Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

DISTRIBUTIONS VIA SEMINORMS AND FRÖLICHER SPACES

PAUL CHERENACK

Abstract. The topology underlying a vector space C on which distributions are defined where semi-norms are used (in the usual way) to define the topology is compared with the initial topology on C defined by the structure functions for C viewed as a Frölicher space. In the Frölicher space sense, such vector spaces C are embedded in a smooth way into the corresponding space of distributions. Finally, appropriate definitions of partial derivative and product for distributions in the case of Frölicher spaces are introduced and are seen to enlarge on the corresponding usual notions arising from the use of semi-norms.

Introduction

The concept Frölicher space is defined in [10]. It was studied under the name "smooth space" at the begining of [8] which constitutes our main reference. We motivate for the use of Frölicher spaces in studying distributions. For notational reasons and easy reference, we define Frölicher spaces. Let $\mathcal{C} = \mathbf{C}^\infty(\mathbb{R}, \mathbb{R})$ be the set of smooth maps from $\mathbb{R}$ to $\mathbb{R}$.

Definition 0.1. *Let $(X, \mathrm{C}_X, \mathrm{F}_X)$ be a triple with X the underlying set of a structure defined by a set*

1. C_X *of curves $c : \mathbb{R} \to X$ called structure curves.*
2. F_X *of functions $f : X \to \mathbb{R}$ called structure functions.*

Let, for a set

1. *F of maps $f : X \to \mathbb{R}$, $\Gamma(F) = \{c : \mathbb{R} \to X | f \circ c \in \mathcal{C}\}$.*
2. *C of curves $c : \mathbb{R} \to X$, $\Phi(C) = \{f : X \to \mathbb{R} | f \circ c \in \mathcal{C}\}$.*

Then, the triple $(X, \mathrm{C}_X, \mathrm{F}_X)$ or just X with the structure supplied by C_X and F_X is called a Frölicher space or just F-space if $\Gamma(F_X) = C_X$ and $\Phi(C_X) = F_X$. A map $f : X \to Y$ of F-spaces is a map such that

$$f \circ C_X = \{f \circ c | c \in \mathrm{C}_X\} \subset C_Y$$

or equivalently, see [8]*, such that*

$$F_Y \circ f = \{g \circ f | g \in \mathrm{F}_Y\} \subset F_X.$$

1991 *Mathematics Subject Classification.* 58C35, 58A40, 57R40.

Financial assistance from the National Research Foundation of South Africa and the University of Cape Town to the Topology and Category Theory Research Group (South Africa.) is acknowledged.

A map of F-spaces will be referred to as F-smooth or just smooth according to need.

Frölicher and Kriegl [8] show that the category FRL of Frölicher spaces has initial and final structures and is thus complete and cocomplete. Let X_1 and X_2 be F-spaces with the same underlying set X. Let $1_X : X_1 \to X_2$ denote the identity map. If 1_X is smooth, then X_1 has a finer structure than X_2 and X_2 has a courser structure than X_1. Then, as with topological spaces, for any collection of F-space structures on a set X, there is a coursest structure on X finer than any of these structures and there is a finest structure on X courser than any of these structures. This enables one to define, as with topological spaces, initial and final structures. However, as opposed to topological spaces, for F-spaces X and Y, there is a natural ([8])structure on

$$\mathrm{HOM}_F(X,Y) = \{f : X \to Y | f \text{ is smooth }\}$$

obtained by setting

$$\mathrm{C}_{\mathrm{HOM}_F(X,Y)} = \{c : \mathbb{R} \to \mathrm{HOM}_F(X,Y) | \hat{c} : \mathbb{R} \times X \to Y \text{ is smooth}\},$$

where $\hat{c}(t,x) = c(t)(x)$ and one has $F_{\mathrm{HOM}_F(X,Y)} = \Phi(\mathrm{C}_{\mathrm{HOM}_F(X,Y)})$. We will write Y^X to denote $\mathrm{HOM}_F(X,Y)$ with this structure. We prove in [3] the following version of the Uniform Boundedness Principle (see [10]):

Theorem 0.1. *Let X and Y be Frölicher spaces. Suppose that F_Y, the set of structure functions on Y, separates points. The structure on Y^X is initial structure induced by the evaluation maps $ev_x : \mathrm{HOM}_F(X,Y) \to Y (x \in X)$ where $ev_x(f) = f(x)$.*

We note that a smooth ($\mathbf{C}^\infty$) manifold, with the usual notion of smooth curve into it and smooth real valued function on it, is a F-space. This assertion uses Boman's Theorem [1] or [2]. We use the reference [6] or [12] for our reference to ordinary distribution theory.

Let K be a compact subset of $\mathbb{R}^n$. Let $\mathbf{C}(\mathbb{R}^n) = \mathbb{R}^{\mathbb{R}^n}$,
$\mathbf{C}_c(\mathbb{R}^n)$ denote the set of all $f \in \mathbf{C}(\mathbb{R}^n)$ such that f has compact support and $\mathbf{C}_{c,K}(\mathbb{R}^n)$ denote the set of all $f \in \mathbf{C}(\mathbb{R}^n)$ such that f has support in K. Let Y be a F-space and $i_X : X \to Y$ an inclusion set map. Then, X becomes a F-subspace of Y on setting $C_X = \{c : \mathbb{R} \to X | i_X \circ c \in \mathrm{C}_Y\}$. Thus, $\mathbf{C}_c(\mathbb{R}^n)$ and $\mathbf{C}_{c,K}(\mathbb{R}^n)$ can be viewed as F-subspaces of $\mathbf{C}(\mathbb{R}^n)$. We show:

Lemma 0.1. *Alternatively, letting $\hat{c}(t,x) = c(t)(x)$ and $M \subset \mathbf{C}(\mathbb{R}^n)$ have the F-subspace structure, $\mathrm{C}_M = \{c : \mathbb{R} \to M | \hat{c} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is smooth $\}$. Here, M could for instance, be $\mathbf{C}_c(\mathbb{R}^n)$ or $\mathbf{C}_{c,K}(\mathbb{R}^n)$.*

Proof. Let $d : \mathbb{R} \to M$ and $\hat{d}$ be snooth. Then, $\widehat{i \circ d}$, for the inclusion $i : M \to \mathbf{C}(\mathbb{R}^n)$, is smooth as $\widehat{i \circ d}(t,x) = i \circ d(t)(x) = \hat{d}(t,x)$. Thus, $i \circ d$ is smooth and hence $d \in \mathrm{C}_M$. The sequence of implications used is readily reversed. □

The paper will be presented in the following manner:

Section 1: F-distributions: Here we introduce the notion of F-distribution using a series of results to place this theory in perspective.

Section 2: F and ordinary distributions: Here we show that, for K a closed interval, the norm $|\cdot|_K$ on a F-subspace $\mathbf{MMC}_{c,K}(\mathbb{R})$ of $\mathbf{C}_{c,K}(\mathbb{R}^n)$ consisting of Morse functions with a unique maximum and minimum in the interior of K, is smooth. Thus, as far as $|\cdot|_K$ is concerned, the topology on the F-space $\mathbf{MMC}_{c,K}(\mathbb{R})$ is finer than the topology which it posseses in ordinary distribution theory.

Section 3: Embeddings: We show that there is a F-space embedding of the collection $\mathbf{C}_c(\mathbb{R})$ of smooth functions on $\mathbb{R}$ with compact support into the space of F-distributions.

Section 4: Operations on F-distributions: We calculate the derivative of F-distributions and partial derivatives of F-distributions. Convolution for F-distributions is seen to be a smooth operation.

1. Section: F-distributions

To distinguish ordinary distribution theory from that occuring with respect to F-spaces we refer, for instance, to F-distribution theory. We endow a F-space X with the initial topology given by the functions $f : X \to \mathbb{R}, f \in \mathrm{F}_X$.This topology has as a basis open sets of the form $f^{-1}(0,1), f \in \mathrm{F}_X$. For smooth manifolds one obtains the usual topology.

Let $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ be a F-space with the same underlying set as $\mathbf{C}_c(\mathbb{R}^n)$ but with a set of structure curves $\underline{\mathbf{C}} = \Gamma \circ \Phi(\underline{\mathbf{C}}_0)$ where $d \in \underline{\mathbf{C}}_0$ if and only if d is a smooth curve $d : \mathbb{R} \to \mathbf{C}_c(\mathbb{R}^n)$ and, for each $a \in \mathbb{R}$, there is an $\epsilon > 0$ such that $d(t)$ has support in some compact set K^* for each $t \in (a-\epsilon, a+\epsilon)$. We say that $\underline{\mathbf{C}}_0$ generates $\underline{\mathbf{C}}$ (see [8]). The set of structure functions on $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ is given by $\underline{\mathbf{F}} = \Phi(\underline{\mathbf{C}}_0)$. The smooth map $\hat{d}$, given by $\hat{d}(t) = e^{-\frac{1}{t^2(1-(t^2x^2))}}$ for $tx < 1, t > 0$ and 0 otherwise, defines a smooth curve into $\mathbf{C}_c(\mathbb{R}^n)$ which is not in $\underline{\mathbf{C}}_0$. As $\underline{\mathbf{C}}_0 \subset \mathbf{C}_c(\mathbb{R}^n)$, the structure on $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ is finer than the structure of $\mathbf{C}_c(\mathbb{R}^n)$. We expect that the structure of $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ will not be the same as the structure of $\mathbf{C}_c(\mathbb{R}^n)$. In ordinary distribution theory, one has, by definition, the following result.

Theorem 1.1. *Let $\underline{\mathcal{K}}$ denote the collection of compact subsets of a F-space X ordered by inclusion. Taking an inductive limit over $\underline{\mathcal{K}}$, one obtains in the category of F-spaces*

$$\underline{\mathbf{C}}_c(\mathbb{R}^n) = \lim_{K \in \underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n).$$

Proof. One need only show that the final structure on the underlying set

$$\mathbf{C}_c(\mathbb{R}^n) = \bigcup_{K \in \underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n) = \lim_{K \in \underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n),$$

for the inclusions $i_K : \mathbf{C}_{c,K}(\mathbb{R}^n) \to \mathbf{C}_c(\mathbb{R}^n)$, agrees with the given structure on $\underline{\mathbf{C}}_c(\mathbb{R}^n)$. Since the inclusions i_K are smooth when viewed as maps $i_K : \mathbf{C}_{c,K}(\mathbb{R}^n) \to \underline{\mathbf{C}}_c(\mathbb{R}^n)$, the definition of direct limits implies that the identity map

$\lim_{K\in\underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n) \to \underline{\mathbf{C}}_c(\mathbb{R}^n)$ is smooth. Now, for $d \in \underline{\mathbf{C}}_0$, let $d(b) = g$. Suppose that g has support K. There is a compact set K^* containing K such that, for some $\epsilon > 0$, $d(t)$ has compact support in K^* for $t \in (b-\epsilon, b+\epsilon)$. This implies that d is a smooth curve into $\mathbf{C}_{c,K^*}(\mathbb{R}^n)$ and hence $\lim_{K\in\underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n)$ for $t \in (b-\epsilon, b+\epsilon)$. Since b is arbitrary, $d(t)$ is smooth. Since $\underline{\mathbf{C}}_0$ generates $\underline{\mathbf{C}}$, we are done (see [8]). $\square$

Thus, in some ways, $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ is a better candidate than $\mathbf{C}_c(\mathbb{R}^n)$ for the F-space on which F-distributions will be defined.

One also can show:

Lemma 1.1. *For F-space topologies, $\mathbf{C}_{c,K}(\mathbb{R}^n)$ is a closed subset of $\mathbf{C}_c(\mathbb{R}^n)$ and $\underline{\mathbf{C}}_c(\mathbb{R}^n)$.*

Proof. Let $g \in \mathbf{C}_c(\mathbb{R}^n) - \mathbf{C}_{c,K}(\mathbb{R}^n) = D$. There is a $x \notin K$ such that $ev_x(g) = g(x) \neq 0$. Thus, $g \in ev_x^{-1}(\mathbb{R} - \{0\}) \subset D$. Since $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ has more structure functions than $\mathbf{C}_c(\mathbb{R}^n)$, it has a finer topology than $\mathbf{C}_c(\mathbb{R}^n)$. $\square$

We would like to prove or find a counter example to the following:

Conjecture 1. *Suppose that $\mathbf{C}_{c,K}(\mathbb{R}^n)$ has its F-space topology. If $\lim_{K\in\underline{\mathcal{K}}} \mathbf{C}_{c,K}(\mathbb{R}^n)$ is taken in topological spaces, then the resulting topology is the same as the F-space topology of $\underline{\mathbf{C}}_c(\mathbb{R}^n)$.*

Of course one would like to show that $\underline{\mathbf{C}}_c(\mathbb{R}^n)$ with its F-space topology, as for ordinary distributions, is sequentially compact. One can show that every Cauchy sequence in $\underline{\mathbf{C}}_c(\mathbb{R})$ and the corresponding sequences of derivatives converge pointwise. But this is only a begining.

It is easy to define F-distributions:

Definition 1.1. *A F-distribution D on $\mathbb{R}^n$ is a smooth map $D : \mathbf{C}_c(\mathbb{R}^n) \to \mathbb{R}$, i.e., a map such that $D \circ d$ is smooth for each smooth curve $d : \mathbb{R} \to \mathbf{C}_c(\mathbb{R}^n)$ and d is smooth if the corresponding map $\hat{d} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by $\hat{d}(s, x) = d(s)(x)$ is smooth in the usual sense. A linear F-distribution (linearity is normal for ordinary distributions) is a F-distribution that is a linear map. Let $\mathcal{D}_n = \mathbb{R}^{\mathbf{C}_c(\mathbb{R}^n)}$ denote the F-space of F-distributions on $\mathbb{R}^n$.*

Clearly, this definition can be extended to distributions on arbitrary F-spaces.

Examples:

1. The evaluation maps ev_x, which are also the Dirac delta functions, defining the F-space structure on $\mathbf{C}_c(\mathbb{R}^n)$.
2. The maps $D : \mathbf{C}_c(\mathbb{R}^n) \to \mathbb{R}$ sending g to a partial derivative $\frac{\partial g}{\partial x^j}(\vec{x}_0), j = 1, \cdots, n$ and $\vec{x}_0$ a fixed point.
3. Let $I(h)(g) = \int_{\mathbb{R}^n} h(\vec{x})g(\vec{x})dV$ where $h \in \mathbf{C}_c(\mathbb{R}^n)$. Then, $I(h)$ is a F-distribution.
4. If $I_1, \cdots, I_n$ are F-distributions and $g \in \mathbf{C}(\mathbb{R}^n)$, then $g(I_1, \cdots, I_n)$ is a F-distribution. This contrasts markedly with the situation ordinarily assumed for distributions.

2. Section: F versa ordinary distributions

For simplicity and since this suffices, we assume in this section that our compact subsets K of $\mathbb{R}$ are bounded closed intervals. By definition a function $f : \mathbb{R} \to \mathbb{R}$ with support K is a Morse function on the compact set K if and only if it is a Morse function on the interior of K (see [11] for this last notion). Let $\mathbf{MMC}_{c,K}(\mathbb{R})$ denote the subset of $\mathbf{C}_{c,K}(\mathbb{R})$ consisting of Morse functions having a maximum at a unique point and minimum at a unique point. Note that $0 \notin \mathbf{MMC}_{c,K}(\mathbb{R})$. The open subsets of $\mathbf{C}_{c,K}(\mathbb{R})$ in (ordinary) distribution theory are in part defined by the semi-norm $|\cdot|_K$ where $|g|_K = \sup_{x\in K} |g(x)|$.

Lemma 2.1. *The set* $\{x \in \mathbf{MMC}_{c,K}(\mathbb{R}) | |x|_K < \epsilon\}$ *is open in the F-space topology on the subspace* $\mathbf{MMC}_{c,K}(\mathbb{R})$ *of* $\mathbf{C}_{c,K}(\mathbb{R})$.

Proof. Let $d : \mathbb{R} \to \mathbf{MMC}_{c,K}(\mathbb{R})$ be a smooth curve and $d(a) = g$. Suppose that $\hat{d}(t, x) = d(t)(x)$. We show that the assignment $\alpha : t \hookrightarrow \sup_{x\in K} \hat{d}(t, x)$ is smooth. Since d has an image in $\mathbf{MMC}_{c,K}(\mathbb{R})$,
$\sup_{x\in K} \hat{d}(t, x) = d(t)(a_t)$ with a_t unique for each $t \in \mathbb{R}$ and $d(t)(a_t) \neq 0$. Since $d(t)$ is a Morse function for each $t \in \mathbb{R}$, $d(t)''(a_t) \neq 0, t \in \mathbb{R}$. It follows that $d(t)'(x) = 0$ can be solved for $x = a_t$, a smooth function in t. Thus, $\alpha(t) = d(t)(a_t)$ is a smooth function in t. Thus, the assignment $g \to \sup_{x\in K} g$ defines a smooth real valued map on $\mathbf{MMC}_{c,K}(\mathbb{R})$. Similarly, the assignment $g \to \inf_{x\in K} g$ defines a smooth real valued map on $\mathbf{MMC}_{c,K}(\mathbb{R})$. Finally, note that $|g|_K < \epsilon$ is equivalent to $\sup_{x\in K} g < \epsilon$ and $-\epsilon < \inf_{x\in K} g$. □

It is reasonably clear that the last result extends from $\mathbb{R}$ to $\mathbb{R}^n$. Also, one expects that, in the Whitney strong topology, $\mathbf{MMC}_{c,K}(\mathbb{R})$ is an open dense subset of $\mathbf{C}_{c,K}(\mathbb{R})$.

Let $\mathbf{FMC}_{c,K}(\mathbb{R})$ denote the subset of the set $\mathbf{MC}_{c,K}(\mathbb{R})$ of Morse functions with support K consisting of functions h (see [11]) which are Morse functions on K; thus, if $h'(b) = 0$ for b in the interior of K, then $h''(b) \neq 0$. We prove an analogous version of Lemma 2.1. First, we need:

Lemma 2.2. *Let* $k \in \mathbf{MMC}_{c,K}(\mathbb{R})$ *have a unique maximum at* b *and let* $d : \mathbb{R} \to \mathbf{MC}_{c,K}(\mathbb{R})$ *be a smooth curve with* $d(a) = k$. *Then for* t *in a neighborhood of* a, $d(t)$ *has a maximum at a unique point.*

Proof. Consider the smooth map $\hat{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\hat{d}(t, s) = d(t)(s)$. As $\frac{\partial^2 \hat{d}}{\partial s^2}(a, b) = d(a)''(b) = k''(b) < 0$, $d(t)''(s) = \frac{\partial^2 \hat{d}}{\partial s^2}(t, s) < 0$ for $a - \delta_1 \leq t \leq a + \delta_1, b - \rho_1 \leq s \leq b + \rho_1$ for suitable $\delta_1, \rho_1 > 0$. Thus, $d(t)$ is concave downward if $a - \delta_1 \leq t \leq a + \delta_1$ in the interval $b - \rho_1 \leq s \leq b + \rho_1$ with $b - \rho_1, b + \rho_1 \in K$. Let $M = k(b) = \hat{d}(a, b)$. Using the fact that k is locally increasing to the left of b and decreasing to the right of b, there is a value $b - \delta_1 < s_1 < b$ and a value $b + \delta_1 > s_2 > b$ such that $m_1 = k(s_1) \geq k(s)$ for $s \leq s_1$ and $k(s_2) \geq k(s)$ for $s \geq s_2$. Let $q = \frac{min\{(M-m_1),(M-m_2)\}}{8}$. Using the continuity of $\hat{d}$ there are $0 < \delta_2 < \delta_1, 0 < \rho_2 < \rho_1$ such that $\hat{d}(s, t) > M - q$ if

$a-\delta_2 \le t \le a+\delta_2, b-\rho_2 \le s \le b+\rho_2$. Using the compactness of $K - K \cap (s_1, s_2)$, one can find $0 < \delta_3 < \delta_2, \gamma > 0$ such that $\hat{d}(s,t) < M - q$ if $a-\delta_3 < t < a+\delta_3$ and either $s \ge s_2 + \gamma$ or $s \le s_1 - \gamma$ but with $s_2+\gamma < b+\rho_1$ and $s_1 - \gamma > b - \rho_1$. From construction, it follows that if $a-\delta_3 \le t \le a+\delta_3$, then $d(t)(s)$ has a maximum and then a unique maximum which lies in the interval $(b-\rho_1, b+\rho_1)$, because of concavity. □

Lemma 2.3. *Let $\epsilon > 0$ and $\mathbf{BUP}(\epsilon) = \{g \in \mathbf{FMC}_{c,K}(\mathbb{R}) | sup_{x\in K} g(x) < \epsilon\}$. Let $\mathbf{FMC}(n)$ be the set of all $h \in \mathbf{FMC}_{c,K}(\mathbb{R})$ such that h has local maxima at exactly n different points in the interior of K. Then, $\mathbf{FMC}_{c,K}(\mathbb{R})$ is the coproduct of the F-subspaces $\mathbf{FMC}(n)$. For any real number ϵ there are smooth maps H_i : $\mathbf{FMC}_{c,K}(\mathbb{R}) \to \mathbb{R}, i = 1, \cdots, n$, such that*

$$\mathbf{BUP}(\epsilon) \cap \mathbf{FMC}(n) = \bigcap_{i=1,\cdots,n} (H_i^{-1}(-\infty, \epsilon) \cap \mathbf{FMC}(n)).$$

Proof. Consider an element $g \in \mathbf{FMC}_{c,K}(\mathbb{R})$ which has local maxima except at the endpoints of K (finite in number as K is compact and g is a Morse function) at $x = b_1 < \cdots < b_n$. Let $d : \mathbb{R} \to \mathbf{FMC}_{c,K}(\mathbb{R})$ be a smooth curve and $d(a) = g$. Localizing to a neighborhood of each b_i and, applying the technique of Lemma 2.1, which is possible because of Lemma 2.2 applied locally, one obtains smooth curves $x^i(t) = a_t^i$ defined on an open interval $\mathbf{I}$ with $x^i(a) = b_i$ and such that $d(t)(x)$, for fixed t, has local maxima at $x^i(t)$. The curves $x^i(t), i = 1, \cdots, n$ never meet for $t \in \mathbf{I}$ since there is always a point of inflection between them. Since $d(t)'(x^i(t)) = 0$ for $t \in \mathbf{I}$, $d(t)'(x^i(t)) = 0$ at the endpoints of $\mathbf{I}$. Since d maps into $\mathbf{FMC}_{c,K}(\mathbb{R})$, $d(t)''(x^i(t)) < 0$ at the endpoints of $\mathbf{I}$. Thus, one can continue the $x^i(t)$ until $\mathbf{I} = \mathbb{R}$. Clearly, every local maximum of every curve $d(t), t \in \mathbb{R}$, lies on some curve $x^i(t), i = 1, \cdots, n$. Since each curve $d(t)$ has its image in exactly one $\mathbf{FMC}(n)$, it follows that $\mathbf{FMC}_{c,K}(\mathbb{R})$ is the coproduct of the F-subspaces $\mathbf{FMC}(n)$.

For an arbitrary element $k \in \mathbf{FMC}_{c,K}(\mathbb{R})$ which has local maxima at $x = d_1 < \cdots < d_n$, define $H_i(k) = k(d_i)$. For $i > n$, define $H_i(k) = 0$. Then, as $H_i(d(t)) = d(t)(x^i(t))$, $H_i(d(t))$ is smooth. Since $d(t)$ is arbitrary, H_i is smooth. Let $g \in \mathbf{BUP}(\epsilon) \cap \mathbf{FMC}(n)$. Then, $H_i(g) = g(d_i) < \epsilon$ for $i = 1, \cdots, n$ and hence $g \in \bigcap_{i=1,\cdots,n}(H_i^{-1}(-\infty, \epsilon) \cap \mathbf{FMC}(n))$. Since the reverse implication clearly holds, we are done. □

From the above result one readily obtains:

Theorem 2.1. *The set $\{g \in \mathbf{FMC}_{c,K}(\mathbb{R}) || g|_K < \epsilon\}$ is open for the F-space structure on $\mathbf{FMC}_{c,K}(\mathbb{R})$.*

Remark: It seems likely that the topology of the F-space structure on $\mathbf{C}_c(\mathbb{R})$ is finer than the usual topology. The other direction requires some better understanding of the smooth scalars on $\mathbf{C}_c(\mathbb{R})$, i.e., the F-distributions. We note (see [11]) that the set of Morse functions on the interior K° of K is open and dense in $\mathrm{C}(K^\circ) = \mathbb{R}^{K^\circ}$.

3. Embeddings

Let $\iota : \mathbb{R}^n \to \mathrm{C}(\mathbb{R}^n)$ be a one-one map defined by setting $\iota(x_1, \cdots, x_n) = x_1y_1 + \cdots + x_ny_n$, a linear polynomial in $(y_1, \cdots, y_n)$. We show:

Proposition 3.1. *The map ι is an embedding identifying $\mathbb{R}^n$ with the F-subspace $\iota(\mathbb{R}^n)$ of $\mathbf{C}(\mathbb{R}^n)$.*

Proof. Since the assignment $((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \to x_1y_1 + \cdots + x_ny_n$ is smooth, the Cartesian closedness of $\underline{\mathrm{FRL}}$ implies that ι is smooth. Let $d : \mathbb{R} \to \iota(\mathbb{R}^n)$ be a smooth map. One has $d(t)(y_1, \cdots, y_n) = a_1(t)y_1 + \cdots + a_n(t)y_n$ since $d(t) \in \iota(\mathbb{R}^n)$ for each t. But, $d(t)(y_1, \cdots, y_n) = \hat{d}(t, (y_1, \cdots, y_n))$ is smooth in t. It follows that $\iota^{-1}(d(t)) = (a_1(t), \cdots, a_n(t))$ is smooth. Hence $\iota^{-1} : \iota(\mathbb{R}^n) \to \mathbb{R}^n$ is smooth and we are done. $\square$

We prove the result in Proposition 3.1 for $\mathrm{C}_c(\mathbb{R}^n)$ instead of $\mathbb{R}^n$.

Proposition 3.2. *The non-degenerate inner product*

$$\langle\ ,\ \rangle : \mathrm{C}_c(\mathbb{R}^n) \times \mathrm{C}_c(\mathbb{R}^n) \to \mathbb{R}$$

defined by setting $\langle f, g\rangle = \int_{\mathbb{R}^n} fg dx_1 \cdots dx_n$ induces a smooth embedding $\mathcal{I}$ of $\mathrm{C}_c(\mathbb{R}^n)$ into the set $\mathcal{D}_n$ of F-distributions on $\mathbb{R}^n$.

Proof. Let $n = 1$. The inner product $\langle\ ,\ \rangle$ is smooth since, for each smooth curve $d : \mathbb{R} \to \mathrm{C}_c(\mathbb{R}) \times \mathrm{C}_c(\mathbb{R})$ with $d(t) = (d_1(t), d_2(t))$, the assignment

$$t \to \int_{\mathbb{R}} d_1(t)d_2(t)dx$$

is a smooth function of t (see below). The map $\mathcal{I}$ satisfies $\mathcal{I}(g)(f) = \langle f, g\rangle$. The Cartesian closedness of $\underline{\mathrm{FRL}}$ implies that $\mathcal{I}$ is smooth.

To show that $\mathcal{I}$ is an embedding, let $d : \mathbb{R} \to \mathcal{I}(\mathrm{C}_c(\mathbb{R}))$ be a smooth map. Using the fact that $\mathcal{I}$ is one-one, $d(t)(f) = \int_{\mathbb{R}} g_t(x)f dx$ for some unique $g_t \in \mathrm{C}_c(\mathbb{R})$. But, $g_t(x) = \hat{g}(t, x)$. Thus, to show that the map $t \to g_t$ is smooth in t, we need to know that if

$$V(t, \omega) = \int_{\mathbb{R}} \hat{g}(t, x)\hat{f}(\omega, x)dx \tag{1}$$

is smooth in t and ω for all smooth functions $\hat{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then $\hat{g}$ is smooth. In equation (1) we let $\hat{f}(\omega, x) = \frac{1}{2\pi}e^{i\omega x}$ and one obtains a smooth function

$$F(t, \omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(t, x)e^{i\omega x}dx \tag{2}$$

where, for each t, $F(t, \omega)$ is the Fourier transform of $\hat{g}(t, x)$. See [9]. Applying the inverse Fourier transform, one obtains the equation

$$\hat{g}(t, x) = \int_{\mathbb{R}} F(t, \omega)e^{-i\omega x}d\omega \tag{3}$$

Using results (8.11.1) and (8.11.2) in Dieudonne [7], which in particular allow us to take derivatives under the integral sign, we are done.

For $n > 1$, one needs to show that, corresponding to equation (1) for $n = 1$ and using Boman's Theorem, that $V(t, (\omega_1, \cdots, \omega_n))$ is smooth if and only if $f((\omega_1, \cdots, \omega_n), x)$ is smooth. Finally, on applying multi-dimensional Fourier and inverse Fourier transforms together with (8.11.1) and (8.11.2) in [7], the proof is complete. □

4. Operations on F-distributions

4.1. **Derivatives.** Using the Cartesian closedness of FRL, one readily shows:

Proposition 4.1. *Let D_i denote the i-th partial of a function of n variables and $D = D_1^{m_1} \cdots D_n^{m_n}$. Then the map $D : \mathrm{C}_c(\mathbb{R}^n) \to \mathrm{C}_c(\mathbb{R}^n)$ is smooth.*

This result doesn't hold for differential spaces. See [3].

One has the relation $\mathcal{I}(h)(D(g)) = \int_{\mathbb{R}} hD(g)dx_1 \cdots dx_n$ and, integrating repeatedly by parts, the last quantity equals
$\int_{\mathbb{R}^n} (-1)^{m_1+\cdots+m_n} gD(h)dx_1 \cdots dx_n$. Hence, we have:

Proposition 4.2. *Viewing functions via their embedding $\mathcal{I}$, one can define the partial D of the F-distribution $\mathcal{I}(h)$ by setting*
$D(\mathcal{I}(h))(g) = \mathcal{I}(D(h))(g) = (-1)^{m_1+\cdots+m_n}\mathcal{I}(h)(D(g))$.

We will define the derivative of F-distributions via the chain rule (which we assume to hold). Let $d : \mathbb{R} \to \mathrm{C}_c(\mathbb{R}^n)$ be a smooth curve with $d(0) = g, h = d'(0) = \frac{\partial \hat{d}}{\partial t}|_{t=0}$ and, as usual, $\hat{d}(t, x) = d(t)(x)$. If the chain rule holds, then one expects that

$$J'(g)(h) = (J \circ d)'(0). \tag{4}$$

For a smooth map $F : \mathbb{R}^n \to \mathbb{R}$, one can view the map F' as a map $F' : \mathbb{R}^n \to \mathbb{R}^{\mathbb{R}^n}$. Similarly, J' is a map $J' : \mathrm{C}_c(\mathbb{R}^n) \to \mathbb{R}^{\mathrm{C}_c(\mathbb{R}^n)}$ and, given that $(J \circ d)'(0)$ can be calculated, J' is defined by (4). In [6], the product $\bullet$ of an element $g \in \mathrm{C}_c(\mathbb{R}^n)$ and a distribution T is defined by setting $g \bullet T(h) = T(gh)$.

Examples:

1. Let $\mathtt{T}_2(k)(f) = \int_{\mathbb{R}} kf^2dx$. Then, if $k \in \mathrm{C}_c(\mathbb{R})$, $\mathtt{T}_2(k)$ is a F-distribution on $\mathbb{R}$. Using the conventions above, one has
$(\mathtt{T}_2(k))'(g)(h) = \int_{\mathbb{R}} k \cdot 2d(0)d'(0)dx = \int_{\mathbb{R}} k \cdot 2ghdx$. Thus,
$(\mathtt{T}_2(k))'(g) = I(2gk) = 2g \bullet I(k)$. The F-distribution $\mathtt{T}_2$ is defined by inner squaring and this example is clearly extendable to higher dimension.
2. Using outer squaring, define the F-distribution $\mathtt{T}^2(k)$ by $\mathtt{T}^2(k) = (\int_{\mathbb{R}} kfdx)^2$. Then, $(\mathtt{T}^2(k))'(g)(h) = 2\int_{\mathbb{R}} kgdx \cdot \int_{\mathbb{R}} khdx = I(k)(h)I(k)(g)$. Thus, $(\mathtt{T}^2(k))'$ is the map sending $g \to 2I(k)(g) \cdot I(k)(-)$.

With our definition of derivative, one can prove the product rule:

Proposition 4.3. *For F-distributions T_1 and T_2 on $\mathbb{R}$, one has*

$$(T_1T_2)' = T_1T_2' + T_1'T_2 : \mathrm{C}_c(\mathbb{R}) \to \mathbb{R}.$$

Proof. On the one hand, using the notation above, $(T_1T_2)'(g)(h) = ((T_1T_2) \circ d)'(t) = ((T_1 \circ d)(T_2 \circ d))'(t) = (T_1 \circ d)(0)(T_2(t) \circ d)'(0) + (T_1(t) \circ d)'(0)(T_2 \circ d)(0) = T_1(g)(T_2)'(g)(h) + (T_1)'(g)(h)T_2(g)$. On the other hand, $(T_1T_2' + T_1'T_2)(g)(h) = (T_1(g)T_2'(g) + T_1'(g)T_2(g))(h) = T_1(g)T_2'(g)(h) + T_1'(g)T_2(g)(h)$ and we are done. □

In the same way, one can show the quotient rule.

Using the embedding ι defined in Section 3, one might reasonably define, for a F-distribution T on $\mathbb{R}^n$, the j-th partial at $\hat{0} \in \mathrm{C}_c(\mathbb{R}^n)$ via the equality $\frac{\partial T}{\partial x_j}(\hat{0}) = (T \circ d)'(0)$ where $d(t)$ is the smooth curve $d : \mathbb{R} \to \mathrm{C}_c(\mathbb{R}^n)$ satisfying $d(t) = tx_j$. This definition makes sense if T extends to $\mathrm{C}(\mathbb{R}^n)$.

Example: For the F-distribution $I(k)$ on $\mathbb{R}^n$ defined above, with k having compact support, the definition of partial derivative makes sense and $\frac{d(I(k)(tx_j))}{dt} = \int_{\mathbb{R}} k\frac{dtx_j}{dt} dx_1 \cdots dx_n = \int_{\mathbb{R}} kx_j dx_1 \cdots dx_n = (x_j \bullet I(k))(1) =$
$-I(\frac{\partial k}{\partial x_j})(1)$.

4.2. **Operations.** Consider the map (looked at above)

$$\bullet : \mathrm{C}_c(\mathbb{R}^n) \times \mathcal{D}_n \to \mathcal{D}_n$$

defined by sending (f, T) to $f \bullet T$ where $f \bullet T(h) = T(fh)$.

Lemma 4.1. *The map $\bullet$ is smooth.*

Proof. Using the Cartesian closedness of FRL, one must show that if T_t is a smooth family of F-distributions and f_t is a smooth family in $\mathrm{C}_c(\mathbb{R}^n)$, then, for each $g \in \mathrm{C}_c(\mathbb{R}^n)$, the map $t \to T_t(f_tg)$ is smooth. This is the case as multiplication of smooth functions and evaluation in FRL, since FRL is Cartesian closed, are smooth operations. □

Since one can differentiate under the integral sign, one can show:

Lemma 4.2. *Convolution acting on $\mathrm{C}_c(\mathbb{R}^n)$ is a smooth operation.*

Let now T_1 be a F-distribution on $\mathbb{R}^n$ and T_2 be a F-distribution on $\mathbb{R}^m$. One defines the direct product $T_1 \times T_2$ (see [12]) of T_1 and T_2 by setting, for $g \in \mathrm{C}_c(\mathbb{R}^{n+m})$, $T_1 \times T_2(g) = T_1(h)$ where $h(x_1, \cdots, x_n) = T_2(g_{(x_1,\cdots,x_n)})$ and $g_{(x_1,\cdots,x_n)}(y_1, \cdots, y_m) = g(x_1, \cdots, x_n, y_1, \cdots, y_m)$. Note that since g has compact support, so does $g_{(x_1,\cdots,x_n)}$. For this definition to make sense, one needs to see that h is smooth. Thus, if $(x_1^t, \cdots, x_n^t)$ is a smooth family of points in $\mathbb{R}^n$, one needs to see that $k(t) = T_2(g_{(x_1^t,\cdots,x_n^t)})$ where $g_{(x_1^t,\cdots,x_n^t)}(y_1, \cdots, y_m) = g(x_1^t, \cdots, x_n^t, y_1, \cdots, y_m)$, is a smooth function of t. But, $k(t)$ is smooth since evaluation is a smooth map (from the Cartesian closedness of FRL), the map sending t to $g_{(x_1^t,\cdots,x_n^t)}$ is smooth and the composite of smooth maps is smooth. See [12].

We define the notion of convolution for distributions:

Definition 4.1. *Let T_1 and T_2 be two distributions on $\mathbb{R}^n$ with compact support (see* [12]*) and $\phi \in \mathrm{C}_c(\mathbb{R}^n)$. Then,*

$$T_1 * T_2(\phi) = T_1 \times T_2(\bar{h})$$

where $\bar{h} = \phi((x_1, \cdots, x_n) + (y_1, \cdots, y_n)))$.

Let $\mathcal{D}_{n,c}$ be the collection of F-distributions on $\mathbb{R}^n$ with compact support. Finally, we show, as one would want:

Proposition 4.4. *Convolution $* : \mathcal{D}_{n,c} \times \mathcal{D}_{n,c} \to \mathcal{D}_{n,c}$ is a smooth operation.*

Proof. One needs to show that if one has F-distributions with compact support $T_1(t), T_2(t)$ smooth in t, then the map

$$t \to T_1(t)(T_2(t)(\phi(\vec{x} + \vec{y}))$$

for $\phi \in \mathrm{C}_c(\mathbb{R}^n)$ is a smooth function of t. Since evaluation is a smooth map in FRL, one need only show that the map

$$w : t \to T_2(t)(\phi(\vec{x} + \vec{y})) \in \mathrm{C}_c(\mathbb{R}^n)$$

is smooth. But w is smooth if the map $\hat{w} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where $\hat{w}(t, \vec{x}) = T_2(t)(\phi(\vec{x} + \vec{y}))$, is smooth. One has however $\hat{w}(t, \vec{x}) = ev \circ (T_2 \times \bar{\phi})(t, \vec{x})$ where ev is evaluation and $\bar{\phi}(t, \vec{x})(\vec{y}) = \phi(\vec{x} + \vec{y})$ is clearly smooth in $t, \vec{x}$ and $\vec{y}$. It follows that T_2 and $\bar{\phi}$ are both smooth in t and $\vec{x}$. Since evaluation is a smooth map, $\hat{w}$ is smooth and we are done. □

References

[1] H. L. Bentley and P. Cherenack, Boman's Theorem: strengthened and applied, to appear.
[2] J. Boman, Differentiability of a Function and of its Compositions with Functions of One Variable, *Math. Scand.* **20**(1967),249-268.
[3] P. Cherenack and P. Multarzyński, Smooth exponential objects and smooth distributions, to appear.
[4] P. Cherenack. Applications of Frölicher spaces to cosmology, *Annales Univ. Sci. Budapest***41**(1998)63-91.
[5] P. Cherenack. Fröhlicher versus Differential spaces: A Prelude to Cosmology, *Applied Categorical Systems***8/1-2**(2000), 391-413.
[6] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics.* (North-Holland, Amsterdam, 1982).
[7] J.Dieudonné, *Foundations of Modern Analysis.* (Academic Press, New York, 1960).
[8] A. Frölicher and A. Kriegl, Linear Spaces and Differentiation Theory (Wiley-Interscience, New York, 1971).
[9] R. Haberman, *Elementay Applied Partial Differential Equations: with Fourier series and Boundary Value Problems.* (Prentice Hall, Upper Saddle River,N.J., 1998).
[10] A. Kriegl and P. Michor. *The Convenient Setting of Global Analysis.* (AMS, Providence, 1997).
[11] M. Hirsch, Differential Topology (Springer-Verlag, Berlin, 1976).
[12] A. H. Zemanian, Distribution Theory and Transform Analysis (Dover, New York, 1965).

Department of Mathematics and Applied Mathematics, University of Cape Town, 7700 Rondebosch, South Africa