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REPRESENTATION OF THE VARIATIONAL SEQUENCE IN FIELD THEORY

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Abstract. The aim of this paper is to discuss some aspects of local and global properties of classical concepts of calculus of variations in the $r-$th order field theory on fibered manifolds within the framework of the variational sequence, which is the quotient of the De Rham sequence with respect to its subsequence of contact differential forms. Such a discussion is, in general, based on the concept of sheaves of differential forms. In the paper a globally defined representation of the variational sequence by forms is constructed for its part closely related to the standard concepts of the calculus of variations. The extended definition of the Euler-Lagrange form as a representative of the class of $(n+2)$-forms is considered and the definition of the so called Helmholtz-Sonin form as a representative of the class of $(n+2)$-forms is presented. The properties of corresponding terms in the variational sequence, considered as the generalized Euler-Lagrange mapping and the Helmholtz-Sonin mapping, are studied. There is a close relationship between elements of the quotient sheaves (*classes* of forms) and the quotient mappings on one hand and the standard objects of the calculus of variations, such as lagrangian, Euler–Lagrange form and Helmholtz–Sonin expressions defining the so called Helmholtz–Sonin form, on the other hand.

1. Introduction

One of the most important questions in the calculus of variations is the characterization of local and global properties of the Euler–Lagrange and Helmholtz–Sonin mappings, especially their kernels and images. The general solution of this problem on an $r-$jet prolongation of a given fibered manifold can give the answers concerning the variationally trivial lagrangians and variational equations of motion in the $r-$th order field theory or mechanics. The close relationship between the exterior

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derivative of a differential form and the Euler–Lagrange mapping in the classical sense, formulated by Lepage and Dedecker, has been developed during last two decades by many authors (Anderson, Betounes, Duchamp, Gotay, Krupka, Krupková, Kuperschmidt, Olver, Pommaret, Saunders, Takens, Tulczyjew, Vinogradov etc.) and it then led to the concept of the variational sequence on finite jet prolongations of fibered manifolds, introduced and systematically studied by Krupka [6–8]. The variational sequence is constructed as the quotient of the well-known De Rham exact sequence of spaces of differential forms with respect to its subsequence of certain spaces of contact forms. This subsequence is chosen in such a way that the Euler–Lagrange and Helmholtz–Sonin mappings, considered in the generalized concept, are contained in the corresponding quotient sequence of mappings. The theoretical background for the study of the variational sequence is, among others, the theory of sheaves which was presented in details and elaborated for the purposes of the variational sequence calculus by Krupka [9]. Some aspects of the variational sequence were studied by some authors pertaining to Krupka's school: Štefánek [15] found a "non- physical" local representation of the $r-$th order variational sequence in mechanics. Musilová [13] and Musilová and Krbek [14] described the (global) "physical" representation of the physically relevant part of the $r-$th order variational sequences in mechanics, including the reconstruction of classes of forms from their representatives. Kašparová [2–4] has been studying the first order variational sequence in field theory and she found the global representatives of physically relevant classes of forms. The problem of variationally trivial lagrangians was completely solved by Krupka and Musilová [10]. Some problems concerning the variational sequence in field theory were recently discussed also by Vitolo in [16] and by Francaviglia, Palese and Vitolo in [1].

In this paper we discuss some properties of the $r-$th order variational sequence on fibered manifolds over n–dimensional base. We construct its representation for classes of $q-$forms, $1 \leq q \leq n+2$, especially for the physically relevant part, i.e. for classes of $n-$forms, $(n+1)-$forms and $(n+2)-$forms. Following the ideas of Krupka [8] for mechanics, we give the generalized definition of the Euler–Lagrange form and Helmholtz–Sonin form as well as the Euler–Lagrange and Helmholtz–Sonin mapping. We show that our representatives are global for $1 \leq q \leq n+2$.

2. Basic notations

Throughout the paper we use the following standard notation, used by Krupka (see e.g. [8,11]): Y is a $(n+m)-$dimensional *fibered manifold* with the $n-$*dimensional base* X and *projection* π. For an arbitrary integer $r \geq 0$, J^rY is the $r-$*jet prolongation* of Y, π^r and $\pi^{r,s}$ for $r \geq s \geq 0$ being the *canonical projections* of J^rY on X and J^sY, respectively, $N_r = \dim J^rY = n + \sum_{j=0}^{r} M_j = n + m\binom{n+r}{n}$, where $M_j = m\binom{n+j-1}{j}$. Moreover, we denote $P_r = \sum_{j=0}^{r-1} M_j + 2n - 1$. By γ and $J^r_x\gamma$ we denote a *section of the fibered manifold* Y (or *section of* π) and its $r-$*jet at* x, respectively. The mapping $J^r\gamma : x \to J^r\gamma(x) = J^r_x\gamma$ is the $r-$*jet prolongation* of γ. $\Gamma_\Omega(\pi)$ is the set of all sections of π defined on $\Omega \subset X$. Let $1 \leq \sigma \leq m$ and (V, ψ),

$\psi = (x^i, y^\sigma)$, $1 \le i \le n$, be a *fibered chart* on Y. Then we denote (U, φ) and (V^r, ψ^r) the *associated chart* on X and *associated fibered chart* on J^rY, respectively. Here $U = \pi(V)$, $\varphi = (x^i)$, $1 \le i \le n$, $V^r = (\pi^{r,0})^{-1}(V)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_r})$, $1 \le j_1, \dots, j_r \le n$. The variables $y^\sigma_{j_1 \dots j_k}$ are completely symmetrical in all indices contained in the multiindex $J = (j_1 \dots j_k)$. The integer $k = |J|$ is the *length of the multiindex* J. (For y^σ the corresponding multiindex is considered to be of zero length.) Other kinds of multiindices used in the paper are of the form $\binom{\sigma}{J} = \binom{\sigma}{j_1 \dots j_k}$, $0 \le |J| \le r$.

Let $\Omega^r_0 V$ be the ring of smooth functions on V^r. Denote by $\Omega^r_q V$ the $\Omega^r_0 V-$*module of smooth differential* $q-$*forms* on V^r, $\Omega^r_{q,c} V \subset \Omega^r_q V$, the submodule of *contact* $q-$*forms* (for $1 \le q \le n$) and *strongly contact* $q-$*forms* (for $n+1 \le q \le N_r$), and $\mathrm{d}\Omega^r_{q-1,c} V \subset \Omega^r_q V$ the subset of exterior derivatives of contact (strongly contact) $(q-1)-$forms. Let $\Theta^r_q V = \mathrm{d}\Omega^r_{q-1,c} V + \Omega^r_{q,c} V$. For $2 \le q \le n$ it holds $\mathrm{d}\Omega^r_{q-1,c} V \subset \Omega^r_{q,c} V$, i.e. $\Theta^r_q V = \Omega^r_{q,c} V$, and of course, $\Theta^r_1 V = \Omega^r_1 V$. $\Theta^r_q V$ is trivial for $q > P_r$. In addition we denote by $\omega^\sigma_J = \mathrm{d}y^\sigma_J - y^\sigma_{Ji}\mathrm{d}x^i$, $0 \le |J| \le r-1$, contact 1–forms, and by $\omega_i = (-1)^{i-1}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \dots \wedge \mathrm{d}x^n$, $\omega_0 = \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$ the most frequently used horizontal forms. It holds $\mathrm{d}x^i \wedge \omega_i = \omega_0$ (without summation over i) and $\mathrm{d}\omega^\sigma_{j_1 \dots j_{k-1}} \wedge \omega_{j_k} = -\omega^\sigma_{j_1 \dots j_k} \wedge \omega_0$.

Any $q-$form $\varrho \in \Omega^r_q V$ is generated by forms $(\mathrm{d}x^i, \omega^\sigma_J, \mathrm{d}y^\sigma_I)$, $1 \le i \le n$, $0 \le |J| \le r-1$, $|I| = r$. The notation ω^σ_J and $\mathrm{d}y^\sigma_I$ means that $\omega^\sigma_J = \omega^\sigma_{j_1 \dots j_k}$ for $|J| = k$ and $\mathrm{d}y^\sigma_I = \mathrm{d}y^\sigma_{j_1 \dots j_r}$.

3. Contact forms

This section presents a brief review of definitions and basic properties of contact and strongly contact forms on J^rY, adapted for practical purposes of our calculations. For the more detailed desription and proofs the reader is referred to the fundamental papers of Krupka [11,12]. The forms

$$(1) \qquad (\mathrm{d}x^i, \omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1 \dots j_{r-1}}, \mathrm{d}y^\sigma_{j_1 \dots j_r}), \quad \text{where } \omega^\sigma_{j_1 \dots j_k} = \mathrm{d}y^\sigma_{j_1 \dots j_k} - y^\sigma_{j_1 \dots j_k i}\, \mathrm{d}x^i,$$

define the *contact base* of 1-forms on V^r. For a function $f \in \Omega^r_0 V$ we denote by $\mathrm{d}_i f$ its total derivative with respect to the variable x^i,

$$\mathrm{d}_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji} = \mathrm{d}'_i f + \frac{\partial f}{\partial y^\sigma_I}\, y^\sigma_{Ii}, \quad 0 \le |J| \le r,\ |I| = r.$$

Lemma 1. *Let $W \subset Y$ be an open set, $q \ge 1$ an integer, and $\varrho \in \Omega^r_q W$ a q-form. Let (V, ψ) be a fibered chart on Y for which $V \subset W$. Let ϱ have the chart expression*

$$(2) \quad \varrho = \sum_{s=0}^{q} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s,\, i_{s+1} i_{s+2} \dots i_q}\, \mathrm{d}y^{\sigma_1}_{I_1} \wedge \mathrm{d}y^{\sigma_2}_{I_2} \wedge \dots \wedge \mathrm{d}y^{\sigma_s}_{I_s} \wedge \mathrm{d}x^{i_{s+1}} \wedge \mathrm{d}x^{i_{s+2}} \wedge \dots \wedge \mathrm{d}x^{i_q}$$

with coefficients antisymmetrical in all multiindices $\left(\binom{I_1}{\sigma_1}, \dots, \binom{I_s}{\sigma_s}\right)$, $0 \le |I_p| \le r$, $1 \le p \le s$, *antisymmetrical in all indices* $(i_{s+1}, \dots, i_q)$ *and symmetrical in all* *indices within each multiindex I_p. Then there exists the unique decomposition*

$$(\pi^{r+1,r})^* \varrho = h\varrho + p\varrho = h\varrho + p_1\varrho + \dots + p_q\varrho, \tag{3}$$

in which for every $1 \le k \le q$ it holds

$$p_k\varrho = C^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k,\, i_{k+1} i_{k+2} \dots i_q}\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \mathrm{d}x^{i_{k+2}} \wedge \dots \wedge \mathrm{d}x^{i_q},$$

(4)

$$C^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k,\, i_{k+1} i_{k+2} \dots i_q} = \sum_{s=k}^{q} \binom{s}{k} A^{I_1 I_2}_{\sigma_1\sigma_2} \cdots {}^{I_k}_{\sigma_k} \cdots {}^{I_s}_{\sigma_s,\, i_{s+1} i_{s+2} \dots i_q}\, y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} \cdots y^{\sigma_s}_{I_s i_s},$$
$$alt(i_{k+1} i_{k+2} \dots i_q).$$

(Note, that the summations are taken over all independent choices of indices in each multiindex, e.g. $(j_1 \dots j_p) = J$, $|J| = p$). The proof can be found in [11]. The term $h\varrho = p_0\varrho$ is the *horizontal* or $0-$*contact component* of the form ϱ, the terms $p_k\varrho$ for $1 \le k \le q$ are its $k-$*contact* components. A form $\varrho \in \Omega^r_q V$ is called π^r-*horizontal* if $\pi^{r+1,r}\varrho = h\varrho$, or *contact* if $h\varrho = 0$. Every $q-$form for $q > n$ is contact. A $q-$form ϱ, $n < q \le N$ is called *strongly contact*, if $p_{q-n}\varrho = 0$. A form ϱ is called $k-$contact, if $p_s\varrho = 0$ for $0 \le s \le k-1$.

In our calculations we frequently use the $(q-n)-$contact component of a form ϱ for $n < q \le N_r$. For $k = q - n$ the equation (4) gives

$$\begin{aligned} p_{q-n}\varrho &= C^{I_1}_{\sigma_1} \cdots {}^{I_{q-n}}_{\sigma_{q-n},\, i_{q-n+1} \dots i_q} \varepsilon^{i_{q-n+1} \dots i_q}\, \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{q-n}}_{I_{q-n}} \wedge \omega_0 = \\ &= B^{I_1}_{\sigma_1} \cdots {}^{I_{q-n}}_{\sigma_{q-n}}\, \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_{q-n}}_{I_{q-n}} \wedge \omega_0, \end{aligned} \tag{5}$$

where $\varepsilon^{j_1 \cdots j_n}$, $1 \le j_1, \dots, j_n \le n$ is the generalized Levi-Civita symbol.

The following lemma describes the local structure of contact forms. (For the proof see [11,12].)

Lemma 2. *Let $W \subset Y$ be an open set and $\varrho \in \Omega^r_q W$ a $q-$form. Let (V, ψ) be any fibered chart on Y for which $V \subset W$. Then*

(a) for $1 \le q \le n$ the form ϱ is contact if and only if it can be expressed as

$$\varrho = \Phi^J_\sigma\, \omega^\sigma_J \text{ for } q = 1, \text{ and } \varrho = \omega^\sigma_J \wedge \Psi^J_\sigma + d\Psi \text{ for } 2 \le q \le n, \quad 0 \le |J| \le r-1, \tag{6}$$

where $\Phi^J_\sigma \in \Omega^r_0 V$ are some functions, $\Psi^J_\sigma \in \Omega^r_{q-1} V$ some $(q-1)-$forms, and $\Psi \in \Omega^r_{q-1} V$ is a contact $(q-1)-$form which can be expressed as $\omega^\sigma_I \wedge \chi^I_\sigma$ for some $(q-2)-$forms $\chi^I_\sigma \in \Omega^r_{q-2} V$, $|I| = r-1$.

(b) for $n < q \le N_r$ the form ϱ is strongly contact if and only if it can be expressed as

$$\varrho = \omega^{\sigma_1}_{J_1} \wedge \dots \wedge \omega^{\sigma_p}_{J_p} \wedge \mathrm{d}\omega^{\sigma_{p+1}}_{I_{p+1}} \wedge \dots \wedge \mathrm{d}\omega^{\sigma_{p+s}}_{I_{p+s}} \wedge \Phi^{J_1 \dots J_p I_{p+1} \dots I_{p+s}}_{\sigma_1 \dots \sigma_p \sigma_{p+1} \dots \sigma_{p+s}}, \tag{7}$$

where $\Phi^{J_1 \dots J_p I_{p+1} \dots I_{p+s}}_{\sigma_1 \dots \sigma_p \sigma_{p+1} \dots \sigma_{p+s}} \in \Omega^r_{q-p-2s} V$, $0 \le |J_l| \le r-1$, $1 \le l \le p$, $|I_j| = r-1$, $p+1 \le j \le p+s$, and summation is made over such all p and s for which $p+s \ge q-n+1$, $p+2s \le q$.

4. Variational sequence

For the case of field theory we follow in this section the general ideas of Krupka [6] and basic concepts presented in [7,8] for mechanics. Let Ω^r_q, $q\geq 0$, be the *direct image* of the sheaf of smooth $q-$forms over J^rY by the jet projection $\pi^{r,0}$ (functions are considered as $0-$forms). Denote

(8)
$$\Omega^r_{q,c} = \ker p_0 \text{ for } 1 \leq q \leq n, \;\; \Omega^r_{q,c} = \ker p_{q-n} \text{ for } q > n \;\text{ and }\; \Theta^r_q = \Omega^r_{q,c} + \mathrm{d}\Omega^r_{q-1,c},$$

where p_0 and p_{q-n} are morphisms of sheaves induced by mappings p_0 and p_{q-n}, assigning to a form ϱ its horizontal and p_{q-n} contact component, respectively. $\mathrm{d}\Omega^r_{q-1,c}$ is the *image sheaf* of $\Omega^r_{q-1,c}$ by d. For every open set $W \subset Y$, $\Omega^r_q W$ is the Abelian group of $q-$forms on $W^r = (\pi^{r,0})^{-1}(W)$ and $\Omega^r_{q,c}W$ is the Abelian group of contact and strongly contact $q-$forms for $1\leq q\leq n$ and $q>n$, respectively, expressed locally by lemma 2. $\mathrm{d}\Omega^r_{q-1,c}W$ is the subgroup of $\Omega^r_q W$ given as $\{\varrho \in \Omega^r_q W | \varrho = \mathrm{d}\eta,\, \eta \in \Omega^r_{q,c}W\}$. Let us consider the sequence

$$\{0\} \to \Theta^r_1 \to \cdots \to \Theta^r_n \to \Theta^r_{n+1} \to \Theta^r_{n+2} \to \cdots \to \Theta^r_{P_r} \to \{0\}, \tag{9}$$

with arrows (except the first one) given by exterior derivatives d. The following lemma describes a basic property of this sequence. It can be proved in coordinates directly.

Lemma 3. *Let $W \subset Y$ be an open set, and let $\varrho \in \Theta^r_q W$ be a form, $1\leq q\leq N_r$. Then there exists the unique decomposition $\varrho = \varrho_1 + \mathrm{d}\varrho_2$, where $\varrho_1 \in \Omega^r_{q,c}W$ and $\varrho_2 \in \Omega^r_{q-1,c}W$.*

Thus, the sequence (9) is an *exact subsequence* of the de Rham sequence

$$\{0\} \to \Omega^r_1 \to \cdots \to \Omega^r_n \to \Omega^r_{n+1} \to \Omega^r_{n+2} \to \cdots \to \Omega^r_{N_r} \to \{0\}.$$

The quotient sequence

$$\{0\} \to \mathbf{R}_Y \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \cdots \to \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1} \to \Omega^r_{n+2}/\Theta^r_{n+2} \to$$
$$\to \cdots \to \Omega^r_{P_r}/\Theta^r_{P_r} \to \Omega^r_{P_r+1} \to \cdots \to \Omega^r_{N_r} \to \{0\} \tag{10}$$

is called the *variational sequence of the $r-$th order.* It is, of course, also exact. We denote quotient mappings as follows

$$E^r_q : \Omega^r_q/\Theta^r_q \ni [\varrho] \longrightarrow E^r_q([\varrho]) = [\mathrm{d}\varrho] \in \Omega^r_{q+1}/\Theta^r_{q+1}. \tag{11}$$

The mappings E^r_n and E^r_{n+1} generalize the classical concept of Euler-Lagrange mapping and Helmholtz-Sonin mapping of calculus of variations, respectively. They represent "physically relevant" terms of the variational sequence.

Using the chart expressions of forms we can prove the following lemma:

Lemma 4. *Let $W \subset Y$ be an open set, and let $\varrho \in \Theta^{r+1}_q W$ be a form, $1\leq q\leq N_r$. Let ϱ be $(\pi^{r+1,r})-$projectable, i.e. $\varrho = (\pi^{r+1,r})^*\eta$ for a form $\eta \in \Omega^r_q$. Then η is an element of $\Theta^r_q W$.*

(The proofs of lemmas 3 and 4 are based on technical coordinate calculations and we do not present them here.)

Let us consider the following scheme:

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & \Theta_q^{r+1} & \longrightarrow & \Omega_q^{r+1} & \longrightarrow & \Omega_q^{r+1}/\Theta_q^{r+1} & \longrightarrow & \{0\} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \{0\} & \longrightarrow & \Theta_q^{r} & \longrightarrow & \Omega_q^{r} & \longrightarrow & \Omega_q^{r}/\Theta_q^{r} & \longrightarrow & \{0\} \end{array}$$

in which the first two "uparrows" represent the immersions by pullbacks and the third one defines the quotient mapping

$$Q_q^{r+1,r} : \Omega_q^r/\Theta_q^r \longrightarrow \Omega_q^{r+1}/\Theta_q^{r+1}.$$

Using lemma 4 we can immediately see that the mapping $Q_q^{r+1,r}$ is injective. The (injective) mappings

$$Q_q^{s,r} : \Omega_q^r/\Theta_q^r \longrightarrow \Omega_q^s/\Theta_q^s, \quad r < s \tag{12}$$

can be defined in a quite analogous way.

The study of global properties of the variational sequence is based on the following facts proved by Krupka [6,8]:

1. *Each sheaf* Ω_q^r *is fine.*
2. *The variational sequence (in the shortened notation denoted by* $\{0\} \to \mathbf{R}_Y \to \mathcal{V}$*) is an acyclic resolution od the constant sheaf* $\mathbf{R}_Y$ *over* Y*.*
3. *For every* $q \geq 0$ *it holds* $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V})) = H^q(Y, \mathbf{R})$*, where*

$$\Gamma(Y, \mathcal{V}) : \{0\} \to \Gamma(Y, \mathbf{R}_Y) \to \Gamma(Y, \Omega_0^r) \to \Gamma(Y, \Omega_1^r) \to \cdots \to \Gamma(Y, \Omega_{N_r}^r) \to \{0\}$$

is the cochain complex of global sections and $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V}))$ *denotes its q-th cohomology group.*

5. Representation of the variational sequence

In this section we use the injectivity of mappings $Q_q^{s,r}$ to discuss the problem of the representation of the variational sequence by the appropriately chosen (exact) sequence of mappings of spaces of forms. Let W be an open subset of Y. Two $q-$forms ϱ, $\eta \in \Omega_q^r W$ belonging to the same class $\Omega_q^r W/\Theta_q^r W$ are called *equivalent.* Two $q-$forms $\varrho \in \Omega_q^r W$ and $\eta \in \Omega_q^t W$ are called *equivalent in the generalized sense* if there exists an integer $s \geq r, t$ for which $(\pi^{s,r})^* \varrho - (\pi^{s,t})^* \eta \in \Theta_q^s W$. Any mapping

$$\Phi_q^{s,r} : \Omega_q^r W/\Theta_q^r W \ni [\varrho] \longrightarrow \Phi_q^{s,r}([\varrho]) = \varrho_0 \in \Omega_q^s W$$

with $\varrho_0 \in [(\pi^{s,r})^* \varrho]$ (i.e. ϱ is equivalent with ϱ_0 in the generalized sense), is called *representation* of $\Omega_q^r W/\Theta_q^r W$. Because of the injectivity of mappings $Q_q^{s,r}$ (see definition (12) and lemma 4) the representation mappings $\Phi_q^{s,r}$ are injective too.

This injectivity enables us to define the *representation of the variational sequence by forms* as the lower row of the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega_q^r/\Theta_q^r & \longrightarrow & \Omega_{q+1}^r/\Theta_{q+1}^r & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \Omega_q^s & \longrightarrow & \Omega_{q+1}^s & \longrightarrow & \cdots \end{array}$$

in which the upper row is the variational sequence, the "downarrows" represent the mappings $\Phi_q^{s,r}$ and mappings of the lower row are defined by

$$E_q^{s,r} : \Omega_q^s \longrightarrow \Omega_{q+1}^s, \quad E_q^{s,r} = \Phi_{q+1}^{s,r} \circ E_q^r \circ (\Phi_q^{s,r})^{-1}, \quad E_0^{s,r} = \Phi_1^{s,r} \circ E_0^r. \tag{13}$$

In the following we shall show that there exists such a representation of the variational sequence by forms (i.e. the integer $s \geq r$ and mappings $E_q^{s,r}$) for which $E_n^{s,r}$ assigns to every lagrangian of the $r-$th order its Euler-Lagrange form and $E_{n+1}^{s,t}$, $r \leq t \leq s$, assigns to every dynamical form on J^tY a form expressed locally by means of Helmholtz-Sonin expressions. We shall concentrate our attention to the case $1 \leq q \leq n+2$, especially to the "physically relevant" terms of the variational sequence, i.e. $q = n, n+1, n+2$.

Theorem 1. *Let $W \subset Y$ be an open set, and let $q \geq 1$ be an integer. Let (V, ψ) be a fibered chart on Y for which $V \subset W$.*

(a) *Let $1 \leq q \leq n$ and let $\varrho \in \Omega_q^r W$ be a form. Then the mapping*

$$\Phi_q^{s,r} : \Omega_q^r V / \Theta_q^r V \ni \varrho \longrightarrow \Phi_q^{s,r}([\varrho]) = (\pi^{s,r})^* h\varrho \in \Omega_q^s V, \quad s \geq r+1 \tag{14}$$

is the representation of $\Omega_q^r V / \Theta_q^r V$.

(b) *Let $q = n+1$ and let $\varrho \in \Omega_{n+1}^r W$ be a form expressed in the fibered chart (V, ψ) by the relation*

$$p_1 \varrho = B_\sigma^J \, \omega_J^\sigma \wedge \omega_0, \tag{15}$$

in which coefficients $B_\sigma^J \in \Omega_0^{r+1} V$, $0 \leq |J| \leq r$, are given by the chart expression of ϱ following eqs. (2-5). Then the mapping

$$\Phi_{n+1}^{s,r} : \Omega_{n+1}^r V / \Theta_{n+1}^r V \ni \varrho \longrightarrow \Phi_{n+1}^{s,r}([\varrho]) = \varrho_0 \in \Omega_{n+1}^s V, \quad s \geq 2r+1$$

assigning to the class $[\varrho]$ the form

$$\varrho_0 = (\pi^{s,2r+1})^* \left(\sum_{l=0}^{r} (-1)^l \mathrm{d}_{j_1} \ldots \mathrm{d}_{j_l} B_\sigma^{j_1 \ldots j_l} \right) \omega^\sigma \wedge \omega_0 \tag{16}$$

is the representation of $\Omega_{n+1}^r V / \Theta_{n+1}^r V$.

(c) *Let $q = n+2$ and let $\varrho \in \Omega_{n+2}^r W$ be a form expressed in the fibered chart (V, ψ) by the relation*

$$p_2 \varrho = B_{\sigma\nu}^{JK} \, \omega_J^\sigma \wedge \omega_K^\nu \wedge \omega_0, \tag{17}$$

in which coefficients $B_{\sigma\nu}^{JK} \in \Omega_0^{r+1} V$, $0 \leq |J| \leq r$, are given by the chart expression of ϱ following eqs. (2-5). Then the mapping

$$\Phi_{n+2}^{s,r} : \Omega_{n+2}^r V / \Theta_{n+2}^r V \ni \varrho \longrightarrow \Phi_{n+2}^{s,r}([\varrho]) = \varrho_0 \in \Omega_{n+2}^s V, \quad s \geq 2r+1$$

assigning to the class $[\varrho]$ the form

$$\varrho_0 = \tag{18}$$

$$= (\pi^{s,2r+1})^* \sum_{j=0}^{2r} \left[\sum_{p=0}^{j} \sum_{l=j-p}^{r} (-1)^l \binom{l}{j-p} \mathrm{d}_{i_{j+1}} \ldots \mathrm{d}_{i_{p+l}} B_{\sigma\nu}^{i_1 \ldots i_p, i_{p+1} \ldots i_{p+l}} \right] \omega_{i_1 \ldots i_j}^\sigma \wedge \omega^\nu \wedge \omega_0,$$

$\mathrm{sym}(i_1 \dots i_j)$, $s \geq 2r+1$, *is the representation of* $\Omega^r_{n+2}V/\Theta^r_{n+2}V$.

Proof: For the proof of theorem 1 the equivalence $\Phi^{s,r}_q([\varrho]) = 0 \Leftrightarrow \varrho \in \Theta^r_q V$ is to be proved.

(a) Let $1 \leq q \leq n$. As it holds $\Theta^r_q = \Omega^r_q$ it is evident that $h\varrho = 0 \Leftrightarrow \varrho \in \Theta^r_q$.

(b) Let $q = n+1$. Let (V, ψ) be a fibered chart on Y and let $\varrho \in \Theta^r_{n+1}V$. Then ϱ is of the form $\varrho = \varrho_1 + \mathrm{d}\varrho_2$, where $\varrho_1 \in \Omega^r_{n+1,c}V$ and $\varrho_2 \in \mathrm{d}\Omega^r_{n,c}V$ (see lemma 3). We can immediately see that $\Phi^{s,r}_{n+1}([\varrho_1]) = 0)$. Thus, only the equation $\Phi^{s,r}_{n+1}([\mathrm{d}\varrho_2]) = 0$ needs proof. Taking into account equation (6) we have

$$\varrho_2 = \omega^\sigma_J \wedge \Psi^J_\sigma + \mathrm{d}\Psi, \quad 0 \leq |J| \leq r-1$$

for some $(n-1)-$forms Ψ^J_σ and some contact $(n-1)-$form $\Psi = \omega^\sigma_I \wedge \chi^I_\sigma$, $|I| = r-1$, $\chi^I_\sigma \in \Omega^r_{q-2}V$. Then it holds

$$\mathrm{d}\varrho_2 = \mathrm{d}(\omega^\sigma_J \wedge \Psi^J_\sigma) \Rightarrow p_1 \mathrm{d}\varrho_2 = -\omega^\sigma_{Ji} \wedge \mathrm{d}x^i \wedge h\Psi^J_\sigma - \omega^\sigma_J \wedge h\mathrm{d}\Psi^J_\sigma.$$

Using the chart expressions of Ψ^J_σ in agreement with (3) and (4), i.e.

$$(\pi^{r+1,r})^* \Psi^J_\sigma = \sum_{l=0}^{n-1} (P^J_\sigma)^{J_1}_{\sigma_1} \cdots ^{J_l}_{\sigma_l, i_{l+1} \dots i_{n-1}} \wedge \omega^{\sigma_1}_{J_1} \wedge \dots \omega^{\sigma_l}_{J_l} \wedge \mathrm{d}x^{i_{l+1}} \wedge \dots \mathrm{d}x^{i_{n-1}},$$

we obtain the coefficients B^J_σ in the corresponding chart expression (15) for $p_1 \mathrm{d}\varrho_2$. Putting them into (16) we can conclude, after some technical steps, that $\Phi^{s,r}_{n+1}([\mathrm{d}\varrho_2]) = 0$.

Conversely, let $\Phi^{s,r}_{n+1}([\varrho]) = 0$. Using lemma 4 we obtain again by coordinate calculations the expected result $\varrho \in \Theta^r_{n+1}V$.

(c) For $q = n+2$ the proof is quite analogous with that presented in (b). Since it involves tedious coordinate calculations we do not present it here.

Theorem 2 presented in the following section of the paper shows that the local expressions (16,18) for representatives of a classes of $(n+1)-$forms and $(n+2)-$forms give globally defined forms. These forms are called the *Euler-Lagrange form* and *Helmholtz-Sonin form*, respectively. Following the relation (13) which defines the representation of the variational sequence we can use theorem 1 for a form $\mathrm{d}\varrho$, $\varrho \in \Omega^r_n W$ or $\varrho \in \Omega^r_{n+1}W$, for obtaining the chart expressions of so called *Euler-Lagrange* and *Helmholtz-Sonin* mappings $E^{s,r}_n$ and $E^{s,r}_{n+1}$, respectively. The first of them represent the generalization of the well-known "classical" Euler-Lagrange mapping of the calculus of variations which is closely related to the trivial variational problem. The second one is connected with the solution of the inverse variational problem.

6. Global properties of the representation

The construction of the representative mappings $\Phi_q^{s,r}$ in the previous section for $1\leq q\leq n$ is given by the horizontalization h, and thus, it is global. For $q=n+1$ the globality of the definition of the representatives of the type (16) is mentioned in [6] with the reference to a proof using an integration method. For the 1−st order variational sequence the globality of representatives (16) and (18) was proved in [2,4], with the use of the integration of appropriately chosen forms. In this section we follow the idea of the integration method to prove the correctness (globality) of higher order representatives (16) and, as a new result, (18).

Theorem 2. *Let (V,ψ) be a fibered chart on Y. Let $1\leq q\leq n+2$ and $\varrho\in\Omega_q^r Y$ be a form. Then the class $[\varrho]$ is represented by eqs. (14), (16) and (16) globally, for $1\leq q\leq n$, $q=n+1$ and $q=n+2$, respectively.*

Proof: Because of globality of the horizontalization mapping h only the cases $q=n+1$, $n+2$ need proof. Let $(\overline{V},\overline{\psi})$ be a fibered chart on Y such that $\overline{V}\cap V\neq\emptyset$ and let $\Omega\subset\pi(\overline{V}\cap V)$ be a compact piece of manifold X.

Let $q=n+1$ and let $\varrho\in\Omega_{n+1}^r W$ be a form with the chart expression given by eqs. (2-5), (15), i.e.

$$(\pi^{r+1,r})^*\varrho = B_\sigma^J\,\omega_J^\sigma\wedge\omega_0+\sum_{k=2}^{n+1}p_k\varrho,\quad \text{summation over}\quad 0\leq|J|\leq r. \tag{19}$$

Let ξ be a $\pi-$vertical vector field such that $\operatorname{supp}\xi\subset\pi^{-1}(\Omega)$, and let $\xi=\xi^\sigma\frac{\partial}{\partial y^\sigma}$ and $\xi=\overline{\xi}^\sigma\frac{\partial}{\partial\overline{y}^\sigma}$ be its chart expressions in (V,ψ) and $(\overline{V},\overline{\psi})$, respectively. Let us define (for $s\geq r$, in general)

$$\eta_\Omega=\int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^*hi_{J^r\xi}\varrho, \tag{20}$$

where $\gamma\in\Gamma_\Omega(\pi)$ is a section of π. (The function η_Ω is independent of the choice of s, of course.) Using the decomposition (19) and the well-known chart expression of the $s-$th prolongation of ξ, $s\geq1$,

$$J^s\xi=\xi_J^\sigma\frac{\partial}{\partial y_J^\sigma},\quad \xi_J^\sigma=\xi_{j_1\dots j_k}^\sigma=\mathrm{d}_{j_k}\dots\mathrm{d}_{j_1}\xi^\sigma,\quad 0\leq|J|\leq s, \tag{21}$$

we obtain

$$\eta_\Omega=\int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^*hi_{J^r\xi}\varrho=\int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^*i_{J^{r+1}\xi}p_1\varrho=$$

$$\int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^*\left(B_\sigma^J\cdot\mathrm{D}_J\xi^\sigma\right)\omega_0=\int_\Omega(B_\sigma^J\cdot\mathrm{D}_J\xi^\sigma)(J^{r+1}\gamma)\,\omega_0, \tag{22}$$

where we have denoted by D_J the symbol $\mathrm{d}_{j_1} \dots \mathrm{d}_{j_k}$ for $J = (j_1 \dots j_k)$, $1 \le k \le r$. Due to the properties of total derivative, the operator D_J is symmetrical in all indices contained in multiindex J. Using recursively the relation

$$
\begin{aligned}
((f \mathrm{d}_j g) \circ J^{r+1}\gamma)\, \omega_0 &= \big((\mathrm{d}_j(fg) - g \mathrm{d}_j f) \circ J^{r+1}\gamma\big)\, \omega_0 = \\
&= \big(\mathrm{d}_i(fg) \circ J^{r+1}\gamma\big)\, \delta^i_j\, \omega_0 - \big((g \mathrm{d}_j f) \circ J^{r+1}\gamma\big)\, \omega_0 = \\
&= \big(\mathrm{d}_i(fg) \circ J^{r+1}\gamma\big)\, \mathrm{d}x^i \wedge \omega_j - ((g \mathrm{d}_j f) \circ J^{r+1}\gamma)\, \omega_0 = \\
&= J^{r+1}\gamma^* \left((\pi^{r+1,r})^* \mathrm{d}((fg) \wedge \omega_j) - (g \mathrm{d}_j f)\omega_0\right)
\end{aligned}
$$

for functions f, g, Stokes theorem and the assumption concerning the support of ξ we have finally

$$
\eta_\Omega = \int_\Omega \left(\xi^\sigma \cdot \sum_{l=1}^{r} (-1)^l \mathrm{d}_{j_1} \dots \mathrm{d}_{j_l} B^{j_1 \dots j_l}_\sigma \right) (J^{2r+1}\gamma)\, \omega_0. \tag{23}
$$

The same holds for the case of the representative obtained from the chart expression of ϱ in $(\overline{V}, \overline{\psi})$, i.e.

$$
\eta_\Omega = \int_\Omega \left(\overline{\xi}^\sigma \cdot \sum_{l=1}^{r} (-1)^l \overline{\mathrm{d}}_{j_1} \dots \overline{\mathrm{d}}_{j_l} \overline{B}^{j_1 \dots j_l}_\sigma \right) (J^{2r+1}\gamma)\, \overline{\omega}_0.
$$

Taking into account the transformation between components of the vector field ξ in charts (V, ψ) and $(\overline{V}, \overline{\psi})$ and the transformation relations for forms ω^σ and ω_0,

$$
\overline{\xi}^\sigma = \xi^\nu \frac{\partial \overline{y}^\sigma}{\partial y^\nu}, \quad \overline{\omega}^\sigma = \frac{\partial \overline{y}^\sigma}{\partial y^\nu} \omega^\nu, \quad \overline{\omega}_0 = \det \left(\frac{\partial \overline{x^j}}{\partial x^i} \right) \omega_0, \quad \overline{\mathrm{d}_j} f = \frac{\partial x^i}{\partial \overline{x}^j}\, \mathrm{d}_i f
$$

we can see that the expression (16) for the representative fulfills the transformation rules for a form. (The detailed transformation calculations for the 1−st order see in [2].)

Let finally $q = n+2$ and let $\varrho \in \Omega^r_{n+2} Y$ be a form, for which

$$
\pi^{r+1,r} \varrho = B^{JK}_{\sigma\nu}\, \omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_0 + \sum_{k=3}^{n+2} p_k \varrho, \tag{24}
$$

with coefficients $B^{JK}_{\sigma\nu}$ given by (2-5). Let ζ be another vector field which fulfills the same conditions as ξ. We define

$$
\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* h i_{J^r\xi} i_{J^r\zeta} \varrho. \tag{25}
$$

Then

$$
\begin{aligned}
\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* i_{J^{r+1}\xi} i_{J^{r+1}\zeta} p_2 \varrho &= \int_\Omega J^s\gamma^* (\pi^{s,r+1})^* \left(2\xi^\sigma_J \zeta^\nu_K\, B^{JK}_{\sigma\nu} \right) \omega_0 = \\
&= \int_\Omega (\mathrm{D}_J \xi^\sigma) \left(2B^{JK}_{\sigma\nu} \cdot \mathrm{D}_K \zeta^\nu \right) (J^{r+1}\gamma)\, \omega_0,
\end{aligned}
$$

with the operator D_J previously defined as $\mathrm{d}_{j_1}\dots\mathrm{d}_{j_k}$, $J=(j_1\dots j_k)$. Applying the procedure used for $q=n+1$ in the first part of the proof to the $n-$form $(2B^{JK}_{\sigma\nu}D_K\zeta^\nu\,\omega_0)$ we have

$$\eta_\Omega=\int_\Omega 2(-1)^{|J|}\left(\xi^\sigma \mathrm{D}_J(B^{JK}_{\sigma\nu}\cdot \mathrm{D}_K\zeta^\nu)\right)(J^{2r+1}\gamma)\,\omega_0,$$

summation over $|J|$, $|K|\leq r$. Calculating the expression $\mathrm{D}_J\left(B^{JK}_{\sigma\nu}\,\mathrm{D}_K\zeta^\nu\right)$ step by step we obtain

$$\eta_\Omega=\int_\Omega\left(2\xi^\sigma\sum_{|J|\leq r}\sum_{|K|\leq r}(-1)^{|J|}\sum_{|J_1|+|J_2|=|J|}\mathrm{D}_{K+J_1}\zeta^\nu\,\mathrm{D}_{J_2}B^{JK}_{\sigma\nu}\right)\circ(J^{2r+1}\gamma)\,\omega_0,$$

summation over $|J|$, $|K|\leq r$.

$$\eta_\Omega=$$

$$=\int_\Omega\left(2\xi^\sigma\sum_{j=1}^{r}\sum_{k=0}^{r}(-1)^j\sum_{p+l=j}\binom{j}{p}\mathrm{d}_{i_1}\dots\mathrm{d}_{i_{k+p}}\zeta^\nu\cdot\mathrm{d}_{i_{k+p+1}}\dots\mathrm{d}_{i_{k+j}}B^{i_{k+1}\dots i_{k+j},i_1\dots i_k}_{\sigma\nu}\right)(J^s\gamma)\omega_0,$$

$\mathrm{sym}(i_1,\dots i_{k+j})$. Rearranging the summations we finally get

(26) $$\eta_\Omega=$$

$$=2\int_\Omega\left(\sum_{j=0}^{2r}\sum_{k=0}^{r}\sum_{l=j-k}^{r}(-1)^{-l}\binom{l}{j-k}\mathrm{d}_{i_{j+1}}\dots\mathrm{d}_{i_{k+l}}\,B^{i_1\dots i_k,i_{k+1}\dots i_{k+l}}_{\sigma\nu}\cdot\xi^\sigma_{i_1\dots i_j}\zeta^\nu\right)(J^{2r+1}\gamma)\,\omega_0.$$

Now, the argumentation leading to the conclusion that the representative (18) is defined correctly (globally) is quite analogous to that presented in the first part of the proof.

It remains to discuss the following problem: Find the criteria for recognizing the representatives of classes of forms in the $r-$th order variational sequence and the reconstruction of classes from their representatives. This problem is solved for the physically relevant part of the variational sequence in mechanics (see [5] and [14]). For the field theory the calculations are technically difficult and are not finished up to now.

7. Examples

Finally, let us present two important examples.

Example 1. Let $W\subset Y$ be an open set. Let $\lambda\in\Omega^r_nW$ be a lagrangian given in a fibered chart (V,ψ), $V\subset W$, by the expression

$$\lambda=\mathcal{L}\,\omega_0,\quad \mathcal{L}\in\Omega^r_0V.$$

Using theorem 1(b) we obtain immediately

(27) $$\mathcal{E}_\lambda=\Phi^{2r,r}_{n+1}([\mathrm{d}\lambda])=\left(\sum_{l=0}^{r}(-1)^l\mathrm{d}_{j_1}\dots\mathrm{d}_{j_l}\frac{\partial\mathcal{L}}{\partial y^\sigma_{j_1\dots j_l}}\right)\omega^\sigma\wedge\omega_0$$

which is evidently the Euler-Lagrange form of the lagrangian λ.

More generally,let $\varrho \in \Omega^r_n W$ be a form and $[\varrho]$ its class represented by the horizontal form $\lambda_\varrho = \Phi^{r+1,r}_n([\varrho])$. λ_ϱ has the chart expression

$$\lambda_\varrho = h\varrho = \mathcal{L}_\varrho\, \omega_0, \quad \mathcal{L}_\varrho \in \Omega^{r+1}_0 V,$$

where $\mathcal{L}_\varrho$ is affine in variables y^σ_{r+1}. Using lemma 4 and theorem 1(b) we obtain immediately

$$\Phi^{2r+1,r}_{n+1}([\mathrm{d}\varrho]) = \Phi^{2r+1,r+1}_{n+1}([\mathrm{d}\lambda_\varrho]) = \mathcal{E}_{\lambda_\varrho},$$

where $\mathcal{E}_{\lambda_\varrho}$ is determined by the function $\mathcal{L}_\varrho$ following the equation (27) for $s = 2r+1$ instead of $2r$.

Example 2. Let $W \subset Y$ be an open set. Let $\mathcal{E} \in \Omega^r_{n+1} W$ be a dynamical form given in the fibered chart (V, ψ), $V \subset W$, by the expression

$$\mathcal{E} = \varepsilon_\sigma\, \omega^\sigma \wedge \omega_0, \quad \varepsilon_\sigma \in \Omega^r_0 V.$$

Then

$$\varrho = \mathrm{d}\mathcal{E} = \sum_{0 \le |J| \le r} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_J}\, \omega^\sigma_J \wedge \omega^\nu \wedge \omega_0.$$

On the other hand, in general, we have

$$p_2\varrho = B^{JK}_{\sigma\nu}\, \omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_0, \qquad B^{JK}_{\sigma\nu} + B^{KJ}_{\nu\sigma} = 0.$$

Thus,

$$B^{0J}_{\sigma\nu} = -B^{J0}_{\nu\sigma} = -\frac{1}{2}\frac{\partial \varepsilon_\sigma}{\partial y^\nu_J}, \qquad J = (j_1 \dots j_k),\ \ 1 \le k \le r,$$

$$B^{00}_{\sigma\nu} = -B^{00}_{\nu\sigma} = \left(\frac{\partial \varepsilon_\nu}{\partial y^\sigma}\right)_{\mathrm{alt}(\sigma\nu)},$$

other coefficients $B^{JK}_{\sigma\nu}$ being zero. Using theorem 1(c) we obtain

$$\mathcal{H}_\mathcal{E} = \Phi^{2r+1,r}_{n+1}([\mathrm{d}\mathcal{E}]) = \tag{28}$$

$$= \frac{1}{2}\left[\sum_{j=0}^{2r}\left(\frac{\varepsilon_\nu}{\partial y^\sigma_{i_1\dots i_j}} - (-1)^j \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{i_1\dots i_j}} - \right.\right.$$

$$\left.\left. - \sum_{l=j+1}^{r} (-1)^l \binom{l}{j} \mathrm{d}_{i_{j+1}} \dots \mathrm{d}_{i_l} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{i_1\dots i_l}}\right)\right] \omega^\sigma_{i_1\dots i_j} \wedge \omega^\nu \wedge \omega_0,$$

which is the Helmholtz-Sonin form of the dynamical form $\mathcal{E}$.

More generally, let $\varrho \in \Omega^r_{n+1} W$ be a form and $[\varrho]$ its class represented by the dynamical form

$$\mathcal{E}_\varrho = \Phi^{2r+1,r}_{n+1}([\varrho]) = (\varepsilon_\varrho)_\sigma \omega^\sigma \wedge \omega_0, \quad (\varepsilon_\varrho)_\sigma \in \Omega^{2r+1}_0 V,$$

given by (16). Using lemma 4 and theorem 1(c) we can obtain

$$\Phi^{s,r}_{n+2}([\mathrm{d}\varrho]) = \Phi^{s,2r+1}_{n+2}([\mathrm{d}\mathcal{E}_\varrho]) = \mathcal{H}_{\mathcal{E}_\varrho}, \quad s \ge 2r+1.$$

These results are in agreement with those of Krupka (see [6]) and Kašparová ([4] for the 1−st order variational sequence).

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