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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH INHOMOGENEOUS DENSITY

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Abstract. In this paper we study the large time behaviour for solutions to the Cauchy problem for degenerate parabolic equations with inhomogeneous density. Under the suitable assumptions on the data of the problem and on the behaviour of the density at infinity we establish new sharp bound of solutions for a large time. One of the main tool of the proof is new weighted embedding result which is of independent interest. In addition, the proof of uniform estimates of the solution is carried out by modified version of the classical method of De-Giorgi–Ladyzhenskaya–Uraltseva–DiBenedetto. Similar results in the case of power-like density was obtained by one of the author [10]. The approach of this work can be applied for example when studying the qualitative properties of solutions to the Neumann problem for a doubly nonlinear parabolic equation with inhomogeneous density in domains with non-compact boundaries.

Key words: degenerate parabolic equation, inhomogeneous density, weighted embedding, large time behavior.

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1. Introduction

We consider the Cauchy problem for degenerate parabolic equations in the form

$$\rho(|x|)u_t(x,t) = \Delta_{m,p}u(x,t), \quad (x,t) \in S_T = \mathbf{R}^N \times (0,T),\ N \geqslant 1, \tag{1.1}$$

$$u(x,0) = u_0(x) \geqslant 0, \quad \text{a. e. for } x \in \mathbf{R}^N. \tag{1.2}$$

Here

$$\Delta_{m,p}u(x,t) := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(u^{m-1}\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

We assume that $p > 1$ and $p+m-3 > 0$ that is (1.1) is of the slow diffusion type. Additionaly, we assume that $p < N$. The function $\rho(s)$ is assumed to be positive nonincreasing, continuous function in $[0,\infty)$, satisfying the condition $\rho(0) = 1$, and

H_1: there exists a positive constant l: $l < p$, such that the function $s^l\rho(s)$ is nonincreasing for all $s \in (1, \infty)$,

H_2: there exists a positive constant $\alpha < p$, such that the function $s^\alpha\rho(s)$ is nondecreasing for all $s \in (1, \infty)$.

Note that H_1 and H_2 imply that there exists $\gamma > 1$ independent of s, such that $\gamma^{-1}s^{-\alpha} \leqslant \rho(s) \leqslant \gamma s^{-l}$ for all $s \in (1, \infty)$.

The qualitative theory of degenerate parabolic equations with variable coefficients and, in particular, with inhomogeneous density has attracted much attention. This is explained by the fact that the asymptotic properties of solutions essentially depend on the nature of the behavior of the coefficients at infinity. In particular, the typical properties of degenerate equations with constant coefficients, such as the compactness of the support, the behavior at large values of time may not take place depending on the degree of degeneracy of the inhomogeneous density at infinity. Firstly the surprising properties of solutions for degenerate parabolic equations with inhomogeneous density were established in [1]. Namely, in one dimensional Cauchy problem for the general porous media equation (PME) with inhomogeneous density, provided the density decays fast enough the solution of the corresponding Cauchy problem tends to the constant as $t \to \infty$. The latter means that the solution of the mentioned Cauchy problem as $t \to \infty$ behaves as a solution of the corresponding Neumann problem in a bounded domain. The paper [2] is devoted to the asymptotic behaviour of solutions to the Cauchy problem for inhomogeneous PME for slowly decaying density. These results were extended and developed in [3–7]. Another nonstandard property of degenerate parabolic equation with inhomogeneous density is the possible absence of the finite speed of propagation globally in time see [8]. Results of [8] were generalized in [9, 10] (see also [11]), where in particular, new critical exponents have been found for a doubly degenerate parabolic equations with inhomogeneous density. The paper [12] was devoted to the studying of large time behavior for degenerate Neumann problem in domains with noncompact boundaries. We recommend to the reader interested the qualitative theory of degenerate parabolic equations the survey [13]

The goal of the paper is to obtain the precise rate of stabilization of $\|u(t)\|_{\infty,\mathbf{R}^N}$ as $t \to \infty$. To this end we need the precise form of the nonpower weighted Sobolev–Gagliardo–Nirenberg which is of independent interest. Here we note that the classical weighted Caffarelli–Kohn–Nirenberg inequality [14] deals with the power-like weights.

Let us start with definition of the weak solution of (1.1), (1.2).

Definition 1.1. By a weak solution of problem (1.1), (1.2) in S_T we mean a non-negative measurable function $u(x,t)$ such that for $\sigma = (p-1)/(p+m-2)$, and any $t > 0$, $u(x,t)^{1/\sigma}$ belongs to the class $L_p((t,T)\times W_p^1(\mathbf{R}^N))\cap C([t,T]; L_{1+\sigma,\rho}(\mathbf{R}^N))$ and (1.1), (1.2) is satisfied in the distributional sense. Moreover, $\rho u(\cdot,t) \to \rho u_0$ as $t \to 0$ in $L_1(\mathbf{R}^N)$. Here, by $W_p^1(\mathbf{R}^N)$ and $L_{1+\sigma,\rho}$ $(\mathbf{R}^N)$ we denote the Sobolev space and weighted Lebesgue integral correspondingly.

Before formulating results of the paper, we define

$$\omega(s) := \rho(s)s^p, \quad \psi_{p,a}(s) := \omega(s)s^{\frac{N-p}{p}(p-a)}. \tag{1.3}$$

Let also

$$E_{q,\rho}(f) := \int\limits_{\mathbf{R}^N} \rho(|x|)\,|f(x)|^q\,dx \quad \text{and} \quad D_p(f) := \int\limits_{\mathbf{R}^N} |\nabla f(x)|^p\,dx.$$

Our first result reads as follows.

Theorem 1.1. *Let $D_p(f)$, $E_{a,\rho}(f) < \infty$, where $0 < q \leqslant p$ and $0 < a < q$, then we have*

$$E_{q,\rho}(f) \leqslant C(N,p)\big(D_p(f)\big)^{\frac{q-a}{p-a}}$$

$$\times\left[\omega\left(\psi^{(-1)}\left(\frac{E_{a,\rho}(f)}{(D_p(f))^{\frac{a}{p}}}\right)\right)\right]^{\frac{q-a}{p-a}}\big(E_{a,\rho}(f)\big)^{\frac{p-q}{p-a}}, \tag{1.4}$$

where $\psi^{(-1)}$ is inverse function to ψ.

The optimal decay rate is given by the following theorem.

Theorem 1.2. *Let $u(x,t)$ be a weak solution of (1.1), (1.2). Assume that $\|\rho u_0\|_{L_1(\mathbf{R}^N)} < \infty$, then there exists globaly in time solution $u(x,t)$ in S_∞ and for any $t > 0$ the following estimate holds true*

$$\big\|u(t)\big\|_{L_\infty(\mathbf{R}^N)} \leqslant \gamma t^{-\frac{1}{p+m-3}}$$

$$\times\Big[\omega\Big(\phi_1^{(-1)}\Big(\big(\|u(t)\|^{-1}_{L_\infty(\mathbf{R}^N)}\|u_0\rho\|_{1,\mathbf{R}^N}\big)^{\frac{1}{N}}\Big)\Big)\Big]^{\frac{1}{p+m-3}}. \tag{1.5}$$

Remark 1.1. Note that if $\rho(s) = (1+|x|)^{-\alpha}$, then (1.5) implies that (see [10])

$$\big\|u(t)\big\|_{L_\infty(\mathbf{R}^N)} \leqslant \gamma\,\big\|\rho u_0\big\|_{L_1(\mathbf{R}^N)}^{\frac{p-\alpha}{(N-\alpha)(p+m-3)+p-\alpha}}\, t^{-\frac{N-\alpha}{(N-\alpha)(p+m-3)+p-\alpha}}.$$

Remark 1.2. If $\alpha = 0$ and $m = 1$, then the latter coincides with the classical result (see, for example, [15]).

In the proof of Theorem 1.2. we use the classical De Giorgi–Ladyzhenskaya–Uraltseva–Di Benedetto approach in the form of [16, 17]. The rest of the paper is organized as follows. In the Chapter 2 we prove Theorems 1.1 and 1.2. In what follows, we use the symbols $\gamma > 0$, $b > 1$ for the constants depending on the parameters of the problem, p, m, N only and which may vary from line to line. Moreover, for simplicity we will understand the equation almost everywhere.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1.

$\lhd$ Denote

$$E_{q,\rho}(f) := \int\limits_{\mathbf{R}^N} |f|^q\rho(|x|)\,dx, \quad D_p(f) := \int\limits_{\mathbf{R}^N} |\nabla f|^p dx.$$

We have

$$E_{q,\rho}(f) = \int\limits_{B_r} |f|^q\rho(|x|)\,dx + \int\limits_{\mathbf{R}^N\setminus B_r} |f|^q\rho(|x|)\,dx. \tag{2.1}$$

By the Hölder inequality, we get

$$\int\limits_{\mathbf{R}^N\setminus B_r} |f|^q\rho(|x|)\,dx \leqslant \left(\int\limits_{\mathbf{R}^N} |f|^{p^*}dx\right)^{\frac{q-a}{p^*-a}} \left(\int\limits_{\mathbf{R}^N\setminus B_r} |f|^a\rho\big(|x|\big)^{\frac{p^*-a}{p^*-q}}\,dx\right)^{1-\frac{q-a}{p^*-a}}, \tag{2.2}$$

$$p^* = \frac{Np}{N-p}.$$

Making use the S. L. Sobolev inequality:

$$\left(\int\limits_{\mathbf{R}^N} |f|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant \big(S(N,p)D_p(f)\big)^{\frac{1}{p}},$$

where $S(N,p)$ is the sharp constant in this inequality, and owing to the monotonicity of ρ:

$$\int\limits_{\mathbf{R}^N\setminus B_r} |f|^a \rho\big(|x|\big)^{\frac{p^*-a}{p^*-q}}\, dx \leqslant \rho(r)^{\frac{p^*-a}{p^*-q}-1} E_{a,\,\rho}(f),$$

we get from (2.2) that

$$\int\limits_{\mathbf{R}^N\setminus B_r} |f|^q\, \rho(|x|)\, dx \leqslant \big(SD_p(f)\big)^{\frac{p^*}{p}\frac{q-a}{p^*-a}}\, \rho(r)^{\frac{q-a}{p^*-a}}\, E_{a,\rho}(f)^{\frac{p^*-q}{p^*-a}}. \tag{2.3}$$

Next, using the classical Hardy inequality

$$\int\limits_{\mathbf{R}^N} |f|^p\, |x|^{-p}\, dx \leqslant \Big(\frac{p}{N-p}\Big)^p D_p(f),$$

and the Hölder inequality, we obtain

$$\begin{aligned}\int\limits_{B_r} |f|^q\, \rho(|x|)\, dx &\leqslant \left(\int\limits_{\mathbf{R}^N} |f|^p\, |x|^{-p}\, dx\right)^{\frac{q-a}{p-a}} \left(\int\limits_{B_r} |f|^a\, |x|^{p\frac{q-a}{p-q}}\, \rho\big(|x|\big)^{\frac{p-a}{p-q}}\, dx\right)^{\frac{p-q}{p-a}} \\ &\leqslant C(N,p)\, \big(D_p(f)\big)^{\frac{q-a}{p-a}}\, \big(r^p\rho(r)\big)^{\frac{q-a}{p-a}}\, \big(E_{a,\rho}(f)\big)^{\frac{p-q}{p-a}}.\end{aligned} \tag{2.4}$$

Combining now (2.1), (2.3) and (2.4), we get

$$\begin{aligned}E_{q,\rho}(f) &\leqslant \big(SD_p(f)\big)^{\frac{p^*}{p}\frac{q-a}{p^*-a}}\, \rho(r)^{\frac{q-a}{p^*-a}}\, E_{a,\rho}(f)^{\frac{p^*-q}{p^*-a}} \\ &+ C(N,p)\, \big(D_p(f)\big)^{\frac{q-a}{p-a}}\, \big(r^p\rho(r)\big)^{\frac{q-a}{p-a}}\, \big(E_{a,\rho}(f)\big)^{\frac{p-q}{p-a}}.\end{aligned}$$

Finally, choosing the free parameter r from the relation:

$$\big(SD_p(f)\big)^{\frac{p^*}{p}\frac{q-a}{p^*-a}} \rho(r)^{\frac{q-a}{p^*-a}} E_{a,\rho}(f)^{\frac{p^*-q}{p^*-a}} = C(N,p)\big(D_p(f)\big)^{\frac{q-a}{p-a}}\, \big(r^p\rho(r)\big)^{\frac{q-a}{p-a}}\, \big(E_{a,\rho}(f)\big)^{\frac{p-q}{p-a}},$$

we have

$$E_{q,\rho}(f) \leqslant C_1(N,p)\big(D_p(f)\big)^{\frac{q-a}{p-a}} \left[\omega\left(\psi^{(-1)}\left(\frac{E_{a,\rho}(f)}{(D_p(f))^{\frac{a}{p}}}\right)\right)\right]^{\frac{q-a}{p-a}} \big(E_{a,\rho}(f)\big)^{\frac{p-q}{p-a}}.$$

If $q=p$, we get from the last inequality:

$$E_{p,\rho}(f) \leqslant C_1(N,p)\, D_p(f) \left[\omega\left(\psi^{(-1)}\left(\frac{E_{a,\rho}(f)}{(D_p(f))^{\frac{a}{p}}}\right)\right)\right].$$

The proof of (1.4) is similar. We give the sketch of the proof only. We have for any R fixed. Applying the Hölder, Hardy and Sobolev inequalities, we obtain

$$E_{p,\rho}(f) = \int\limits_{B_R} |f|^p \rho\, dx + \int\limits_{\mathbf{R}^N\setminus B_R} |f|^p \rho\, dx \leqslant \rho(R)\mathbf{R}^p \int\limits_{\mathbf{R}^N} |f|^p\, |x|^{-p} dx$$

$$+ \rho^{\frac{p-a}{p^*-a}} \Bigg(\int\limits_{\mathbf{R}^N\setminus B_R} |f|^{p^*} dx \Bigg)^{\frac{p-a}{p^*-a}} \Bigg(\int\limits_{\mathbf{R}^N\setminus B_R} |f|^a \rho\, dx \Bigg)^{\frac{p^*-p}{p^*-a}}$$

$$\leqslant \Big(\frac{p}{N-p}\Big)^p \rho(R)\, R^p D_p(f) + S_1\, \rho^{\frac{p-a}{p^*-a}}\, D_p(f)^{\frac{p^*(p-a)}{p(p^*-a)}}\, E_{a,\rho}(f)^{\frac{p^*-p}{p^*-a}},$$

where $S_1 = S(N,p)^{p^*(p-a)/(p*-a)}$. Let us choose R from the equality

$$R^{\frac{p(p^*-a)}{p^*-p}} \rho(R) = D_p(f)^{-\frac{a}{p}}\, E_{a,\rho}(f),$$

that is

$$\psi(R) := \omega(R) R^{\frac{(N-p)(p-a)}{p}} = D_p(f)^{-\frac{a}{p}} E_{a,\rho}(f).$$

Therefore,

$$E_{p,\rho}(f) \leqslant \gamma\omega\bigg(\psi^{(-1)}\Big(\frac{E_{a,\rho}(f)}{D_p(f)^{\frac{a}{p}}}\Big)\bigg) D_p(f).$$

Theorem 1.1 is proved. $\triangleright$

We need the following Caccioppoli type inequality.

Lemma 2.1. *Let $\theta > 0$, and $\theta > 2-m$ if $m < 1$, be fixed, and define $s = (p+m+\theta-2)/p$. Fix also $a_1 > a_2 > 0$, $\tau_1 > \tau_2 > 0$, $r_2 > r_1$. Then*

$$\sup_{\tau_1<\tau<t} \int\limits_{B_{r_1}} \big(u(\tau) - a_1\big)_+^{\theta+1} \rho(|x|)\, dx + \int\limits_{\tau_1}^{t}\int\limits_{B_{r_1}} \big|\nabla(u-a_1)_+^s\big|^p dx d\tau \leqslant \gamma\Big(\frac{a_1}{a_1-a_2}\Big)^{|m-1|}$$
$$\times\bigg((\tau_1-\tau_2)^{-1}\int\limits_{\tau_1}^{t}\int\limits_{B_{r_2}} (u-a_2)_+^{\theta+1}\rho(|x|)\, dx d\tau + (r_2-r_1)^{-p}\int\limits_{\tau_1}^{t}\int\limits_{B_{r_2}\setminus B_{r_1}} (u-a_2)_+^{sp}\, dx d\tau\bigg). \tag{2.5}$$

For the proof of (2.5) we refer the reader to [16]. Passing $r_2 \to \infty$, $r_1 \to \infty$ in (2.5), we arrive at

$$\sup_{\tau_1<\tau<t} \int\limits_{\mathbf{R}^N} \big(u(\tau) - a_1\big)_+^{\theta+1} \rho(|x|)\, dx + \int\limits_{\tau_1}^{t}\int\limits_{\mathbf{R}^N} \big|\nabla(u-a_1)_+^s\big|^p\, dx d\tau$$
$$\leqslant \gamma\Big(\frac{a_1}{a_1-a_2}\Big)^{|m-1|} (\tau_1-\tau_2)^{-1}\int\limits_{\tau_2}^{t}\int\limits_{\mathbf{R}^N} (u-a_2)_+^{\theta+1}\rho(|x|)\, dx d\tau. \tag{2.6}$$

Proof of Theorem 1.2.

$\triangleleft$ Define for $h_0 > h_\infty > 0$, $\tau_0 > \tau_\infty > 0$, and $i = 0, 1, 2, \dots$,

$$k_i = h_\infty + (h_0 - h_\infty)2^{-i}, \quad t_i = \tau_\infty + (\tau_0 - \tau_\infty)2^{-i}, \quad f_i = (u - k_i)_+^{\frac{p+m+\theta-2}{p}}.$$

Now using the Hölder inequality and the embedding (1.4) with $f = f_i$ and $q = p(1+\theta)/(p+m+\theta-2)$, $a = p/(p+m+\theta-2)$ we obtain that

$$\begin{aligned}\gamma\,\frac{2^i}{(\tau_0-\tau_\infty)}\int\limits_{\mathbf{R}^N} f_i^q\rho(|x|)\,dx &\leqslant \gamma\,\frac{2^i}{(\tau_0-\tau_\infty)}\left(\int\limits_{\mathbf{R}^N} f_i^p\rho(|x|)\,dx\right)^{\frac{q}{p}}\mu_i(\tau)^{1-\frac{q}{p}}\\ &\leqslant C_1\gamma\,\frac{2^i}{(\tau_0-\tau_\infty)}\left(D_p(f_i)\right)^{\frac{q}{p}}\left[\omega\left(\phi_1^{(-1)}\left(c_1\mu_i(\tau)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p}}\mu_i(\tau)^{1-\frac{q}{p}}\\ &\leqslant \sigma^{\frac{p}{q}}D_p(f_i)+\gamma\sigma^{-\frac{p}{p-q}}\left\{C_1\gamma\,\frac{2^i}{(\tau_0-\tau_\infty)}\left[\omega\left(\phi_1^{(-1)}\left(c_1\mu_i(\tau)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p}}\mu_i(\tau)^{1-\frac{q}{p}}\right\}^{\frac{p}{p-q}}.\end{aligned}\tag{2.7}$$

Here it is denoted

$$\mu_i(\tau) := \int\limits_{u(\tau)>k_i}\rho(|x|)\,dx.$$

Integrating in time (2.7) and denoting $M_i(t) = \sup_{0<\tau<t}\mu_i(\tau)$ we get

$$\begin{aligned}\gamma\,\frac{2^i}{(\tau_0-\tau_\infty)}\int\limits_{t_{i+1}}^{t}\int\limits_{\mathbf{R}^N} f_{i+1}^q\rho(|x|)\,dx &\leqslant \varepsilon\int\limits_{t_{i+1}}^{t}\int\limits_{\mathbf{R}^N}|\nabla f_{i+1}|^p\,dxd\tau\\ &+\gamma\,\varepsilon^{-\frac{p}{p-q}}\,t(\tau_0-\tau_\infty)^{-\frac{p}{p-q}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p|m-1|}{p-q}}\\ &\times\left[\omega\left(\phi_1^{(-1)}\left(cM_{i+1}(t)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p-q}}M_{i+1}(t).\end{aligned}\tag{2.8}$$

Combining now (2.8) and (2.6) with $a_1 = k_i$, $a_2 = k_{i+1}$, $\tau_1 = t_i$, $\tau_2 = t_{i+1}$, we get

$$\begin{aligned}J_i := \sup_{t_i<\tau<t}\int\limits_{\mathbf{R}^N} f_i^q\rho(|x|)\,dx + \int\limits_{t_i}^{t}\int\limits_{\mathbf{R}^N}|\nabla f_i|^p\,dxd\tau &\leqslant \varepsilon\int\limits_{t_{i+1}}^{t}\int\limits_{\mathbf{R}^N}|\nabla f_{i+1}|^p\,dxd\tau\\ &+\gamma\,b^i\varepsilon^{-\frac{p}{p-q}}\,t(\tau_0-\tau_\infty)^{-\frac{p}{p-q}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p|m-1|}{p-q}}\\ &\times\left[\omega\left(\phi_1^{(-1)}\left(cM_{i+1}(t)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p-q}}M_{i+1}(t).\end{aligned}$$

Iterating this inequality, we get

$$\begin{aligned}J_0 &\leqslant \varepsilon^i J_i + \gamma\,\varepsilon^{-\frac{p}{p-q}}\,t(\tau_0-\tau_\infty)^{-\frac{p}{p-q}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p|m-1|}{p-q}}\\ &\times\left[\omega\left(\phi_1^{(-1)}\left(cM_\infty(t)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p-q}}M_\infty(t)\sum_{k=0}^{i}(b\varepsilon)^k.\end{aligned}$$

Choosing ε so small that $\varepsilon b < 1$ and letting $i\to\infty$, we have

$$\sup_{\tau_0<\tau<t}\int\limits_{\mathbf{R}^N} f_0^q\rho(|x|)\,dx \leqslant \gamma\,t(\tau_0-\tau_\infty)^{-\frac{p}{p-q}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p|m-1|}{p-q}}$$

$$\times \left[\omega\left(\phi_1^{(-1)}\left(cM_\infty(t)^{\frac{1}{N}}\right)\right)\right]^{\frac{q}{p-q}} M_\infty(t). \tag{2.9}$$

To complete the proof, we need the second iteration. Let $k > 0$, $n = 0, 1, 2, \dots$, and

$$K_n = k\big(1 - 2^{-n-1}\big), \quad \overline{K}_n = \frac{K_n + K_{n+1}}{2}, \quad t_n^{'} = t\big(1 - 2^{-n-1}\big).$$

Applying (2.9) with $\tau_0 = t_{n+1}^{'}$, $\tau_\infty = t_n^{'}$, $h_0 = \overline{K}_n$, $h_\infty = K_n$, we deduce from (2.9) that

$$\begin{aligned} Y_{n+1} := \sup_{t_{n+1}^{'}<\tau<t} \int\limits_{u(\tau)>K_{n+1}} \rho(|x|)\,dx \leqslant \gamma\, b^n k^{-(1+\theta)} \\ \times \sup_{t_{n+1}^{'}<\tau<t} \int\limits_{u(\tau)>K_{n+1}} \big(u - \overline{K}_n\big)_+^{1+\theta} \rho(|x|)\,dx \\ \leqslant \gamma\, b^n k^{-(1+\theta)}\, t^{-\frac{q}{p-q}} \left[\omega\Big(\phi_1^{(-1)}\Big(cY_n^{\frac{1}{N}}\Big)\Big)\right]^{\frac{q}{p-q}} Y_n, \quad b > 1. \end{aligned} \tag{2.10}$$

Taking into account the property H_1 and H_2, which imply that $\phi_1^{(-1)}(s\lambda) \leqslant \lambda^{N/(N-l)} \phi_1^{(-1)}(s)$ for $0 < \lambda \leqslant 1$, $\phi_1^{(-1)}(s\lambda) \leqslant \phi_1^{(-1)}(s)$ for $\lambda > 1$, and $Y_n \leqslant Y_0$, we derive from (2.10) that

$$Y_{n+1} \leqslant \gamma\, b^n k^{-(1+\theta)}\, t^{-\frac{1+\theta}{p+m-3}} \left[\omega\Big(\phi_1^{(-1)}\Big(Y_0^{\frac{1}{N}}\Big)\Big)\right]^{\frac{1+\theta}{p+m-3}} Y_0^{-a}\, Y_n^{1+a}, \quad a = \frac{N(1+\theta)}{(N-l)(p+m-3)}.$$

Thus the last inequality has a form

$$Y_{n+1} \leqslant b^n C Y_n^{1+\varepsilon}, \quad C = \gamma k^{-(1+\theta)}\, t^{-\frac{1+\theta}{p+m-3}} \left[\omega\Big(\phi_1^{(-1)}\Big(Y_0^{\frac{1}{N}}\Big)\Big)\right]^{\frac{1+\theta}{p+m-3}} Y_0^{-a}, \quad \varepsilon = a.$$

Then, using the iterative lemma (see [18, Chap. 2, Lemma 5.6]), we conclude that $Y_n \to 0$ as $n \to \infty$, provided $Y_0 C^{1/\varepsilon} \leqslant b^{-1/\varepsilon^2}$, that is

$$k^{-1}\, t^{-\frac{1}{p+m-3}} \left[\omega\Big(\phi_1^{(-1)}\Big(c\, Y_0^{\frac{1}{N}}\Big)\Big)\right]^{\frac{1}{p+m-3}} \leqslant \delta, \tag{2.11}$$

where δ is a sufficiently small constant depending on the data of the problem. Thus $u \leqslant k$. Next, note that by the Chebyshev inequality we have

$$Y_0 \leqslant \sup_{0<\tau<t} \int\limits_{u(\tau)>k} \rho(|x|)\,dx \leqslant \frac{1}{k} \sup_{0<\tau<t} \int\limits_{u(\tau)>k} u(x,\tau)\,\rho(|x|)\,dx \leqslant \frac{1}{k} \int\limits_{\mathbf{R}^N} u_0\,\rho(|x|)\,dx,$$

where we also have used the estimate $\|u(t)\rho\|_{1,\mathbf{R}^N} \leqslant \|u_0\rho\|_{1,\mathbf{R}^N}$ (see [10]). Thus, it is enough to check that

$$k^{-1}\, t^{-\frac{1}{p+m-3}} \left[\omega\Big(\phi_1^{(-1)}\Big(\big(k^{-1}\|u_0\rho\|_{1,\mathbf{R}^N}\big)^{1/N}\Big)\Big)\right]^{\frac{1}{p+m-3}} \leqslant \delta.$$

To this end, we can choose the free parameter k as follows

$$k = \frac{2}{\delta}\, t^{-\frac{1}{p+m-3}} \left[\omega\Big(\phi_1^{(-1)}\Big(\big(k^{-1}\|u_0\rho\|_{1,\mathbf{R}^N}\big)^{1/N}\Big)\Big)\right]^{\frac{1}{p+m-3}}.$$

Then, after taking into account that $\|u(t)\|_\infty \leqslant k$ and using the monotonicity arguments, we arrive at the desired result. Theorem 1.2 is proved. $\rhd$

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АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЯ ДВАЖДЫ ВЫРОЖДАЮЩИХСЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С НЕОДНОРОДНОЙ ПЛОТНОСТЬЮ

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Аннотация. В данной работе мы изучаем поведение решений задачи Коши для вырожденных параболических уравнений с неоднородной плотностью при неограниченном возрастании времени. При определенных условиях на параметры задачи и поведения плотностной функции на бесконечности устанавливаются новые точные оценки решений при неограниченном возрастании времени. Одним из основных моментом в доказательстве является новая теорема вложения, представляющая независимый интерес. Кроме того, в доказательстве равномерных оценок решения используется модифицированная версия классического метода Де-Джорджи — Ладыженская — Уральцева — Ди Бенедетто. Аналогичные результаты для неоднородной плотности степенного роста были получены одним из авторов в [10]. Подход данной работы может быть использован также при качественном изучении решений задачи Неймана для дважды нелинейного параболического уравнения в областях с некомпактными границами.

Ключевые слова: вырождающееся параболическое уравнение, неоднородная плотность, весовые вложения, поведение при неограниченном возрастании времени.

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