexing in floating beam. In [6] the authors used discontinuous Galekin method to approximate a biharmonic problem. They also gave an a priori analysis of the error in L^2 norm. In [7] the author has studied a problem p-biharmonic using discontinuous Galerkin finite element Hessian. To solve the problem, the authors used a fixed point iterative method. In [8], a nonlinear (in space and time) wave equation with delay term in the internal feedback is considered. By multiplier method and general weighted integral inequalities, the authors treated the question of asymptotic behavior of solutions. For more details, please see [9–14].

The present paper is organized as follows. In Section 2, we present some assumptions and preliminaries. Section 3 is devoted to the blow-up result.

2. Spaces and Operator Settings

Definition 2.1 [15]. We define the function spaces of our problem as follows

$$\mathscr{D}^{1,p}(\mathbb{R}^n) = \big\{u \in L^{np/(n-p)}(\mathbb{R}^n) : \nabla_x u \in (L^p(\mathbb{R}^n))^n\big\}, \tag{2.1}$$

with respect to the norm

$$\|u\|_{\mathscr{D}^{1,p}} = \left(\int\limits_{\mathbb{R}^n} |\nabla_x u|^p dx\right)^{\frac{1}{p}},$$

and the spaces $L^2_\nu(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(u,v)_{L^2_\nu(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} \nu uv\,dx.$$

For $1 < q < \infty$, if u is a measurable function on $\mathbb{R}^n$, we define

$$\|u\|_{L^q_\nu(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} \nu|f|^q dx\right)^{\frac{1}{q}}, \tag{2.2}$$

and that $\mathscr{D}^{1,p}(\mathbb{R}^n)$ can be embedded continuously in $L^{np/(n-p)}(\mathbb{R}^n)$, i. e., there exists $k > 0$ such that

$$\|u\|_{L^{np/(n-p)}} \leqslant k\|u\|_{\mathscr{D}^{1,p}}. \tag{2.3}$$

We shall frequently use the following version of the generalized Poincaré's inequality.

Lemma 2.1 [16]. *Suppose $\nu \in L^{n/p}(\mathbb{R}^n)$. Then there exists $\gamma > 0$ such that*

$$\int\limits_{\mathbb{R}^n} |\nabla_x u|^p dx \geqslant \gamma \int\limits_{\mathbb{R}^n} \nu|u|^p dx, \tag{2.4}$$

for all $u \in \mathscr{D}^{1,p}(\mathbb{R}^n)$.

Lemma 2.2 [16]. *For any $u \in \mathscr{D}^{1,p}(\mathbb{R}^n)$*

$$\|u\|_{L^2_\nu(\mathbb{R}^n)} \leqslant \|\nu\|_{L^{n/p}(\mathbb{R}^n)}\|\nabla_x u\|_{L^p(\mathbb{R}^n)}. \tag{2.5}$$

Lemma 2.3 [16]. *Assume that $\nu \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for any $1 \leqslant q \leqslant p < \infty$*

$$L^p_\nu(\mathbb{R}^n) \subset L^q_\nu(\mathbb{R}^n), \tag{2.6}$$

that is $\exists\, c > 0$ such that $\|u\|_{L^q_\nu(\mathbb{R}^n)} \leqslant c\|u\|_{L^p(\mathbb{R}^n)}$, where $c = \|\nu\|_1^{(p-q)/pq}$.

Lemma 2.4. *Suppose that $\infty > p \geqslant s \geqslant r \geqslant 1$. Then there exists a positive constant c' such that*

$$\|u\|^s_{L^r_\nu(\mathbb{R}^n)} \leqslant c'\big(\|u\|^r_{L^r_\nu(\mathbb{R}^n)} + \|u\|^p_{\mathscr{D}^{1,p}}\big),$$

for any $u \in \mathscr{D}^{1,p}(\mathbb{R}^n)$.

$\lhd$ If $\|u\|_{L^r_\nu(\mathbb{R}^n)} \leqslant 1$, then $\|u\|^s_{L^r_\nu(\mathbb{R}^n)} \leqslant \|u\|^r_{L^r_\nu(\mathbb{R}^n)}$ for $s \geqslant r$. If $\|u\|_{L^r_\nu(\mathbb{R}^n)} \geqslant 1$, then $\|u\|^s_{L^r_\nu(\mathbb{R}^n)} \leqslant \|u\|^p_{L^r_\nu(\mathbb{R}^n)}$ for $p \geqslant s$. By using Lemma 2.3 and Lemma 2.1, we obtain $\|u\|^s_{L^r_\nu(\mathbb{R}^n)} \leqslant \bar{c}\|u\|^p_{\mathscr{D}^{1,p}}$. Together with the two cases, we get Lemma 2.4. $\rhd$

Lemma 2.5. *If x and y are nonnegative real numbers and $p, q > 0$ such that $1/p+1/q = 1$, then for any nonnegative real number β*

$$xy \leqslant \frac{\beta^p}{p}\,x^p + \frac{p-1}{p}\,\beta^{-\frac{p}{p-1}}\,y^{\frac{p}{p-1}}.$$

Now, we define the energy associated to the solution of the system (1.1)–(1.2) by

$$\mathscr{E}(t) = \frac{1}{2}\,\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\,\|u\|^p_{\mathscr{D}^{1,p}} - \frac{b}{r}\,\|u\|^r_{L^r_\nu(\mathbb{R}^n)}. \tag{2.7}$$

Lemma 2.6. *Let u be the solution of* (1.1)–(1.2). *Then*

$$\frac{d}{dt}\mathscr{E}(t) = -\mu\|\partial_t u\|^m_{L^m_\nu(\mathbb{R}^n)} \leqslant 0. \tag{2.8}$$

$\lhd$ Multiplying (1.1) by $\nu(x)\partial_t u$ and integrating over $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} \nu(x)\partial_t u \partial_{tt} u\,dx - \int_{\mathbb{R}^n} \partial_t u \Delta_p u\,dx + \mu \int_{\mathbb{R}^n} \nu(x)|\partial_t u|^m\,dx = b\int_{\mathbb{R}^n} \nu(x)|u|^{r-2}u\partial_t u\,dx. \tag{2.9}$$

Using equation (1.1) and integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\frac{d}{dt}\|u\|^p_{\mathscr{D}^{1,p}} + \mu\|\partial_t u\|^m_{L^m_\nu(\mathbb{R}^n)} = \frac{1}{r}\frac{d}{dt}\|u\|^r_{L^r_\nu(\mathbb{R}^n)}.$$

The proof is completed. $\rhd$

3. Blow Up Results

We define

$$\mathscr{H}(t) = -\mathscr{E}(t).$$

Our main result is reads as follows.

Theorem 3.1. *Suppose that $p \geqslant r \geqslant m$ and $\nu \in L^1(\mathbb{R}^n)$ and $E(0) < 0$. Then, any weak solution u of the problem* (1.1)–(1.2) *blows up in finite time, i. e.,*

$$\limsup_{t\to T^-}\left(\frac{1}{2}\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\|u\|^p_{\mathscr{D}^{1,p}}\right) = \infty.$$

$\lhd$ Assume that there exists some positive constants C such that u solution of (1.1) satisfies

$$\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\|u\|^p_{\mathscr{D}^{1,p}} \leqslant C, \tag{3.1}$$

and

$$0 < \mathscr{H}(0) \leqslant \mathscr{H}(t) \leqslant -\frac{1}{2}\left[\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\|u\|^p_{\mathscr{D}^{1,p}}\right] + \frac{b}{r}\|u\|^r_{L^r_\nu(\mathbb{R}^n)}. \tag{3.2}$$

Hence

$$\mathscr{H}(0) \leqslant \mathscr{H}(t) \leqslant \frac{b}{r}\|u\|^r_{L^r_\nu(\mathbb{R}^n)},$$

we define

$$\mathscr{L}(t) = \mathscr{H}^{1-\alpha}(t) + 2\varepsilon\int_{\mathbb{R}^n} \nu(x)u\partial_t u\,dx, \tag{3.3}$$

for small ε to be chosen later and for

$$\max\left(\frac{r-m}{r(m-1)}, \frac{r-2}{2}\right) \leqslant \alpha \leqslant \min\left(\frac{1}{2}, \frac{p-2}{2}, \frac{p-m}{r(m-1)}\right).$$

Taking a derivative of (3.3), we obtain

$$\mathscr{L}'(t) = -\alpha\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) + 2\varepsilon\int_{\mathbb{R}^n} \nu(x)u\partial_{tt} u\,dx + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)}. \tag{3.4}$$

On the other hand, from (1.1), we have

$$\begin{aligned}\mathscr{L}'(t) = (1-\alpha)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \varepsilon b\|u\|^r_{L^r_\nu(\mathbb{R}^n)} \\ - \varepsilon\|u\|^p_{\mathscr{D}^{1,p}} - \mu\varepsilon \int\limits_{\mathbb{R}^n} \nu(x) u\partial_t u|\partial_t u|^{m-2}dx.\end{aligned} \tag{3.5}$$

Now, we estimate the term $\int_{\mathbb{R}^n} \nu(x)u\partial_t u|\partial_t u|^{m-2}dx$, by using Young's inequality with the conjugate exponents m and $m/(m-1)$

$$\mu \int\limits_{\mathbb{R}^n} \nu(x)u\partial_t u|\partial_t u|^{m-2}\, dx \leqslant \mu \int\limits_{\mathbb{R}^n} \nu(x) \left(\frac{\beta^m}{m}\,|u|^m + \frac{m-1}{m}\,\beta^{-\frac{m}{m-1}}|\partial_t u|^m\right) dx$$

$$\leqslant \frac{\mu\beta^m}{m}\,\|u\|^m_{L^m_\nu(\mathbb{R}^n)} + \frac{\mu(m-1)}{m}\,\beta^{-\frac{m}{m-1}}\|\partial_t u\|^m_{L^m_\nu(\mathbb{R}^n)} \leqslant \frac{\mu\beta^m}{m}\,\|u\|^m_{L^m_\nu(\mathbb{R}^n)} + \frac{m-1}{m}\,H'(t)\beta^{-\frac{m}{m-1}}.$$

By substitution in (3.5), we get

$$\begin{aligned}\mathscr{L}'(t) \geqslant \left[(1-\alpha)\mathscr{H}^{-\alpha}(t) + \varepsilon\,\frac{m-1}{m}\,\beta^{-\frac{m}{m-1}}\right]\mathscr{H}'(t) \\ + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + 2\varepsilon b\|u\|^r_{L^r_\nu(\mathbb{R}^n)} - 2\varepsilon\|u\|^p_{\mathscr{D}^{1,p}} + \frac{\mu\beta^m}{m}\,\|u\|^m_{L^m_\nu(\mathbb{R}^n)},\end{aligned} \tag{3.6}$$

it remains valid even if β is time dependent. Therefore by taking β so that

$$\beta^{-\frac{m}{m-1}} = K\mathscr{H}^{-\alpha}(t),$$

for large K to be specified later and substituting in (3.6), we obtain

$$\begin{aligned}\mathscr{L}'(t) &\geqslant (1-\alpha)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + 2\varepsilon b\|u\|^r_{L^r_\nu(\mathbb{R}^n)} - 2\varepsilon\|u\|^p_{\mathscr{D}^{1,p}} \\ &- 2\varepsilon\,\frac{\mu}{m}\,K^{1-m}\mathscr{H}^{\alpha(m-1)}\|u\|^m_{L^m_\nu(\mathbb{R}^n)} - 2\varepsilon\,\frac{m-1}{m}\,K\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) \\ &\geqslant \left((1-\alpha) - 2\varepsilon K\,\frac{m-1}{m}\right)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \\ &+ 2\varepsilon b\|u\|^r_{L^r_\nu(\mathbb{R}^n)} - 2\varepsilon\|u\|^p_{\mathscr{D}^{1,p}} - 2\varepsilon\,\frac{\mu}{m}\,K^{1-m}\mathscr{H}^{\alpha(m-1)}\|u\|^m_{L^m_\nu(\mathbb{R}^n)}.\end{aligned} \tag{3.7}$$

On the other hand, by using (3.2) and the inequality $\|u\|_{L^m_\nu(\mathbb{R}^n)} \leqslant c\|u\|_{L^r_\nu(\mathbb{R}^n)}$, we get

$$\mathscr{H}^{\alpha(m-1)}(t)\|u\|^m_{L^m_\nu(\mathbb{R}^n)} \leqslant c\left(\frac{b}{r}\right)^{\alpha(m-1)}\|u\|^{r\alpha(m-1)+m}_{L^r_\nu(\mathbb{R}^n)}. \tag{3.8}$$

Inserting (3.8) in (3.7), using Lemma 2.4 for $p \geqslant s = r\alpha(m-1)+m \geqslant r$, to deduce that

$$\begin{aligned}\mathscr{L}'(t) \geqslant \left((1-\alpha) - 2\varepsilon K\,\frac{m-1}{m}\right)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) + 2\varepsilon\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \\ + 2\varepsilon r\left(\frac{1}{2}\,\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p}\,\|u\|^p_{\mathscr{D}^{1,p}} + \mathscr{H}(t)\right) - 2\varepsilon\|u\|^p_{\mathscr{D}^{1,p}} \\ - 2c\varepsilon\,\frac{\mu}{m}\,K^{1-m}c'\left(\frac{b}{r}\right)^{\alpha(m-1)}\left(\frac{r}{2b}\,\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \left(1+\frac{r}{pb}\right)\|u\|^p_{\mathscr{D}^{1,p}} + \mathscr{H}(t)\right).\end{aligned}$$

Consequently, we obtain for $0 < \theta < 1$

$$\begin{aligned}
\mathscr{L}'(t) \geqslant & \left((1-\alpha) - 2\varepsilon K\,\frac{m-1}{m}\right)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) \\
+ \varepsilon\,\frac{b}{r}\,(1-\theta)p\|u\|^r_{L^r_\nu(\mathbb{R}^n)} & + \varepsilon\left(2r - (1-\theta)p - 2C_1K^{1-m}\right)\mathscr{H}(t) \\
& + \varepsilon\left(2 + r - \frac{r}{b}\,C_1K^{1-m} - \frac{1}{2}\,(1-\theta)p\right)\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \\
& + \varepsilon\left(2\,\frac{r}{p} - (1-\theta) - 2\left(1 + \frac{r}{pb}\right)C_1K^{1-m}\right)\|u\|^p_{\mathscr{D}^{1,p}},
\end{aligned} \tag{3.9}$$

where

$$C_1 = \frac{C\mu}{m}\left(\frac{b}{r}\right)^{\alpha(m-1)}.$$

At this point, we choose K large enough so that (3.9) becomes

$$\begin{aligned}
\mathscr{L}'(t) \geqslant & \left((1-\alpha) - 2\varepsilon K\,\frac{m-1}{m}\right)\mathscr{H}'(t)\mathscr{H}^{-\alpha}(t) \\
& + \varepsilon\beta\Big[\mathscr{H}(t) + \|u\|^r_{L^r_\nu(\mathbb{R}^n)} + \|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \|u\|^p_{\mathscr{D}^{1,p}}\Big],
\end{aligned} \tag{3.10}$$

where $\beta > 0$ is the minimum of the coefficients in (3.9).

We choose $\frac{1}{2}(1-\theta)p < \min(r, 2+r)$, that is $\frac{1}{2}(1-\theta)p - r < 0$.

Finally, we pick ε so small so that $(1-\alpha) - 2\varepsilon K\,\frac{m-1}{m} > 0$. Therefore, (3.10) takes on the form

$$\mathscr{L}'(t) \geqslant \delta\Big(\mathscr{H}(t) + \|u\|^r_{L^r_\nu(\mathbb{R}^n)} + \|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \|u\|^p_{\mathscr{D}^{1,p}}\Big). \tag{3.11}$$

We conclude that $\mathscr{L}$ is a nondecreasing function of t

$$L(t) \geqslant \mathscr{L}(0) = \mathscr{H}^{1-\alpha}(0) + 2\varepsilon\int_{\mathbb{R}^n}\nu(x)u_0(x)u_1(x)\,dx > 0 \quad \text{for all } t > 0.$$

Now, we estimate the term $\int_{\mathbb{R}^n}\nu(x)u\partial_t u\,dx$ as follows

$$\left|\int_{\mathbb{R}^n}\nu(x)u\partial_t u\,dx\right| \leqslant \int_{\mathbb{R}^n}\nu^{\frac{r-2}{2r}}(x)\Big(\nu^{\frac{1}{r}}(x)|u|\Big)\Big(\nu^{\frac{1}{2}}(x)|\partial_t u|\Big)dx.$$

Using Hölder's inequality with the functions

$$f = \nu^{\frac{r-2}{2r}}, \quad g = \nu^{\frac{1}{r}}|u|, \quad h = \nu^{\frac{1}{2}}|\partial_t u|,$$

and the conjugate exponents $a_1 = \frac{2r}{r-2}$, $a_2 = r$, $a_3 = 2$, we get

$$\left|\int_{\mathbb{R}^n}\nu(x)u\partial_t u\,dx\right| \leqslant \|\nu\|^{\frac{r-2}{2r}}_{L^1(\mathbb{R}^n)}\|u\|_{L^r_\nu(\mathbb{R}^n)}\|\partial_t u\|_{L^2_\nu(\mathbb{R}^n)}.$$

Owing to the Young's inequality, with $1/a + 1/b = 1$

$$\left|\int_{\mathbb{R}^n}\nu(x)u\partial_t u\,dx\right| \leqslant C_1\|\nu\|^{\frac{r-2}{2r}}_{L^1(\mathbb{R}^n)}\Big(\|u\|^a_{L^r_\nu(\mathbb{R}^n)} + \|\partial_t u\|^b_{L^2_\nu(\mathbb{R}^n)}\Big).$$

Finally, we obtain

$$\left| \int\limits_{\mathbb{R}^n} \nu(x) u \partial_t u \, dx \right|^{\frac{1}{1-\alpha}} \leqslant C_2 \Big(\|u\|^{\frac{a}{1-\alpha}}_{L^r_\nu(\mathbb{R}^n)} + \|\partial_t u\|^{\frac{b}{1-\alpha}}_{L^2_\nu(\mathbb{R}^n)} \Big).$$

We choose $b = \frac{2}{1-\alpha}$ and $a = 2\frac{1-\alpha}{1-2\alpha}$, then

$$\left| \int\limits_{\mathbb{R}^n} \nu(x) u \partial_t u \, dx \right|^{\frac{1}{1-\alpha}} \leqslant C_2 \left(\|u\|^{\frac{2}{1-2\alpha}}_{L^r_\nu(\mathbb{R}^n)} + \|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \right).$$

Using the Lemma 2.4 for $p \geqslant \frac{2}{1-2\alpha} \geqslant r$, we get

$$\left| \int\limits_{\mathbb{R}^n} \nu(x) u \partial_t u \, dx \right|^{\frac{1}{1-\alpha}} \leqslant C_3 \left(\|u\|^r_{L^r_\nu(\mathbb{R}^n)} + \|u\|^p_{\mathscr{D}^{1,p}} + \|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \right).$$

Therefore, we obtain

$$\begin{aligned} \mathscr{L}^{\frac{1}{1-\alpha}}(t) &= \left(\mathscr{H}^{1-\alpha}(t) + 2\varepsilon \int\limits_{\mathbb{R}^n} \nu(x) u \partial_t u \, dx \right)^{\frac{1}{1-\alpha}} \\ &\leqslant \lambda \left(\mathscr{H}(t) + \|u\|^r_{L^r_\nu(\mathbb{R}^n)} + \|u\|^p_{\mathscr{D}^{1,p}} + \|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} \right), \quad t \geqslant 0. \end{aligned} \tag{3.12}$$

Combining (4.1) and (3.12), we arrive at

$$\mathscr{L}'(t) \geqslant \Lambda \mathscr{L}^{\frac{1}{1-\alpha}}(t), \quad t > 0, \tag{3.13}$$

where Λ is a positive constant depending only on λ and C.

A simple integration of (3.13) over $(0,t)$ yields

$$\mathscr{L}^{\frac{\alpha}{1-\alpha}}(t) \geqslant \frac{1}{\mathscr{L}^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\Lambda \alpha t}{1-\alpha}}, \quad t > 0.$$

Therefore, $\mathscr{L}(t)$ blows up in time

$$T_0 \leqslant \frac{1-\alpha}{\Lambda \alpha \mathscr{L}^{\frac{\alpha}{1-\alpha}}(0)}.$$

Furthermore, we have

$$\lim_{t \to T_0^-} \left(\|\partial_t u\|^2_{L^2_\nu(\mathbb{R}^n)} + \frac{1}{p} \|u\|^p_{\mathscr{D}^{1,p}} \right) = \infty.$$

This leads to a contradiction with (3.1). Thus, the solution of problem (1.1) blows up in finite time. This completes the proof. $\triangleright$

Conclusion

In this paper, we studied the global nonexistence of solutions for a class of nonlinear wave equation with p-Laplacian. Under suitable assumptions on the variable exponents p, m, r and the density function, it is proved that the solutions with negative initial energy blow up in finite time. The paper can be viewed as an extension of the previous works to p-Laplace type in unbounded domain $\mathbb{R}^n$. Additional difficulties due to the constant exponents in $\mathbb{R}^n$ are treated in order to obtain the main conclusion. We used a contradiction argument to obtain a condition on initial data such that the solution extinct at finite time.

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РЕЗУЛЬТАТ О ВЗРЫВЕ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ p-ЛАПЛАСА С НЕЛИНЕЙНОЙ ДИССИПАЦИЕЙ В $\mathbb{R}^n$

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Аннотация. Уравнение Лапласа изучалось в несколько этапов и получило бурное развитие в течение последних десятилетий. Начиная с хорошо известного стандартного уравнения $\Delta u = 0$, которое хорошо изучено во всех аспектах, были усилены многие результаты и найдены новые постановки. Переход к p-уравнению Лапласа $\Delta_p u = 0$ с постоянным параметром, будь то в стационарных или эволюционных системах, привел к беспрецедентному развитию и почти исчерпывающему исследованию. В данной статье мы рассматриваем начальную задачу для нелинейного волнового уравнения, содержащего p-лапласиан. Методом от противного доказано, что класс решений с отрицательной начальной энергией взрывается за конечное время, если $p \geqslant r \geqslant m$. Чтобы получить основной вывод, необходимо обойти дополнительные трудности, связанные с постоянными показателями в $\mathbb{R}^n$. Получено условие на начальные данные, при которых решение исчезает за конечное время. В отсутствие функции плотности наша система сводится к нелинейному уравнению затухающей волны, которое в ограниченной области активно изучалось многими математиками.

Ключевые слова: взрыв, конечное время, нелинейное затухание, уравнение p-Лапласа, весовые пространства.

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