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A NOTE ON PERIODIC RINGS[#]

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Abstract. We obtain a new and non-trivial characterization of periodic rings (that are those rings R for which, for each element x in R, there exists two different integers m, n strictly greater than 1 with the property $x^m = x^n$) in terms of nilpotent elements which supplies recent results in this subject by Cui–Danchev published in (J. Algebra & Appl., 2020) and by Abyzov–Tapkin published in (J. Algebra & Appl., 2022). Concretely, we state and prove the slightly surprising fact that an arbitrary ring R is periodic if, and only if, for every element x from R, there are integers $m > 1$ and $n > 1$ with $m \neq n$ such that the difference $x^m - x^n$ is a nilpotent.

Key words: potent rings, periodic rings, nilpotent elements.

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1. Introduction and Background

Everywhere in the text of this short note, as usual all rings R are assumed to be associative, containing the identity element 1 which differs from the zero element 0 of R. Standardly, $\mathrm{Nil}\,(R)$ stands for the set of all nilpotents in R and $J(R)$ for the Jacobson radical of R.

It is well known that a ring R is called *periodic* if, for each $r \in R$, there exist two distinct natural numbers, say $m \neq n$, both depending on r such that $r^m = r^n$. In particular, when $m = 1$, that is, $r^n = r$, these rings are said to be *potent*.

For example, finite rings are always periodic. These rings are surely non-commutative as the example of the upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$ over the two elements field $\mathbb{Z}_2$ shows — the elements of this ring actually satisfy the equation $r^2 = r^4$. However, in some specific situations, periodic rings are necessarily commutative (see, e.g., [1]). In fact, for instance, the ring R whose elements satisfy the equation $r^2 = r^3$ are always commutative, and even more surprisingly the equality $r^2 = r$ holds for all $r \in R$. Even more generally, if R is a ring for which, for each $x \in R$, there exist two positive integers m, n of opposite parity with $m < n$

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such that $x^m = x^n$, then the equality $x^{n-m+1} = x$ holds for every element x of R, and thus R is definitely potent.

A necessary and sufficient condition for a ring to be periodic in terms of nilpotent elements was obtained in [2], namely it was proved there that *a ring R is periodic if, and only if, for each $a \in R$, the PI relation $a - a^k \in \mathrm{Nil}\,(R)$ holds for some integer $k \geqslant 2$ depending on a.* An important result was established in [3] in order to characterize these rings R for a fixed integer $k > 1$ (it is worthwhile noticing that some further generalizations were established in [4]).

That is why, in view of all facts presented above, it is rather logical in a connection with periodical rings, to initiate the study of those rings R for which the PI-identity $x^n - x^m \in \mathrm{Nil}\,(R)$ is fulfilled for an arbitrary element $x \in R$ and two naturals m and n depending on it (not considering they to be fixed). Concretely, if we consider the fixed case $m = 2$ and $n = 3$, i. e., $x^2 - x^3 \in \mathrm{Nil}\,(R)$ for all $x \in R$, by using the idea from [1] to replace $x \to x + 1$, we will arrive at the PI-identity $x^2 - x \in \mathrm{Nil}\,(R)$ taking into account that $2 \in \mathrm{Nil}\,(R)$ as choosing $x = -1$.

And so, the purpose of the present brief article is to demonstrate that nothing new will *not* arise and, in addition, to give a new surprising criterion for a ring to be periodic in light of the described above results.

2. The Main Result

Our main motivating result is the following curious necessary and sufficient condition for a ring to be periodic. Specifically, the following assertion is true.

Proposition 2.1. *Let R be a ring. Then the following three points are equivalent:*

(1) *R is periodic.*

(2) *For any $x \in R$, there exist two different $i, j \in \mathbb{N} \setminus \{1\}$ such that $x^i - x^j \in J(R)$ and $J(R)$ is nil.*

(3) *For any $x \in R$, there exist two different $i, j \in \mathbb{N} \setminus \{1\}$ such that $x^i - x^j \in \mathrm{Nil}\,(R)$.*

$\lhd$ Clearly (1) implies (2), and (2) implies (3) since $J(R) \subseteq \mathrm{Nil}\,(R)$, so it remains to show the validity of the implication (3) $\Rightarrow$ (1). To that goal, we may without loss of generality assume that $i > j > 1$. So, one observes that the following series of implications is true for any $x \in R$, that is,

$$(x^i - x^j)^t = 0 \Rightarrow (x^{j-1}(x^{i-j+1} - x)^t = 0 \Rightarrow (k := i - j + 1) \vee (x^k - x)^{t+t(j-1)} = 0,$$

which technical verification we leave to the interested reader for a direct check. We, consequently, may apply our aforementioned criterion from [2] to get the truthfulness of the desired claim. $\rhd$

In the spirit of [1], it is quite adequate to ask what happens with the ring R when the integers i and j are of opposite parity.

We end our scientific work with the following question of some interest and importance, which is closely related to Proposition 2.1 and which was partly settled in [4].

PROBLEM 2.1: Suppose that $l, m > 1$ are fixed integers that are different each other. Describe the structure of those rings R for which (either) one of the relation $x^l - x^m \in \mathrm{Nil}\,(R)$ or $x^l + x^m \in \mathrm{Nil}\,(R)$ holds for all $x \in R$.

Maybe using the same or a similar idea for proof as in Proposition 2.1, these rings are relevant to the generalized version of periodic rings, i. e., to rings satisfying the at least one of the PI-identities $x^n = \pm x^m$ for some integers m, n with $m \neq n$ which are greater than 1.

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ОДНО ЗАМЕЧАНИЕ О ПЕРИОДИЧЕСКИХ КОЛЬЦАХ

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Аннотация. В терминах нильпотентных элементов получена новая нетривиальная характеризация периодических колец. (Так называют кольца R, в которых для любого элемента $x \in R$ существуют два различных целых числа m и n, строго большие чем 1, такие, что $x^m = x^n$.) Этот результат содержит в себе результат Цуй — Данчева на эту тему, опубликованный в J. Algebra & Appl., 2020, и результат Абызова — Тапкина, опубликованный в J. Algebra & Appl., 2022. Точнее говоря, установлен такой неожиданный факт: произвольное кольцо R будет периодическим в том и только в том случае, когда для любого элемента x из R, существуют целые числа $m > 1$ и $n > 1$, $m \neq n$, такие, что разность $x^m - x^n$ — нильпотентный элемент.

Ключевые слова: потентные кольца, периодические кольца, нильпотентные элементы.

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