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Hyperbolic Functional-Differential Equations with Unbounded Delay

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Abstract. The phase space for quasilinear equations with unbounded delay is constructed. Carathéodory solutions of initial problems are investigated. A theorem on the existence, uniqueness and continuous dependence upon initial data is given. The method of bicharacteristics and integral inequalities are used.

Keywords: *Unbounded delay, local existence, Carathéodory solutions*

AMS subject classification: 35 L 50, 35 D 05

1. Introduction

For any metric spaces U and V we denote by $C(U,V)$ the class of all continuous functions defined on U and taking values in V. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let

$$E = [-r_0, 0] \times [-r, +r] \subset \mathbb{R}^{1+n}$$

where $r_0 \in \mathbb{R}_+ := [0, +\infty)$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$. Assume that $a > 0$, $(t,x) = (t, x_1, \ldots, x_n) \in [0,a] \times \mathbb{R}^n$ and $z : [-r_0, a] \times \mathbb{R}^n \to \mathbb{R}$. We define a function $z_{(t,x)} : E \to \mathbb{R}^n$ by

$$z_{(t,x)}(\tau, s) = z(t+\tau, x+s) \qquad ((\tau, s) \in E).$$

For each $(t,x) \in [0,a] \times \mathbb{R}^n$ the function $z_{(t,x)}$ is the restriction of z to the set $[t-r_0, t] \times [x-r, x+r]$ and this restriction is shifted to the set E. Suppose that

$$F : [0,a] \times \mathbb{R}^n \times C(E, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$$

is a given function. In this time numerous papers were published concerning various problems for the equation

$$D_t z(t,x) = F\big(t, x, z_{(t,x)}, D_x z(t,x)\big)$$

where $D_x z = (D_{x_1} z, \ldots, D_{x_n} z)$ and for adequate weakly coupled hyperbolic systems. The following questions were considered: functional-differential inequalities, uniqueness

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of initial or initial-boundary value problems, difference-functional inequalities, approximate solutions of initial or initial-boundary value problems, existence of classical or generalized solutions (see [1 - 3, 5 - 8, 10 - 15]). All these problems have the property that the set E is bounded.

In the paper we start the investigation of first order partial functional-differential equations with unbounded delay. We give sufficient conditions for the existence and uniqueness of Carathéodory solutions of initial problems for quasilinear equations with unbounded delay. We consider functional-differential equations in a Banach space. The theory of ordinary functional-differential equations with unbounded delay is given in monographs [4, 9].

We formulate the problem. Let B be a Banach space with norm $\|\cdot\|$ and $D = (-\infty, 0] \times [-r, +r] \subset \mathbb{R}^{1+n}$ $(r \in \mathbb{R}^n_+)$. The norm in $\mathbb{R}^n$ will also be denoted by $\|\cdot\|$. For a function $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(b \geq 0)$ and for a point $(t, x) \in [0, b] \times \mathbb{R}^n$ we define a function $z_{(t,x)} : D \to B$ by

$$z_{(t,x)}(\tau, s) = z(t + \tau, x + s) \qquad ((\tau, s) \in D).$$

The phase space X for equations with unbounded delay is a linear space, with norm $\|\cdot\|_X$ and consisting of functions mapping the set D into B. Let $a > 0$ be fixed and suppose that

$$\begin{aligned} \varrho = (\varrho_1, \ldots, \varrho_n) : [0, a] \times \mathbb{R}^n \times X \to \mathbb{R}^n \\ f : [0, a] \times \mathbb{R}^n \times X \to B \\ \varphi : (-\infty, 0] \times \mathbb{R}^n \to B \end{aligned}$$

are given functions. We consider the quasilinear equation

$$D_t z(t, x) + \sum_{i=1}^{n} \varrho_i(t, x, z_{(t,x)})\, D_{x_i} z(t, x) = f(t, x, z_{(t,x)}) \tag{1}$$

with the intial condition

$$z(t, x) = \varphi(t, x) \qquad \text{on} \;\; (-\infty, 0] \times \mathbb{R}^n. \tag{2}$$

We will deal with Carathéodory solutions of problem (1) - (1.2). A function $\bar{u} : (-\infty, b] \times \mathbb{R}^n \to B$ where $0 < b \leq a$ is a *solution* of the above problem provided:

(i) $\bar{u}$ is continuous on $[0, b] \times \mathbb{R}^n$ and the derivatives $D_t \bar{u}(t, x)$ and $D_x \bar{u}(t, x)$ exist for almost all $(t, x) \in [0, b] \times \mathbb{R}^n$.

(ii) $\bar{u}$ satisfies equation (1.1) almost everywhere on $[0, b] \times \mathbb{R}^n$ and condition (1.2) holds.

We adopt the following notations. If $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(0 < b \leq a)$ is a function such that z is continuous on $[0, b] \times \mathbb{R}^n$, then we put for $(t, x) \in [0, b] \times \mathbb{R}^n$

$$\begin{aligned} \|z\|_{[0,t;x]} &= \max\Big\{ \|z(\tau, s)\| : (\tau, s) \in [0, t] \times [x - r, x + r] \Big\} \\ \|z\|_{[0,t;\mathbb{R}^n]} &= \sup\Big\{ \|z(\tau, s)\| : (\tau, s) \in [0, t] \times \mathbb{R}^n \Big\} \end{aligned}$$

and

$$\operatorname{Lip} z\big|_{[0,t;x]} = \sup\left\{ \frac{\|z(\tau, s) - z(\tau, \bar{s})\|}{\|s - \bar{s}\|} : (\tau, s), (\tau, \bar{s}) \in [0, t] \times [x - r, x + r] \right\}.$$

The fundamental axioms assumed on X are the followings.

Assumption H[X]. Suppose the following:

1) $(X, \|\cdot\|_X)$ is a Banach space.

2) If $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(0 < b \leq a)$ is a function such that $z_{(0,x)} \in X$ for $x \in \mathbb{R}^n$ and z is continuous on $[0,b] \times \mathbb{R}^n$, then $z_{(t,x)} \in X$ for $(t,x) \in (0,b] \times \mathbb{R}^n$ and

(i) for $(t,x) \in [0,b] \times \mathbb{R}^n$ we have $\|z_{(t,x)}\|_X \leq K\|z\|_{[0,t;x]} + L\|z_{(0,x)}\|_X$ where $K, L \in R_+$ are constant independent on z

(ii) the function $(t,x) \to z_{(t,x)}$ is continuous on $[0,b] \times \mathbb{R}^n$.

3) The linear subspace $X_L \subset X$ is such that

(i) X_L endowed with the norm $\|\cdot\|_{X_L}$ is a Banach space

(ii) if $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(0 < b \leq a)$ is a function such that $z_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$, z is continuous on $[0,b] \times \mathbb{R}^n$ and $z(t,\cdot) : \mathbb{R}^n \to B$ satisfies the Lipschitz condition with a constant independent on t $(t \in [0,b])$, then

(α) $z_{(t,x)} \in X_L$ for $(t,x) \in (0,b] \times \mathbb{R}^n$

(β) for $(t,x) \in [0,b] \times \mathbb{R}^n$ we have

$$\|z_{(t,x)}\|_{X_L} \leq K_0\big(\|z\|_{[0,t;x]} + \operatorname{Lip} z|_{[0,t;x]}\big) + L_0\|z_{(0,x)}\|_{X_L}$$

where $K_0, L_0 \in \mathbb{R}_+$ are constants independent on z.

Examples of phase spaces are given in Section 4.

Let us denote by $L([\alpha,\beta],\mathbb{R})$ $([\alpha,\beta] \subset \mathbb{R})$ the class of functions

$$L([\alpha,\beta],\mathbb{R}) = \Big\{\mu : [\alpha,\beta] \to \mathbb{R} : \ \mu \text{ integrable on } [\alpha,\beta]\Big\}.$$

Further, we will use the symbol Θ to denote the set of functions

$$\Theta = \left\{ \gamma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+ \ \middle| \ \begin{array}{l} \gamma(t,\cdot) \text{ is non-decreasing for a.a. } t \in [0,a] \\ \gamma(\cdot,\tau) \in L([0,a],\mathbb{R}_+) \ \text{ for all } \tau \in \mathbb{R}_+ \end{array} \right\}.$$

Further, write

$$X[\kappa] = \big\{w \in X : \ \|w\|_X \leq \kappa\big\}$$
$$X_L[\kappa] = \big\{w \in X_L : \ \|w\|_{X_L} \leq \kappa\big\}$$

where $\kappa \in \mathbb{R}_+$.

2. Bicharacteristics of functional-differential equations

We start with assumptions on the initial function φ.

Assumption H$_0$. Suppose that $\varphi : (-\infty, 0] \times \mathbb{R}^n \to B$ and

(i) $\varphi_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$

(ii) there are $\tilde{L}, L \in \mathbb{R}_+$ such that $\|\varphi(0,x)\| \le \tilde{L}$ for $x \in \mathbb{R}^n$ and $\|\varphi_{(0,x)} - \varphi_{(0,\bar{x})}\|_X \le L\|x - \bar{x}\|$ for $x, \bar{x} \in \mathbb{R}^n$.

Assumption H$[\varrho]$. Suppose the following:

1) The function $\varrho(\cdot, x, w) : [0,a] \to \mathbb{R}^n$ is measurable for $(x,w) \in \mathbb{R}^n \times X$ and $\varrho(t,\cdot) : \mathbb{R}^n \times X \to \mathbb{R}^n$ is continuous for almost all $t \in [0,a]$.

2) There exist $\alpha, \beta \in \Theta$ such that $\|\varrho(t,x,w)\| \le \alpha(t,\kappa)$ for $(x,w) \in \mathbb{R}^n \times X[\kappa]$ almost everywhere on $[0,a]$ and

$$\big\|\varrho(t,x,w) - \varrho(t,\bar{x},\bar{w})\big\| \le \beta(t,\kappa)\big[\|x - \bar{x}\| + \|w - \bar{w}\|_X\big] \tag{3}$$

for $(x,w), (\bar{x},\bar{w}) \in \mathbb{R}^n \times X_L[\kappa]$ almost everywhere on $[0,a]$.

Suppose that $\varphi : (-\infty, 0] \times \mathbb{R}^n \to B$ and $\varphi_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$. Let $c \in [0,a]$, $d = (d_0, d_1) \in \mathbb{R}^2_+$ and $\omega \in L([0,a], \mathbb{R}_+)$. The symbol $Y_{c.\varphi}[\omega, d]$ denotes the function class

$$Y_{c,\varphi}[\omega,d] =$$

$$\left\{ z : (-\infty, c] \times \mathbb{R}^n \to B \;\middle|\; \begin{array}{l} z(t,x) = \varphi(t,x) \;\text{ on }\; (-\infty,0] \times \mathbb{R}^n \\ \|z(t,x)\| \le d_0 \;\text{ on }\; [0,c] \times \mathbb{R}^n \\ \|z(t,x) - z(\bar{t},\bar{x})\| \le \left| \int_t^{\bar{t}} \omega(\tau)\, d\tau \right| + d_1 \|x - \bar{x}\| \\ \text{for } (t,x), (\bar{t},\bar{x}) \in [0,c] \times \mathbb{R}^n. \end{array} \right\}$$

For the above φ and for $z \in Y_{c.\varphi}[\omega, d]$ consider the Cauchy problem

$$\left. \begin{array}{l} \eta'(\tau) = \varrho\big(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}\big) \\ \eta(t) = x \end{array} \right\} \tag{4}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$. We consider Carathéodory solutions of problem (4). Denote by $g[z](\cdot, t, x)$ the solution of the above problem. The function $g[z]$ is the bicharacteristic of equation (1) corresponding to $z \in Y_{c.\varphi}[\omega, d]$.

Let $\Delta_c = [0,c] \times [0,c] \times [0,c]$. For φ satisfying Assumption H$_0$ define

$$\|\varphi\|_{(X,\infty)} = \sup\big\{\|\varphi_{(0,x)}\|_X : x \in \mathbb{R}^n\big\}.$$

Lemma 2.1. *Suppose that Assumptions* H$[X]$ *and* H$[\varrho]$ *are satisfied and*

1) *the functions* $\varphi, \bar{\varphi} : (-\infty, 0] \times \mathbb{R}^n \to B$ *satisfy Assumption* H_0

2) $c \in [0, a]$, $z \in Y_{c.\varphi}[\omega, d]$ *and* $\bar{z} \in Y_{c,\bar{\varphi}}[\omega, d]$.

Then the solutions $g[z](\cdot, t, x)$ *and* $g[\bar{z}](\cdot, t, x)$ *are defined on* $[0, c]$ *and they are unique. Moreover, we have the estimates on* Δ_c

$$\big\| g[z](\tau, t, x) - g[z](\tau, \bar{t}, \bar{x}) \big\| \le \left[\|x - \bar{x}\| + \left| \int_t^{\bar{t}} \alpha(\xi, \bar{\kappa})\, d\xi \right| \right] \exp \left[\bar{d} \left| \int_t^{\tau} \beta(\xi, \kappa_0)\, d\xi \right| \right] \tag{5}$$

and

$$\big\| g[z](\tau, t, x) - g[\bar{z}](\tau, t, x) \big\| \le \left| \int_t^{\tau} \beta(\xi, \kappa_0) \Big[K \|z - \bar{z}\|_{[0,\xi;\mathbb{R}^n]} + M \|\varphi - \bar{\varphi}\|_{(X,\infty)} \Big]\, d\xi \, \exp \left[\bar{d} \int_t^{\tau} \beta(\xi, \kappa_0)\, d\xi \right] \right| \tag{6}$$

where

$$\left.\begin{aligned} \bar{d} &= 1 + K d_0 + M L \\ \bar{\kappa} &= K d_0 + M \, \|\varphi\|_{(X,\infty)} \\ \kappa_0 &= K_0 (d_0 + d_1) + L_0 \sup \big\{ \|\varphi_{(0,x)}\|_{X_L} \,:\, x \in \mathbb{R}^n \big\}. \end{aligned}\right\}$$

Proof. Suppose that $(\xi, \eta), (\xi, \bar{\eta}) \in [0, c] \times \mathbb{R}^n$ and a function $\tilde{z} : (-\infty, c] \times \mathbb{R}^n \to B$ is defined by

$$\tilde{z}(\tau, s) = z(\tau, s + \bar{\eta} - \eta) \qquad \big((\tau, s) \in (-\infty, 0] \times \mathbb{R}^n \big).$$

Then $\tilde{z}_{(\xi,\eta)} = z_{(\xi,\bar{\eta})}$. It follows from Assumptions $\mathrm{H}[X]$ and H_0 that

$$\|z_{(\xi,\eta)} - z_{(\xi,\bar{\eta})}\|_X = \|(z - \tilde{z})_{(\xi,\eta)}\|_X \le (K d_0 + M L) \, \|\eta - \bar{\eta}\|.$$

The existence and uniqueness of the solutions of problem (4) follows from classical theorems. On this purpose, note that the right-hand side of the differential system satisfies the Carathéodory assumptions, and the Lipschitz condition

$$\big\| \varrho(\tau, \eta, z_{(\tau,\eta)}) - \varrho(\tau, \bar{\eta}, z_{(\tau,\bar{\eta})}) \big\| \le \bar{d}\, \beta(\tau, \kappa_0) \, \|\eta - \bar{\eta}\|$$

holds on $[0, c] \times \mathbb{R}^n$. The function $g[z](\cdot, t, x)$ satisfies the integral equation

$$g[z](\tau, t, x) = x + \int_t^{\tau} \varrho\big(\xi, g[z](\xi, t, x), z_{(\xi, g[z](\xi,t,x))}\big)\, d\xi.$$

For $(\tau, t, x), (\tau, \bar{t}, \bar{x}) \in \Delta_c$ we have

$$\left.\begin{aligned} \big\| z_{(\tau, g[z](\tau,t,x))} \big\|_X &\le \bar{\kappa} \\ \big\| z_{(\tau, g[z](\tau,t,x))} \big\|_{X_L} &\le \kappa_0 \end{aligned}\right\} \tag{7}$$

and

$$\left\| z_{(\tau,g[z](\tau,t,x))} - z_{(\tau,g[z](\tau,\bar t,\bar x))} \right\|_X \le (Kd_1 + ML) \left\| g[z](\tau,t,x) - g[z](\tau,\bar t,\bar x) \right\|. \tag{8}$$

It follows from Assumption H$[X]$ and from (7) - (8) that the integral inequality

$$\begin{aligned} &\left\| g[z](\tau,t,x) - g[z](\tau,\bar t,\bar x) \right\| \\ &\quad \le \|x - \bar x\| + \left| \int_t^{\bar t} \alpha(\xi,\bar\kappa)\, d\xi \right| + \bar d \left| \int_t^{\tau} \beta(\xi,\kappa_0) \left\| g[z](\xi,t,x) - g[z](\xi,\bar t,\bar x) \right\| d\xi \right| \end{aligned}$$

is sastisfied. Now we obtain (5) by the Gronwall inequality.

For $z \in Y_{c.\varphi}[\omega,d]$ and $\bar z \in Y_{c.\bar\varphi}[\omega,d]$ we have the estimate

$$\begin{aligned} &\left\| z_{(\xi,g[z](\xi,t,x))} - \bar z_{(\xi,g[\bar z](\xi,t,x))} \right\|_X \\ &\quad \le (Kd_1 + M\tilde L) \left\| g[z](\xi,t,x) - g[\bar z](\xi,t,x) \right\| \\ &\qquad + K\|z - \bar z\|_{[0,\xi;\mathbb{R}^n]} + M \left\| \varphi - \bar\varphi \right\|_{(X,\infty)}. \end{aligned} \tag{9}$$

It follows from Assumption H$[X]$ and from (9) that the integral inequality

$$\begin{aligned} &\left\| g[z](\tau,t,x) - g[\bar z](\tau,t,x) \right\| \\ &\quad \le \left| \int_t^{\tau} \beta(\xi,\kappa_0) \left[\|z - \bar z\|_{[0,\xi;\mathbb{R}^n]} + M\|\varphi - \bar\varphi\|_{(X,\infty)} \right] d\xi \right| \\ &\qquad + \bar d \left| \int_t^{\tau} \beta(\xi,\kappa_0) \left\| g[z](\xi,t,x) - g[\bar z](\xi,t,x) \right\| d\xi \right| \end{aligned}$$

is satisfied. Now we obtain (6) by the Gronwall inequality. This completes the proof of the lemma ∎

3. Existence and uniqueness of solutions

Now we construct an integral operator corresponding to problem (1) - (2). Suppose that the function φ satisfies Assumption H_0, $c \in (0,a]$, $z \in Y_{c.\varphi}[\omega,d]$ and $g[z](\cdot,t,x)$ is the bicharacteristic corresponding to z. Let us define the operator U_φ for all $z \in Y_{c.\varphi}[\omega,d]$ be the formulas

$$U_\varphi z(t,x) = \varphi\big(0, g[z](0,t,x)\big) + \int_0^t f\big(\tau, g[z](\tau,t,x), z_{(\tau,g[z](\tau,t,x))}\big)\, d\tau \tag{10}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$ and

$$U_\varphi z(t,x) = \varphi(t,x) \qquad \text{on } \ (-\infty,0] \times \mathbb{R}^n. \tag{11}$$

Remark 3.1. The operator U_φ is obtained by integration of equation (1) along bicharacteristics.

Now we give sufficient conditions for the solvability of the equation $z = U_\varphi z$ on $Y_{c.\varphi}[\omega,d]$.

Assumption H[f]. Suppose the following:

1) The function $f(\cdot, x, w) : [0, a] \to B$ is measurable for $(x, w) \in \mathbb{R}^n \times X$ and $f(t, \cdot) : \mathbb{R}^n \times X \to B$ is continuous for almost all $t \in [0, a]$.

2) For $(x, w) \in \mathbb{R}^n \times X[\kappa]$ and for almost all $t \in [0, a]$ we have

$$\|f(t, x, w)\| \leq \alpha(t, \kappa). \tag{12}$$

3) For $(x, w), (\bar{x}, \bar{w}) \in \mathbb{R}^n \times X_L[\kappa]$ and for almost all $t \in [0, a]$ we have

$$\big\|f(t, x, w) - f(t, \bar{x}, \bar{w})\big\| \leq \beta(t, \kappa) \big[\|x - \bar{x}\| + \|w - \bar{w}\|_X\big].$$

Remark 3.2. We prove a theorem on the existence and uniqueness of solutions of problem (1) - (2). For simplicity of notations, we have assumed the same estimation for ϱ and for f. We have assumed also the Lipschitz condition for these functions with the same coefficient.

Lemma 3.3. *Suppose that Assumptions* H[X], H$_0$, H[ϱ] *and* H[f] *are satisfied. Then there are* $(d_0, d_1) = d \in \mathbb{R}^2_+$, $c \in (0, a]$ *and* $\omega \in L([0, c], \mathbb{R}_+)$ *such that* $U_\varphi : Y_{c.\varphi}[\omega, d] \to Y_{c.\varphi}[\omega, d]$.

Proof. Suppose that the constants $(d_0, d_1) = d$ and $c \in (0, a]$ and the function $\omega \in L([0, c], \mathbb{R}_+)$ satisfy the conditions

$$\left.\begin{aligned} d_0 &\geq \tilde{L} + \int_0^c \alpha(\tau, \bar{\kappa})\, d\tau \\ d_1 &\geq \Gamma_c \\ \omega(t) &\geq (1 + \Gamma_c)\, \alpha(t, \bar{\kappa}) \end{aligned}\right\}$$

where

$$\Gamma_c = \left[L + \bar{d} \int_0^c \beta(\tau, \kappa_0)\, d\tau\right] \exp\left[\bar{d} \int_0^c \beta(\tau, \kappa_0)\, d\tau\right]. \tag{13}$$

Suppose that $z \in Y_{c.\varphi}[\omega, d]$. Then we have

$$\|U_\varphi z(t, x)\| \leq \tilde{L} + \int_0^c \alpha(\tau, \bar{\kappa})\, d\tau \leq d_0 \qquad \text{on } \ [0, c] \times \mathbb{R}^n. \tag{14}$$

If $(t, x), (\bar{t}, \bar{x}) \in [0, c] \times \mathbb{R}^n$, then using Lemma 2.1 and (10) we obtain

$$\begin{aligned} &\big\|U_\varphi z(t, x) - U_\varphi z(\bar{t}, \bar{x})\big\| \\ &\quad \leq \big\|\varphi\big(0, g[z](0, t, x)\big) - \varphi\big(0, g[z](0, \bar{t}, \bar{x})\big)\big\| + \left|\int_t^{\bar{t}} \alpha(\tau, \bar{\kappa})\, d\tau\right| \end{aligned}$$

$$+\int_0^t \beta(\tau,\kappa_0)\Big[\big\|g[z](\tau,t,x)-g[z](\tau,\bar t,\bar x)\big\| + \big\|z_{(\tau,g[z](\tau,t,x))}-z_{(\tau,g[z](\tau,\bar t,\bar x))}\big\|_X\Big]d\tau$$

$$\le \Gamma_c\left[\|x-\bar x\| + \left|\int_t^{\bar t}\alpha(\tau,\bar\kappa)\,d\tau\right|\right] + \left|\int_t^{\bar t}\alpha(\tau,\bar\kappa)\,d\tau\right|.$$

Thus we see that

$$\big\|U_\varphi z(t,x)-U_\varphi z(\bar t,\bar x)\big\| \le d_1\|x-\bar x\| + \left|\int_t^{\bar t}\omega(\tau)\,d\tau\right|. \tag{15}$$

It follows from (14) and (15) that $U_\varphi z \in Y_{c.\varphi}[\omega,d]$ which completes the proof of the lemma ∎

Next we will show that there exists exactly one solution of problem (1) - (2). The solution is local with respect to t.

Theorem 3.4. *Suppose that Assumptions* H$[X]$, H$_0$, H$[\varrho]$ *and* H$[f]$ *are satisfied. Then there are* $(d_0,d_1)=d\in\mathbb{R}^2_+$, $c\in(0,a]$ *and* $\omega\in L([0,c],\mathbb{R}_+)$ *such that problem* $(1)-(2)$ *has exactly one solution* $u\in Y_{c.\varphi}[\omega,d]$.

If $\bar\varphi:(-\infty,0]\times\mathbb{R}^n\to B$ *satisfies Assumption* H$_0$ *and* $\bar u\in Y_{c.\varphi}[\omega,d]$ *is a solution of equation* (1) *with the initial condition* $z=\bar\varphi$ *on* $(-\infty,0]\times\mathbb{R}^n$, *then there is* $\Lambda_c\in\mathbb{R}_+$ *such that*

$$\|u-\bar u\|_{[0,t;\mathbb{R}^n]} \le \Lambda_c\Big[\|\varphi-\bar\varphi\|_{(X,\infty)} + \sup_{y\in\mathbb{R}^n}\|\varphi(0,y)-\bar\varphi(0,y)\|\Big] \tag{16}$$

where $t\in[0,c]$.

Proof. Lemma 3.3 shows that there are $(d_0,d_1)=d$, $c\in(0,a]$ and $\omega\in L([0,c],\mathbb{R}_+)$ such that $U_\varphi : Y_{c.\varphi}[\omega,d]\to Y_{c.\varphi}[\omega,d]$. Write

$$\lambda_c = K(1+\Gamma_c)\int_0^c \beta(\tau,\kappa_0)\,d\tau$$

where Γ_c is given by (13). Let $c\in(0,a]$ be such a constant that $\lambda_c<1$. Now we prove that U_φ is a contraction on $Y_{c.\varphi}[\omega,d]$. If $z,\tilde z\in Y_{c.\varphi}[\omega,d]$, then

$$\begin{aligned}\big\|U_\varphi z(t,x)-U_\varphi\tilde z(t,x)\big\| &\le L\,\big\|g[z](0,t,x)-g[\tilde z](0,t,x)\big\|\\ &\quad+\int_0^t \beta(\tau,\kappa_0)\Big[\big\|g[z](\tau,t,x)-g[\tilde z](\tau,t,x)\big\|\\ &\quad+\big\|z_{(\tau,g[z](\tau,t,x))}-\tilde z_{(\tau,g[z](\tau,t,x))}\big\|_X\Big]d\tau.\end{aligned}$$

The estimate

$$\begin{aligned}&\big\|z_{(\tau,g[z](\tau,t,x))}-\tilde z_{(\tau,g[z](\tau,t,x))}\big\|_X\\ &\quad\le (Kd_1+M\tilde L)\,\big\|g[z](\tau,t,x)-g[\tilde z](\tau,t,x)\big\| + K\|z-\tilde z\|_{[0,\tau;\mathbb{R}^n]}\end{aligned}$$

and Lemma 2.1 imply

$$\big\|U_\varphi z(t,x) - U_\varphi \tilde{z}(t,x)\big\| \le K(1+\Gamma_c)\int_0^t \beta(\tau,\kappa_0)\|z-\tilde{z}\|_{[0,\tau;\mathbb{R}^n]}d\tau$$

for all $(t,x) \in [0,c] \times \mathbb{R}^n$, and consequently

$$\|U_\varphi z - U_\varphi \tilde{z}\|_{[0,c;\mathbb{R}^n]} \le \lambda_c \|z-\tilde{z}\|_{[0,c;\mathbb{R}^n]}.$$

By the Banach fixed point theorem there exists a unique solution $u \in Y_{c.\varphi}[\omega,d]$ of the equation $z = U_\varphi z$.

Now we prove that u is a solution of (1). We have proved that

$$u(t,x) = \varphi\big(0, g[u](0,t,x)\big) + \int_0^t f\big(\tau, g[u](\tau,t,x), u_{(\tau, g[u](\tau,t,x))}\big)d\tau \tag{17}$$

on $[0,c] \times \mathbb{R}^n$. For given $x \in \mathbb{R}^n$ let us put $\eta = g[u](0,t,x)$. It follows that $g[u](\tau,t,x) = g[u](\tau,0,\eta)$ for $\tau \in [0,c]$ and that $x = g[u](t,0,\eta)$. The relations $\eta = g[u](0,t,x)$ and $x = g[u](t,0,\eta)$ are equivalent for $x,\eta \in \mathbb{R}^n$. It follows from (17) that

$$u\big(t, g[u](t,0,\eta)\big) = \varphi(0,\eta) + \int_0^t f\big(\tau, g[u](\tau,0,\eta), u_{(\tau, g[u](\tau,0,\eta))}\big)d\tau \tag{18}$$

where $(t,\eta) \in [0,c] \times \mathbb{R}^n$. By differentiating (18) with respect to t and by using the transformation $\eta = g[u](0,t,x)$ which preserves sets of measure zero, we obtain that u satisfies equation (1) for almost all $(t,x) \in [0,c] \times \mathbb{R}^n$. It follows from (11) that u satisfies also condition (2).

Now we prove relation (16). If $u = U_\varphi u$ and $\bar{u} = U_{\bar{\varphi}}\bar{u}$, then

$$\begin{aligned}
&\big\|u(t,x) - \bar{u}(t,x)\big\| \\
&\quad \le \sup_{y\in\mathbb{R}^n} \big\|\varphi(0,y) - \bar{\varphi}(0,y)\big\| + L\,\big\|g[u](0,t,x) - g[\bar{u}](0,t,x)\big\| \\
&\qquad + \int_0^t \beta(\tau,\kappa_0)\Big[\bar{d}\,\big\|g[u](\tau,t,x) - g[\bar{u}](\tau,t,x)\big\| \\
&\qquad + K\,\|u-\bar{u}\|_{[0,\tau;\mathbb{R}^n]} + M\|\varphi - \bar{\varphi}\|_{(X,\infty)}\Big]d\tau
\end{aligned}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$. Put

$$A_c = (1+\Gamma_c)M\int_0^t \beta(\tau,\kappa_0)\,d\tau \qquad \textit{and} \qquad \gamma(t) = K(1+\Gamma_c)\,\beta(t,\kappa_0).$$

Then we get the integral inequality

$$\begin{aligned}
&\|u-\bar{u}\|_{[0,t;\mathbb{R}^n]} \\
&\quad \le \sup_{y\in\mathbb{R}^n} \big\|\varphi(0,y) - \bar{\varphi}(0,y)\big\| + A_c\|\varphi - \bar{\varphi}\|_{(X,\infty)} + \int_0^t \gamma(\tau)\|u-\bar{u}\|_{[0,\tau;\mathbb{R}^n]}d\tau
\end{aligned}$$

for all $t \in [0,c]$. It follows from the Gronwall inequality that we have estimate (16) for $\Lambda_c = \exp\big[\int_0^c \gamma(\tau)\,d\tau\big]$. This completes the proof of the theorem ∎

4. Phase spaces

We give examples of spaces X satisfying Assumption H$[X]$.

Example 4.1. Let X be the class of all function $w : (-\infty, 0] \times [-r, +r] \to B$ which are uniformly continuous and bounded on $(-\infty, 0] \times [-r, +r]$. For $w \in X$ we write

$$\|w\|_X = \sup\Big\{\|w(\tau, s)\| : (\tau, s) \in (-\infty, 0] \times [-r, +r]\Big\}.$$

Let $X_L \subset X$ denote the set of all $w \in X$ such that

$$|w|_L = \sup\left\{\frac{\|w(\tau, s) - w(\tau, \bar{s})\|}{\|s - \bar{s}\|} : (\tau, s), (\tau, \bar{s}) \in (-\infty, 0] \times [-r, +r]\right\} < +\infty. \tag{19}$$

Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H$[X]$ is satisfied.

Example 4.2. Let X be the class of all functions $w : (-\infty, 0] \times [-r, +r] \to B$ such that

(i) w is continuous and bounded on $(-\infty, 0] \times [-r, +r]$

(ii) the limit $\lim_{t\to-\infty} w(t, x)$ exists uniformly with respect to $x \in [-r, +r]$.

Let

$$\|w\|_X = \sup\Big\{\|w(\tau, s)\| : (\tau, s) \in (-\infty, 0] \times [-r, +r]\Big\}.$$

Let $X_L \subset X$ denote the class of all $w \in X$ such that the Lipschitz condition (19) is satisfied. Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H$[X]$ is satisfied.

Example 4.3. Let $\gamma : (-\infty, 0] \to (0, +\infty)$ be a continuous function. Assume also that γ is non-increasing on $(-\infty, 0]$. Let X be the space of continuous functions $w : (-\infty, 0] \times [-r, +r] \to B$ for which

$$\lim_{\tau\to\infty} \frac{\|w(\tau, x)\|}{\gamma(\tau)} = 0 \qquad (x \in [-r, +r]).$$

Put

$$\|w\|_X = \sup\left\{\frac{\|w(\tau, s)\|}{\gamma(\tau)} : (\tau, s) \in (-\infty, 0] \times [-r, +r]\right\}.$$

Denote by $X_L \subset X$ the set of all $w \in X$ such that

$$|w|_{\gamma.L} = \sup\left\{\frac{\|w(\tau, s) - w(\tau, \bar{s})\|}{\gamma(\tau)\|s - \bar{s}\|} : (\tau, s), (\tau, \bar{s}) \in (-\infty, 0] \times [-r, +r]\right\} < +\infty.$$

For $w \in X_L$ put $\|w\|_{X_L} = \|w\|_X + |w|_{\gamma.L}$. Then Assumption H$[X]$ is satisfied.

Example 4.4. Let $\delta \in \mathbb{R}_+$ and $p \geq 1$ be fixed. Denote by X the class of all functions $w : (-\infty, 0] \times [-r, +r] \to B$ such that

(i) w is continuous on $[-\delta, 0] \times [-r, +r]$

(ii) for $x \in [-r, +r]$ we have $\int_{-\infty}^{-\delta} \|w(\tau, x\|^p \, d\tau < +\infty$

(iii) $w(t,\cdot) : [-r,+r] \to B$ is continuous for $t \in (-\infty,-\delta]$.

Write

$$\|w\|_X = \sup\Big\{\|z(\tau,s)\| : (\tau,s) \in [-\delta,0]\times[-r,+r]\Big\}$$
$$+\sup\Big\{\Big(\int_{-\infty}^{-\delta}\|w(\tau,x\|^p d\tau\Big)^{1/p} : x\in[-r,+r]\Big\}.$$

Let $X_L \subset X$ be the set of functions $w \in X$ such that the Lipschitz condition (19) is satisfied. Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H[X] is satisfied.

Remark 4.5. Differential equations with a deviated argument and differential-integral equations can be obtained from equation (1) by specializing operators ϱ and f.

Remark 4.6. It is important in our considerations that we have assumed the Lipschitz condition for given functions on some special function spaces. More precisely, we have assumed that the functions $\varrho(t,\cdot)$ and $f(t,\cdot)$ satisfy the Lipschitz condition on the space $\mathbb{R}^n \times X_L$ for almost all $t \in [0,a]$, and the condition is local with respect to the functional variable.

Let us consider simplest assumption on ϱ and f. Suppose that there is $P \in \mathbb{R}_+$ such that for almost all $t \in [0,a]$ we have

$$\big\|\varrho(t,x,w)-\varrho(t,\bar{x},\bar{w})\big\| \le P\big[\|x-\bar{x}\|+\|w-\bar{w}\|_X\big] \tag{20}$$
$$\big\|f(t,x,w)-f(t,\bar{x},\bar{w})\big\| \le P\big[\|x-\bar{x}\|+\|w-\bar{w}\|_X\big] \tag{21}$$

where $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X$. Of course, our results are true if we assume (20), (21) instead of (3), (12).

Now we show that formulations (3), (12) are important. We show that there is a class of quasilinear equations satisfying (3), (12) but not satisfying (20), (21). Let X and X_L be the spaces given in Example 4.1. Consider the equation with a deviated argument

$$D_t z(t,x) + \sum_{i=1}^{n} \tilde{\varrho}_i\big(t,x,z(\psi_0(t),\psi(t,x))\big)D_{x_i}z(t,x) = \tilde{f}\big(t,x,z(\psi_0(t),\psi(t,x))\big) \tag{22}$$

where

$$\left.\begin{aligned}\tilde{\varrho}=(\tilde{\varrho}_1,\ldots,\tilde{\varrho}_n) &: [0,a]\times\mathbb{R}^n\times B\to\mathbb{R}^n\\ f &: [0,a]\times\mathbb{R}^n\times B \to B\\ \psi_0 &: [0,a]\to(-\infty,a]\\ \psi &: [0,a]\times\mathbb{R}^n\to\mathbb{R}^n.\end{aligned}\right\}$$

We assume that $\psi(t) \le t$ and $-r \le \psi(t,x)-x \le +r$ for $(t,x)\in[0,a]\times\mathbb{R}^n$. We get (22) by putting in (1)

$$\varrho(t,x,w) = \tilde{\varrho}\big(t,x,w\big(\psi_0(t)-t,\psi(t,x)-x\big)\big)$$
$$f(t,x,w) = \tilde{f}\big(t,x,w\big(\psi_0(t)-t,\psi(t,x)-x\big)\big).$$

From now we consider the function ϱ only. Suppose that there are $\bar{C}, \tilde{C} \in \mathbb{R}_+$ such that

$$\begin{aligned} \big\|\tilde{\varrho}(t,x,\zeta) - \tilde{\varrho}(t,\bar{x},\bar{\zeta})\big\| &\le \bar{C}\,\big[\|x-\bar{x}\| + \|\zeta - \bar{\zeta}\|\big] \\ \big\|\psi(t,x) - \psi(t,\bar{x})\big\| &\le \tilde{C}\|x-\bar{x}\|. \end{aligned}$$

It is evident that for $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X_L[\kappa]$ and for almost all $t \in [0,a]$ we have

$$\big\|\varrho(t,x,w) - \varrho(t,\bar{x},\bar{w})\big\| \le \bar{C}\big[1+\kappa(1+\tilde{C})\big]\,\|x-\bar{x}\| + \bar{C}\,\|w-\bar{w}\|_X.$$

Then condition (3) is satisfied.

We see at once the the function $\varrho(t,\cdot)$ does not satisfy the global Lipschitz condition (20) for $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X$. Similar consideration apply to f.

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