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NONLINEAR VISCOSITY ALGORITHM WITH PERTURBATION FOR NONEXPANSIVE MULTI-VALUED MAPPINGS

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Abstract. The viscosity iterative algorithms for finding a common element of the set of fixed points for nonlinear operators and the set of solutions of variational inequality problems have been investigated by many authors. The viscosity technique allow us to apply this method to convex optimization, linear programming and monoton inclusions. In this paper, based on viscosity technique with perturbation, we introduce a new nonlinear viscosity algorithm for finding an element of the set of fixed points of nonexpansive multi-valued mappings in a Hilbert spaces. Furthermore, strong convergence theorems of this algorithm were established under suitable assumptions imposed on parameters. Our results can be viewed as a generalization and improvement of various existing results in the current literature. Moreover, some numerical examples that show the efficiency and implementation of our algorithm are presented.

Key words: fixed point problem, generalized equilibrium problem, nonexpansive multi-valued mapping, Hilbert space.

Mathematical Subject Classification (2010): 47H09, 47H10, 47J20.

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1. Introduction

Throughout the paper unless otherwise stated, H denotes a real Hilbert space, we denote the norm and inner product of H by $\langle .,. \rangle$ and norm $\|.\|$, respectively. The set C (C being a nonempty closed convex subset of H) is called proximinal if for each $x \in H$, there exists an element $y \in C$ such that $\|x-y\| = d(x,C)$, where $d(x,C) = \inf\{\|x-z\| : z \in C\}$. Let $CB(D)$, $K(C)$ and $P(C)$ be the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad A,B \in CB(C).$$

A multi-valued mapping $T : C \to 2^C$ is said to be nonexpansive if $H(Tx,Ty) \leqslant \|x-y\|$ for all $x, y \in C$. An element $p \in C$ is called a fixed point of $T : C \to 2^C$ if $p \in Tp$. The fixed points set of T is denoted by $\mathrm{Fix}(T)$.

The problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points for single-valued mappings in the framework of Hilbert spaces has been intensively studied by many authors, for instance, see [1–5] and the references cited therein.

Ceng et al. [6], introduced the following generalized equilibrium problem with perturbation: Find $x^* \in C$ such that

$$f(x^*, y) + \langle (A+B)x^*, y - x^* \rangle \geqslant 0 \quad (\forall\, y \in C), \tag{1.1}$$

where $A, B : C \to H$ are nonlinear mappings and $f : C \times C \to \mathbb{R}$ is a bifunction. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others [1, 4, 5, 7, 8].

In 2016, Azhini and Taherian [9], motivated by [6, 10], proposed the following iteration process for finding a common element of the set of solutions of variational inequality (1.1) and the set of common fixed points of infinitely many nonexpansive mappings $\{S_n\}$ of C into itself and proved the strong convergence of the sequence generated by this iteration process to an element of $F(P_C S) = \bigcap_{n=1}^{\infty} F(P_C S_n)$.

$$\begin{gathered} F(u_n, y) + \big\langle (M+N)x_n, y - u_n \big\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geqslant 0 \quad (\forall\, y \in C), \\ x_{n+1} = \beta_n P_C f(x_n) + \gamma_n x_n + \lambda_n P_C S_n[\alpha_n z + (1-\alpha_n)u_n] \quad (\forall\, n \in \mathbb{N}), \end{gathered}$$

where $\beta_n + \gamma_n + \lambda_n = 1$.

In 2019, Sahebi et al. [11] by intuition from [12–15] considered a general viscosity iterative algorithm for finding a common element of the set general equilibrium problem system and the set of fixed points of a nonexpansive semigroup in a Hilbert space as follows:

$$\begin{cases} u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n), \\ w_n = \frac{1}{k}\sum\limits_{i=1}^{k} u_{n,i}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + \big((1-\epsilon_n)I - \beta_n B - \alpha_n A\big)\frac{1}{s_n}\int\limits_0^{s_n} T(s) w_n \, ds. \end{cases} \tag{1.2}$$

They proved that, the sequence generated by this algorithm under the certain conditions imposed on parameters strongly convergence to a common solution of general equilibrium problem system.

Many authors have shown the existence of fixed points of multi-valued mappings in Hilbert spaces (see [16–19]). The study of multi-valued mappings is much more complicated and difficult than that of single-valued mappings.

In this paper, motivated by the research going on in this direction, we introduce the iterative algorithm for finding a common element of the set of fixed point of a nonexpansive multi-valued mapping in a real Hilbert space. Some strong convergence theorems and lemmas of the proposed algorithm are proven under new techniques and some mild assumption on the control conditions. Finally, some numerical examples that show the efficiency and implementation of our algorithm are presented.

The paper is structured as follows. In Section 2, we collect some lemmas, which are essential to prove our main results. In Section 3, we introduce a new algorithm for finding a common element of the set of fixed point of a nonexpansive set-valued mapping in a real Hilbert space. Then, we establish and prove the strong convergence theorem under some proper conditions. In Section 4, we also give some numerical examples to support our main theorem.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leqslant \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping. Also, a mapping $M : C \to H$ is said to be monotone, if

$$\langle Mx - My, x - y\rangle \geqslant 0 \quad (\forall\, x, y \in C).$$

M is called α-inverse-strongly-monotone if there exist a positive real number α such that

$$\langle Mx - My, x - y\rangle \geqslant \alpha\|Mx - My\|^2 \quad (\forall\, x, y \in C).$$

It is obvious that any α-inverse-strongly-monotone mapping M is monotone and Lipschitz continuous.

Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geqslant \|Tx - Ty\|^2 \quad (\forall\, x, y \in H).$$

It is also known that H satisfies Opial's condition [20], i. e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.1}$$

holds for every $y \in H$ with $y \neq x$. The following lemmas will be used for proving the convergence result of this paper in the sequel.

Lemma 2.1 [21]**.** *Let C be a nonempty and weakly compact subset of a Banach space E with the Opial condition and $T : C \to K(E)$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.2 [22]**.** *The following inequality holds in real space H:*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle \quad (\forall\, x, y \in H).$$

Lemma 2.3 [23]**.** *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a nonexpansive multi-valued map with Fix$(T) \neq \varnothing$, and $Tp = \{p\}$ for each $p \in$ Fix(T). Then Fix(T) is a closed and convex subset of C.*

Lemma 2.4 [24]**.** *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 and let T_r^F be defined as in Lemma 2.5, for $r > 0$. Let $x, y \in H$ and $t, s > 0$. Then,*

$$\|T_s^F y - T_t^F x\| \leqslant \|x - y\| + \left|\frac{s - t}{s}\right| \|T_s^F y - y\|.$$

Lemma 2.5 [25]**.** *Let C be a nonempty, closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Then for $r > 0$ and $x \in H$, there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0$ for all $y \in C$. Further define*

$$T_r^F x = \left\{ z \in C :\ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0 \right\} \quad (\forall\, y \in C)$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(i) *T_r^F is single-valued.*

(ii) T_r^F *is firmly nonexpansive, i. e.,*

$$\left\|T_r^F(x) - T_r^F(y)\right\|^2 \leqslant \left\langle T_r^F(x) - T_r^F(y), x - y\right\rangle \quad (\forall\, x, y \in H).$$

(iii) $Fix(T_r^F) = EP(F)$.

(iv) $EP(F)$ *is compact and convex.*

Lemma 2.6 [26]**.** *Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leqslant \|B\|^{-1}$. Then $\|I - \rho B\| \leqslant 1 - \rho\bar{\gamma}$.*

Lemma 2.7 [27, 28]**.** *Let C be a closed and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \leqslant 0 \quad (\forall\, y \in C).$$

Lemma 2.8 [29]**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geqslant 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.9 [10]**.** *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 and let T_r^F be defined as in Lemma 2.5, for $r > 0$. Let $x \in H$ and $s, t > 0$. Then,*

$$\left\|T_s^F x - T_t^F x\right\|^2 \leqslant \frac{s-t}{s}\left\langle T_s^F(x) - T_t^F(x), T_s^F(x) - x\right\rangle.$$

Lemma 2.10 [30]**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leqslant (1-\alpha_n)a_n + \delta_n$, $n \geqslant 0$, where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leqslant 0$ *or* $\Sigma_{n=1}^{\infty}\delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Assumption 2.1. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

1. $F(x,x) \geqslant 0$ $(\forall x \in C)$;
2. F *is monotone, i. e.,* $F(x,y) + F(y,x) \leqslant 0$ $(\forall\, x \in C)$;
3. F *is upper hemicontinuouse, i. e., for each* $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1-t)x, y) \leqslant F(x,y).$$

For each $x \in C$ fixed, the function $x \to F(x,y)$ is convex and lower semicontinuous.

3. A Nonlinear Iterative Algorithm

Let C be a nonempty closed convex subset of real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Let M, N be two $\bar{\alpha}$-inverse strongly monotone and $\bar{\beta}$-inverse strongly monotone mappings from C into H, respectively. Recall that the set of all solutions of problem (1.1) is denoted by GEPP, i. e.

$$\text{GEPP} = \Big\{\bar{x} \in C : \ F(\bar{x}, y) + \langle (M+N)\bar{x}, y - \bar{x}\rangle \geqslant 0 \ \ (\forall\, y \in C)\Big\}.$$

Let T be a nonexpansive multi-valued mapping on C into $K(H)$ such that $\Theta = \text{Fix}(T) \cap \text{GEPP} \neq \varnothing$. Also $f : C \to H$ be a α-contraction mapping and A, B be a strongly positive bounded linear self adjoint operators on H with coefficient $\bar{\gamma}_1 > 0$ and $\bar{\gamma}_2 > 0$ respectively such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, $\bar{\gamma}_1 \leqslant \|A\| \leqslant 1$ and $\|B\| = \bar{\gamma}_2$.

Algorithm 3.1. *For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = T_{r_n}^F(x_n - r_n(M+N)x_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A) z_n, \end{cases} \tag{3.1}$$

where $z_n \in Tu_n$ such that $\|z_{n+1} - z_n\| \leqslant H(Tu_{n+1}, Tu_n)$.

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$ are sequences in $(0,1)$, $\{r_n\} \subset [r, \infty)$ with $r > 0$ satisfied the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\limsup_{n\to\infty} \beta_n \neq 1$;

(C3) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $0 < b < r_n < a < 2\min\{\bar{\alpha}, \bar{\beta}\}$.

Lemma 3.1. *Let $p \in \Theta$. Then the sequence $\{x_n\}$ generated by Algorithm* 3.1 *is bounded.*

$\lhd$ We may assume without loss of generality that $\alpha_n \leqslant (1 - \epsilon_n - \beta_n\|B\|)\|A\|^{-1}$. Since A and B are linear bounded self adjoint operators, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\},$$
$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}$$

observe that

$$\begin{aligned} \big\langle \big((1-\epsilon_n)I - \beta_n B - \alpha_n A\big)x, x\big\rangle &= (1-\epsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle \\ &\geqslant 1 - \epsilon_n - \beta_n\|B\| - \alpha_n\|A\| \geqslant 0. \end{aligned}$$

Therefore, $(1-\epsilon_n)I - \beta_n B - \alpha_n A$ is positive. Then, by strong positivity of A and B, we get

$$\begin{aligned} \|(1-\epsilon_n)I - \beta_n B - \alpha_n A\| &= \sup\Big\{\big\langle\big((1-\epsilon_n)I - \beta_n B - \alpha_n A\big)x, x\big\rangle\ x \in H, \|x\| = 1\Big\} \\ &= \sup\Big\{(1-\epsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\Big\} \\ &\leqslant 1 - \epsilon_n - \beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1 \leqslant 1 - \beta_n\bar{\gamma}_2 - \alpha_n\bar{\gamma}_1. \end{aligned} \tag{3.2}$$

Let $p \in \Theta := \mathrm{Fix}(T) \cap \mathrm{GEPP}$. Since $p \in \mathrm{GEPP}$, from [4, Theorem 3.1] we have

$$\begin{aligned} \|u_n - p\|^2 &\leqslant \|x_n - p\|^2 + r_n(r_n - 2\bar{\alpha})\|Mx_n - Mp\|^2 \\ &+ r_n(r_n - 2\bar{\beta})\|Nx_n - Np\|^2 \leqslant \|x_n - p\|^2. \end{aligned} \tag{3.3}$$

Then $\|u_n - p\| \leqslant \|x_n - p\|$. We obtain

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\epsilon_n)I-\beta_n B-\alpha_n A)z_n-p\big\| \\
&\leqslant \alpha_n\|\gamma f(x_n)-Ap\|+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\|+\big\|((1-\epsilon_n)I-\beta_n B-\alpha_n A)\big\|\|z_n-p\| \\
&\leqslant \alpha_n\big(\|\gamma f(x_n)-\gamma f(p)\|+\|\gamma f(p)-Ap\|\big)+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)d(z_n,Tp) \\
&\leqslant \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\bar{\gamma}_2\|x_n-p\|+\alpha_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)H(Tu_n,Tp) \\
&\leqslant \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\bar{\gamma}_2\|x_n-p\|+\alpha_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|u_n-p\| \\
&\leqslant (1-(\bar{\gamma}_1-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n(\|p\|+\|\gamma f(p)-Ap\|) \\
&\leqslant \max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-Ap\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\right\} \\
&\ldots \leqslant \max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-Ap\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\right\}.
\end{aligned} \tag{3.4}$$

Hence $\{x_n\}$ is bounded. This implies that the sequences $\{u_n\}$, $\{z_n\}$ and $\{f(x_n)\}$ are bounded. ▷

Lemma 3.2. *The following properties are satisfying for the Algorithm* 3.1.
P1. $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.
P2. $\lim_{n\to\infty}\|x_n-t_n\| = 0$.
P3. $\lim_{n\to\infty}\|Mx_n-Mp\| = 0$ *and* $\lim_{n\to\infty}\|Nx_n-Np\| = 0$.
P4. $\lim_{n\to\infty}\|x_n-u_n\| = 0$.

◁ P1: We have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \big\|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-T_{r_n}(x_n-r_n(M+N)x_n)\big\| \\
&\leqslant \big\|(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_n-r_n(M+N)x_n)\big\| \\
&+\frac{|r_{n+1}-r_n|}{r_{n+1}}\big\|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_{n+1}-r_{n+1}(M+N)x_{n+1})\big\| \\
&\leqslant \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\,\|(M+N)(x_{n+1}-x_n)\|+\frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1},
\end{aligned} \tag{3.5}$$

where $\sigma_{n+1} = \sup_{n\in\mathbb{N}}\|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_{n+1}-r_{n+1}(M+N)x_{n+1})\|$.

Setting $x_{n+1} = \epsilon_n x_n + (1-\epsilon_n)e_n$, then we have

$$\begin{aligned}
e_{n+1}-e_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}Bx_{n+1}+((1-\epsilon_{n+1})I-\beta_{n+1}B-\alpha_{n+1}A)z_{n+1}-\epsilon_{n+1}x_{n+1}}{1-\epsilon_{n+1}} \\
&-\frac{\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\epsilon_n)I-\beta_n B-\alpha_n A)z_n-\epsilon_n x_n}{1-\epsilon_n} \\
&= \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}(\gamma f(x_{n+1})-Az_{n+1})+\frac{\alpha_n}{1-\epsilon_n}(Az_n-\gamma f(x_n)) \\
&+\left(\frac{\beta_{n+1}}{1-\epsilon_{n+1}}-\frac{\beta_n}{1-\epsilon_n}\right)B(x_{n+1}-x_n)+(z_{n+1}-z_n) \\
&+\left(\frac{\beta_n}{1-\epsilon_n}-\frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right)B(z_{n+1}-z_n)+\left(\frac{\epsilon_n}{1-\epsilon_n}-\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}}\right)(x_n-x_{n+1}).
\end{aligned}$$

Using (3.5), we have

$$
\begin{aligned}
\|e_{n+1}-e_n\| \leqslant\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\big\|\gamma f(x_{n+1})-Az_{n+1}\big\| + \frac{\alpha_n}{1-\epsilon_n}\big\|\gamma f(x_n)-Az_n\big\| \\
& + \left|\frac{\beta_{n+1}}{1-\epsilon_{n+1}} - \frac{\beta_n}{1-\epsilon_n}\right| \|B\| \|x_{n+1}-x_n\| \\
& + \|z_{n+1}-z_n\| + \left|\frac{\beta_n}{1-\epsilon_n} - \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right| \|B\| \|z_{n+1}-z_n\| + \left|\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}} - \frac{\epsilon_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\| \\
\leqslant\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\big\|\gamma f(x_{n+1})-Az_{n+1}\big\| + \frac{\alpha_n}{1-\epsilon_n}\big\|\gamma f(x_n)-Az_n\big\| \\
& + \left|\frac{\beta_{n+1}}{1-\epsilon_{n+1}} - \frac{\beta_n}{1-\epsilon_n}\right| \|B\| \|x_{n+1}-x_n\| \\
& + H(Tu_{n+1},Tu_n) + \left|\frac{\beta_n}{1-\epsilon_n} - \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right| \|B\| H(Tu_{n+1},Tu_n) \\
& + \left|\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}} - \frac{\epsilon_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\| \\
\leqslant\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\big\|\gamma f(x_{n+1})-Az_{n+1}\big\| + \frac{\alpha_n}{1-\epsilon_n}\big\|\gamma f(x_n)-Az_n\big\| \\
& + \left|\frac{\beta_{n+1}}{1-\epsilon_{n+1}} - \frac{\beta_n}{1-\epsilon_n}\right| \|B\| \|x_{n+1}-x_n\| \\
& + \|u_{n+1}-u_n\| + \left|\frac{\beta_n}{1-\epsilon_n} - \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right| \|B\| \|u_{n+1}-u_n\| \\
& + \left|\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}} - \frac{\epsilon_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\| \\
\leqslant\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\big\|\gamma f(x_{n+1})-Az_{n+1}\big\| + \frac{\alpha_n}{1-\epsilon_n}\big\|\gamma f(x_n)-Az_n\big\| \\
& + \left|\frac{\beta_{n+1}}{1-\epsilon_{n+1}} - \frac{\beta_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\| + \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\, \big\|(M+N)(x_{n+1}-x_n)\big\| \\
& + \frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \left|\frac{\beta_n}{1-\epsilon_n} - \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right| \bar{\gamma}_2 \Big(\|x_{n+1}-x_n\| + |r_{n+1}-r_n| \\
& \times \big\|(M+N)(x_{n+1}-x_n)\big\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \Big) + \left|\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}} - \frac{\epsilon_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\| \leqslant\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\big\|\gamma f(x_{n+1})-Az_{n+1}\big\| \\
& + \frac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\| + \left|\frac{\beta_{n+1}}{1-\epsilon_{n+1}} - \frac{\beta_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\| \\
& + |r_{n+1}-r_n| \big\|(M+N)(x_{n+1}-x_n)\big\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \\
& + \left|\frac{\beta_n}{1-\epsilon_n} - \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\right| \bar{\gamma}_2 \Big(\|x_{n+1}-x_n\| + |r_{n+1}-r_n|\, \|(M+N)(x_{n+1}-x_n)\| \\
& + \frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \Big) + \left|\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}} - \frac{\epsilon_n}{1-\epsilon_n}\right| \|x_{n+1}-x_n\|.
\end{aligned}
$$

Hence, it follows by conditions (C1)–(C4) that

$$\limsup_{n\to\infty}\big(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|\big)\leqslant 0. \tag{3.6}$$

From (3.6) and Lemma 2.8, we get $\lim_{n\to\infty}\|e_n-x_n\|=0$, and then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\epsilon_n)\|e_n-x_n\|=0. \tag{3.7}$$

P2: We can write

$$\begin{aligned}\|x_n-z_n\|&\leqslant\|x_{n+1}-x_n\|+\|\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)z_n-z_n\|\\&\leqslant\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Az_n\|+\beta_n\|Bx_n-Bz_n\|+\epsilon_n\|z_n\|\\&\leqslant\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Az_n\|+\beta_n\bar\gamma_2\|x_n-z_n\|+\epsilon_n\|z_n\|.\end{aligned}$$

Then

$$(1-\beta_n\bar\gamma_2)\|x_n-z_n\|\leqslant\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Az_n\|+\epsilon_n\|z_n\|.$$

Therefore

$$\begin{aligned}\|x_n-z_n\|&\leqslant\frac{1}{1-\beta_n\bar\gamma_2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n\bar\beta}\|\gamma f(x_n)-At_n\|+\frac{\epsilon_n}{1-\beta_n\bar\gamma_2}\|z_n\|\\&\leqslant\frac{1}{1-\beta_n\bar\gamma_2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n\bar\gamma_2}\big(\|\gamma f(x_n)-Az_n\|+\|z_n\|\big).\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and (C2) we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.8}$$

P3: From (3.3), we have

$$\begin{aligned}\|x_{n+1}-p\|^2&=\big\|\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)z_n-p\big\|^2\\&=\big\|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(Bx_n-Bp)+((1-\epsilon_n)I-\beta_nB-\alpha_nA)(z_n-p)-\epsilon_np\big\|^2\\&\leqslant\big\|((1-\epsilon_n)I-\beta_nB-\alpha_nA)(z_n-p)+\beta_n(Bx_n-Bp)-\epsilon_np\big\|^2\\&\quad+2\big\langle\alpha_n(\gamma f(x_n)-Ap),x_{n+1}-p\big\rangle\\&\leqslant\Big(\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)d(z_n,Tp)+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|\Big)^2\\&\quad+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\leqslant\Big(\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)H(Tu_n,Tp)\\&\quad+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|\Big)^2+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\\&\leqslant\Big(\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\|u_n-p\|+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|\Big)^2\\&\quad+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\\&=\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)^2\|u_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2\\&\quad+2\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\&\quad+2\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\&\quad+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\leqslant \big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)^2\Big(\|x_n-p\|^2+\ r_n(r_n-2\bar{\alpha})\|Mx_n-Mp\|^2\\
&+r_n(r_n-2\bar{\beta})\|Nx_n-Np\|^2\Big)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\leqslant\|x_n-p\|^2+\big(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1\big)^2\|x_n-p\|^2\\
&+\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)^2\Big(r_n(r_n-2\bar{\alpha})\|Mx_n-Mp\|^2+r_n(r_n-2\bar{\beta})\|Nx_n-Np\|^2\Big)\\
&+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.
\end{aligned}$$

By (C3), we can write

$$\begin{aligned}
&\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)^2\Big(r_n(2\bar{\alpha}-r_n)\|Mx_n-Mp\|^2+r_n(2\bar{\beta}-r_n)\|Nx_n-Np\|^2\Big)\\
&\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+\big(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1\big)^2\|x_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\alpha_n)^2\|p\|^2\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\alpha_n\|p\|\|u_n-p\|\\
&+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\\
&\leqslant\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+\big(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1\big)^2\|x_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&+(\alpha_n)^2\|p\|^2+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\alpha_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle.
\end{aligned}$$

By $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $\|x_n-z_n\|\to 0$, then we obtain $\|Mx_n-Mp\|\to 0$ and $\|Nx_n-Np\|\to 0$ as $n\to\infty$.

P4: Since $p\in\Theta$, we can obtain

$$\|u_n-p\|^2\leqslant\|x_n-p\|^2-\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\big(\|Mx_n-Mp\|+\|Nx_n-Np\|\big).$$

It follows from (3.9) that

$$\begin{aligned}
\|x_{n+1}-p\|^2&\leqslant\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)^2\|u_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\\
&\leqslant\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)^2\Big(\|x_n-p\|^2-\|u_n-x_n\|^2\\
&+2r_n\|u_n-x_n\|\big(\|Mx_n-Mp\|+\|Nx_n-Np\|\big)\Big)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&+(\epsilon_n)^2\|p\|^2+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)^2\|u_n-x_n\|^2 \leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2+\big(\beta_n\bar\gamma_2+\alpha_n\bar\gamma_1\big)^2\|x_n-p\|^2\\
&+2r_n\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)^2\|u_n-x_n\|\big(\|Mx_n-Mp\|+\|Nx_n-Np\|\big)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&+(\epsilon_n)^2\|p\|^2+2\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle\\
&\leqslant\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+(\beta_n\bar\gamma_2+\alpha_n\bar\gamma_1)^2\|x_n-p\|^2\\
&+2r_n\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)^2\|u_n-x_n\|(\|Mx_n-Mp\|+\|Nx_n-Np\|)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&+(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&+2\big(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1\big)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&+2\alpha_n\big\langle\gamma f(x_n)-Ap,x_{n+1}-p\big\rangle.
\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, $\|Mx_n-Mp\|\to 0$, $\|Nx_n-Np\|\to 0$ and $\|x_n-z_n\|\to 0$ as $n\to\infty$ and we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.10}$$

Using (3.8) and (3.10), we obtain $\|z_n-u_n\|\leqslant\|z_n-x_n\|+\|x_n-u_n\|\to 0$, as $n\to\infty$. Then $\lim_{n\to\infty}\|z_n-u_n\|=0$. ▷

4. Strong Convergence Algorithm

Theorem 4.1. *The Algorithm defined by* (3.1) *convergence strongly to* $z\in Fix(T)\cap GEPP$*, which is a unique solution in of the variational inequality* $\langle(\gamma f-A)z,y-z\rangle\leqslant 0$ *for all* $y\in\Theta$.

◁ Let $s=P_\Theta$. We get

$$\begin{aligned}
\|s(I-A+\gamma f)(x)-s(I-A+\gamma f)(y)\| &\leqslant \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\
\leqslant\|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\| &\leqslant (1-\bar\gamma_1)\|x-y\|+\gamma\alpha\|x-y\|\\
&=(1-(\bar\gamma_1-\gamma\alpha))\|x-y\|.
\end{aligned}$$

Then $s(I-A+\gamma f)$ is a contraction mapping from H into itself. Therefore by Banach contraction principle, there exists $z\in H$ such that $z=s(I-A+\gamma f)z=P_{\mathrm{Fix}(T)\cap\mathrm{EPP}}(I-A+\gamma f)z$.

We show that $\langle(\gamma f-A)z,x_n-z\rangle\leqslant 0$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle(\gamma f-A)z,x_n-z\big\rangle=\lim_{i\to\infty}\big\langle(\gamma f-A)z,x_{n_i}-z\big\rangle. \tag{4.1}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $w\in C$. Without loss of generality, we can assume that $x_{n_i}\rightharpoonup w$. Now, we prove that $w\in\mathrm{Fix}(S)\cap\mathrm{GEPP}$. Let us first show that $w\in\mathrm{Fix}(S)$. From $\|x_n-u_n\|\to 0$, we obtain $u_{n_i}\rightharpoonup w$. On the other hand $\lim_{n\to\infty}\|z_n-u_n\|=0$ and by Lemma 2.1, $I-T$ is demiclosed at 0. Thus, we obtain $w\in\mathrm{Fix}(T)$. We show that $w\in\mathrm{GEPP}$. Since $u_n=T_{r_n}(x_n-r_n(M+N)x_n)$. we have

$$F(u_n,y)+\big\langle(M+N)x_n,y-u_n\big\rangle+\frac{1}{r_n}\big\langle y-u_n,u_n-x_n\big\rangle\geqslant 0\quad(\forall\, y\in C).$$

It follows from the monotonicity of F that

$$\big\langle (M+N)x_n, y-u_n\big\rangle + \frac{1}{r_n}\big\langle y-u_n, u_n-x_n\big\rangle \geqslant F(y,u_n) \quad (\forall\, y\in C)$$

which implies that

$$\big\langle (M+N)x_{n_i}, y-u_{n_i}\big\rangle + \frac{1}{r_{n_i}}\big\langle y-u_{n_i}, u_{n_i}-x_{n_i}\big\rangle \geqslant F(y,u_{n_i}) \quad (\forall\, y\in C).$$

Let $u_t = ty+(1-t)w$ for all $t\in(0,1]$. Since $y\in C$ and $w\in C$, we get $u_t\in C$. It follows that

$$\begin{aligned}
\big\langle u_t-u_{n_i}, (M+N)u_t\big\rangle &\geqslant \big\langle u_t-u_{n_i}, (M+N)u_t\big\rangle - \big\langle u_t-u_{n_i}, (M+N)x_{n_i}\big\rangle\\
-\Big\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle &+ F(u_t,u_{n_i}) = \big\langle u_t-u_{n_i}, (M+N)u_t-(M+N)u_{n_i}\big\rangle\\
+\big\langle u_t-u_{n_i}, (M+N)u_{n_i}-(M+N)x_{n_i}\big\rangle &- \Big\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle + F(u_t,u_{n_i})\\
&= \big\langle u_t-u_{n_i}, Mu_t-Mu_{n_i}\big\rangle + \big\langle u_t-u_{n_i}, Nu_t-Nu_{n_i}\big\rangle\\
&+\big\langle u_t-u_{n_i}, Mu_{n_i}-Mx_{n_i}\big\rangle + \big\langle u_t-u_{n_i}, Nu_{n_i}-Nx_{n_i}\big\rangle\\
&-\Big\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle + F(u_t,u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|Mu_{n_i}-Mx_{n_i}\|\to 0$ and $\|Nu_{n_i}-Nx_{n_i}\|\to 0$.

Further from monotonically of M and N, we obtain

$$\begin{aligned}
\big\langle u_t-u_{n_i}, Mu_t-Mu_{n_i}\big\rangle \geqslant 0,\\
\big\langle u_t-u_{n_i}, Nu_t-Nu_{n_i}\big\rangle \geqslant 0,
\end{aligned}$$

so as $i\to\infty$ from Assumption 2.1, we have $\big\langle u_t-w, (M+N)u_t\big\rangle \geqslant F(u_t,w)$.

Therefore

$$\begin{aligned}
0 = F(u_t,u_t) &\leqslant tF(u_t,y)+(1-t)F(u_t,w)\\
&\leqslant tF(u_t,y)+(1-t)\big\langle u_t-w,(M+N)u_t\big\rangle \leqslant tF(u_t,y)+(1-t)t\big\langle y-w,(M+N)u_t\big\rangle,
\end{aligned}$$

then $0\leqslant F(u_t,y)+(1-t)\big\langle y-w,(M+N)u_t\big\rangle$.

Letting $t\to 0$, we obtain $0\leqslant F(w,y)+\big\langle y-w,(M+N)w\big\rangle$. This implies that $w\in$ GEPP.

Now from Lemma 2.7, we have

$$\limsup_{n\to\infty}\big\langle (\gamma f-A)z, x_n-z\big\rangle \leqslant \limsup_{i\to\infty}\big\langle (\gamma f-A)z, x_{n_i}-z\big\rangle = \big\langle (\gamma f-A)z, w-z\big\rangle \leqslant 0. \quad (4.2)$$

Now we prove that x_n is strongly convergence to z.

It follows from (3.3) that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \alpha_n\big\langle \gamma f(x_n)-Az, x_{n+1}-z\big\rangle + \beta_n\big\langle Bx_n - Bz, x_{n+1}-z\big\rangle \\
&\quad -\epsilon_n\big\langle z, x_{n+1}-z\big\rangle + \Big\langle \big((1-\epsilon_n)I-\beta_n B-\alpha_n A\big)(z_n-z), x_{n+1}-z\Big\rangle \\
&\leqslant \alpha_n\big(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle\big) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\| \\
&\quad -\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_n B-\alpha_n A\|\|z_n-z\|\|x_{n+1}-z\| \\
&\leqslant \alpha_n\big(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle\big) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\| \\
&\quad -\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_n B-\alpha_n A\|d(z_n, Tz)\|x_{n+1}-z\| \\
&\leqslant \alpha_n\big(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle\big) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\| \\
&\quad -\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_n B-\alpha_n A\|H(Tu_n, Tz)\|x_{n+1}-z\| \\
&\leqslant \alpha_n\alpha\gamma\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle + \beta_n\bar{\beta}\|x_n-z\|\|x_{n+1}-z\| \\
&\quad -\epsilon_n\|z\|\|x_{n+1}-z\| + (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|x_n-z\|\|x_{n+1}-z\| \\
&= \big(1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)\big)\|x_n-z\|\|x_{n+1}-z\| - \epsilon_n\|z\|\|x_{n+1}-z\| + \alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle \\
&\leqslant \frac{1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)}{2}\big(\|x_n-z\|^2+\|x_{n+1}-z\|^2\big)-\epsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle \\
&\leqslant \frac{1-\alpha_n(\bar{\delta}-\alpha\gamma)}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2-\epsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
2\|x_{n+1}-z\|^2 &\leqslant \big(1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)\big)\|x_n-z\|^2 + \|x_{n+1}-z\|^2 \\
&\quad -2\alpha_n\|z\|\|x_{n+1}-z\| + 2\alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leqslant \big(1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)\big)\|x_n-z\|^2 - 2\alpha_n\|z\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle = (1-k_n)\|x_n-z\|^2 + 2\alpha_n l_n,
\end{aligned} \tag{4.3}$$

where $k_n = \alpha_n(\bar{\gamma}_1-\alpha\gamma)$ and $l_n = \big\langle \gamma f(z)-Az, x_{n+1}-z\big\rangle - \|z\|\|x_{n+1}-z\|$.

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} k_n = 0$, $\Sigma_{n=0}^{\infty}k_n = \infty$ and $\limsup_{n\to\infty} l_n \leqslant 0$. Hence, from (4.2) and (4.3) and Lemma 2.10, we deduce that $x_n \to z$, where $z = P_\Theta(I-A+\gamma f)z$. ▷

Remark 4.1. Putting $A = B = M = N = 0$, $\gamma = 1$, we obtain methods introduced in [31, Theorem 3.1].

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem.

All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory.

Example 5.1. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [0,2]$; let $F : C\times C \to \mathbb{R}$ be defined by $F(x,y) = (x-4)(y-x)$ for all $x, y \in C$; let $M, N : C \to H$ be defined by $M(x) = x$ and $N(x) = 2x$ for all $x \in C$, such that $\bar{\alpha} = \frac{1}{2}$ and $\bar{\beta} = \frac{1}{3}$ respectively, and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{8}x$, $A(x) = 2x$, $B(x) = \frac{1}{3}x$ and

$$Tx = \begin{cases} \{x\}, & 0 \leqslant x \leqslant 1; \\ \left\{\frac{1}{2}\right\}, & 1 < x \leqslant 2. \end{cases}$$

Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ and $\{u_n\} \subset C$ generated by the iterative schemes

$$u_n = T^F_{r_n}\big(x_n - r_n(M+N)x_n\big); \tag{4.4}$$

$$x_{n+1} = \left(\frac{1}{8n} + \frac{1}{3n^2}\right)x_n + \left(\left(1 - \frac{1}{2n^2-3}\right)I - \frac{1}{n^2}B - \frac{1}{n}A\right)z_n, \tag{4.5}$$

where $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n^2}$, $\epsilon_n = \frac{1}{2n^2-3}$ and $r_n = 1$. Then $\{x_n\}$ converges to $\{1\} \in \text{Fix}(T) \cap \text{GEPP}$.

It is easy to prove that the bifunction F satisfy the Assumption 2.1. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\text{Fix}(T) = [0,1]$ and $\text{GEPP} = \{1\}$. Hence $\text{Fix}(T) \cap \text{GEPP} = \{1\} \neq \varnothing$. After simplification, schemes (4.6) and (4.7) reduce to $u_n = 2 - x_n$.

$$Tu_n = \begin{cases} \{2 - x_n\}, & 0 \leqslant u_n \leqslant 1 \text{ or } (1 \leqslant x_n \leqslant 2); \\ \{\frac{1}{2}\}, & 1 < u_n \leqslant 2 \text{ or } (0 \leqslant x_n < 1). \end{cases}$$

If $z_n = 2 - x_n$ for $x_n \in [1,2]$, we have

$$x_{n+1} = \left(-1 + \frac{17}{8n} + \frac{2}{3n^2} + \frac{1}{2n^2-3}\right)x_n + 2\left(1 - \frac{1}{2n^2-3} - \frac{1}{3n^2} - \frac{2}{n}\right).$$

If $z_n = \frac{1}{2}$ for $x_n \in [0,1)$, we have

$$x_{n+1} = \left(\frac{1}{8n} + \frac{1}{3n^2}\right)x_n + \frac{1}{2}\left(1 - \frac{1}{2n^2-3} - \frac{1}{3n^2} - \frac{2}{n}\right).$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$, $\{u_n\}$ converges strongly to $w = \{1\} \in \text{Fix}(T) \cap \text{GEPP}$ as $n \to \infty$.

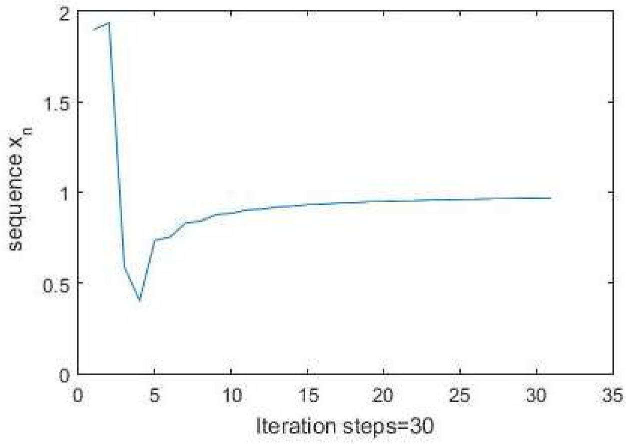

Fig. 1. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

Example 5.2. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-1, 3]$; let $F : C \times C \to \mathbb{R}$ be defined by $F(x,y) = x(y-x)$ for all $x, y \in C$; let $M, N : C \to H$ be defined by $M(x) = 2x$ and $N(x) = 3x$ for all $x \in C$, such that $\bar{\alpha} = \frac{1}{3}$ and $\bar{\beta} = \frac{1}{4}$ respectively, and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{6}x$, $A(x) = \frac{x}{2}$, $B(x) = \frac{1}{10}x$ and

$$Tx = \begin{cases} \{\frac{x}{2}\}, & 0 < x \leqslant 3; \\ \{0\}, & -1 \leqslant x \leqslant 0. \end{cases}$$

Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ and $\{u_n\} \subset C$ generated by the iterative schemes

$$u_n = T^F_{r_n}(x_n - r_n(M+N)x_n); \tag{4.6}$$

$$x_{n+1} = \left(\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2}\right)x_n + \left(\left(1 - \frac{2}{n^2}\right)I - \frac{1}{(n+1)^2}B - \frac{1}{\sqrt{n}}A\right)z_n, \tag{4.7}$$

where $\alpha_n = \frac{1}{\sqrt{n}}$, $\beta_n = \frac{1}{(n+1)^2}$, $\epsilon_n = \frac{2}{n^2}$ and $r_n = 1 + \frac{1}{n}$. Then $\{x_n\}$ converges to $\{0\} \in \mathrm{Fix}(T) \cap \mathrm{GEPP}$.

It is easy to prove that the bifunction F satisfy the Assumption 2.1. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\mathrm{Fix}(T) = \{0\}$ and $\mathrm{GEPP} = \{0\}$. Hence $\mathrm{Fix}(T) \cap \mathrm{GEPP} = \{0\} \neq \varnothing$. After simplification, schemes (4.6) and (4.7) reduce to

$$u_n = \left(\frac{-4n-5}{2n+1}\right)x_n,$$

$$Tu_n = \begin{cases} \{0\}, & -15 \leqslant u_n < 0 \ \text{ or } \ (0 < x_n \leqslant 3); \\ \left\{\left(\frac{-4n-5}{4n+2}\right)x_n\right\}, & 0 \leqslant u_n \leqslant 2 \ \text{ or } \ (-1 \leqslant x_n \leqslant 0). \end{cases}$$

If $z_n = \frac{-4n-5}{4n+2}x_n$ for $x_n \in [-1, 0]$, we have

$$x_{n+1} = \left(\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2}\right)x_n + \left(1 - \frac{2}{n^2} - \frac{1}{10(n+1)^2} - \frac{1}{2\sqrt{n}}\right)\left(\frac{-4n-5}{4n+2}\right)x_n.$$

If $z_n = 0$ for $x_n \in (0, 3]$, we have

$$x_{n+1} = \left(\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2}\right)x_n.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$, $\{u_n\}$ converges strongly to $w = \{0\} \in \mathrm{Fix}(T) \cap \mathrm{GEPP}$ as $n \to \infty$.

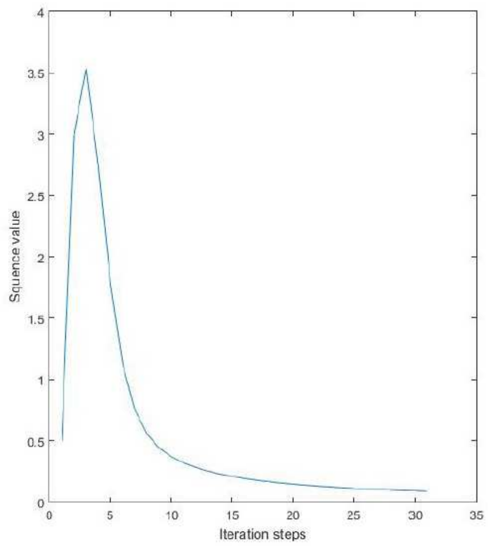

Fig. 2. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

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АЛГОРИТМ НЕЛИНЕЙНОЙ ВЯЗКОСТИ С ВОЗМУЩЕНИЕМ ДЛЯ НЕРАСШИРЯЮЩИХ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ

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Аннотация. Итерационные алгоритмы вязкости для поиска общего элемента множества неподвижных точек нелинейных операторов и множества решений вариационных неравенств исследовались многими авторами. Соответствующая техника позволяет применить этот метод к выпуклой оптимизации, линейному программированию и монотонным включениям. В этой статье на основе метода вязкости с возмущением, мы вводим новый алгоритм нелинейной вязкости для нахождения элемента множества неподвижных точек нерасширяющих многозначных отображений в гильбертовом пространстве. Установлены теоремы о сильной сходимости этого алгоритма при подходящих предположениях относительно параметров. Наши результаты можно рассматривать как обобщение и усиление имеющихся в текущей литературе результатов. Представлены также некоторые числовые примеры, показывающие эффективность и применитость предложенного алгоритма.

Ключевые слова: проблема неподвижной точки, обобщенного проблема равновесия, нерасширяющее многозначное отображение, гильбертово пространство.

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