Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

PROPERTIES OF THE WEYL CONFORMAL CURVATURE OF KÄHLER-NORDEN MANIFOLDS

KARINA SŁUKA

Abstract. Let (M, J, g) be an $n = 2m$-dimensional Kählerian manifold endowed with a Norden metric. It is proved that (M, J, g) is conformally flat if and only if it is holomorphically projectively flat and its scalar curvature vanishes; the $*$-scalar curvature of such a manifold is constant and it is locally symmetric. If (M, J, g) is of recurrent conformal curvature, then it is locally symmetric in case of dimension $n \geqslant 6$, and it is locally symmetric or holomorphically projectively flat in case of dimension $n = 4$. Next, we show that the pseudosymmetry as well as the Weyl-pseudosymmetry and the holomorphically projective-pseudosymmetry of (M, J, g) reduces to the semisymmetry. Moreover, the Ricci-pseudosymmetry of (M, J, g) reduces to the Ricci-semisymmetry. An example of a Ricci-semisymmetric and non-semisymmetric Kähler-Norden structure is stated. Examples of semisymmetric, especially locally symmetric, Kähler-Norden structures are given in [15].

1. Preliminaries

By a Kählerian manifold with Norden metric (Kähler-Norden in short) [9] we mean a triple (M, J, g), where M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1,1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$J^2 = -I, \qquad g(JX, JY) = -g(X, Y), \qquad \nabla J = 0,$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M and ∇ is the Levi-Civita connection of g.

Let (M, J, g) be a Kähler-Norden manifold. Since in dimension 2 such a manifold is flat, we assume in the sequel that $\dim M \geq 4$. Let $R(X, Y)$ be the curvature operator $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and let R be the Riemann-Christoffel curvature tensor, $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. The Ricci tensor S is defined as $S(X, Y) = \operatorname{trace}\{Z \mapsto R(Z, X)Y\}$. These tensors have the following properties [1]

$$(1) \qquad \begin{aligned} &R(JX, JY) = -R(X, Y), && R(JX, Y) = R(X, JY), \\ &S(JY, Z) = \operatorname{trace}\{X \mapsto R(JX, Y)Z\}, && S(JX, Y) = S(JY, X). \end{aligned}$$

1991 *Mathematics Subject Classification.* 53C56, 53C50.

Key words and phrases. Kähler-Norden manifold, locally symmetric manifold, curvature conditions of pseudosymmetric type, Weyl conformal curvature.

Let $\widetilde{S}$ be the Ricci operator. Then we have $g(\widetilde{S}X, Y) = S(X, Y)$ and

$$\widetilde{S}Y = -\sum_i \varepsilon_i R(e_i, Y)e_i.$$

In the above and in the sequel, $(e_1, e_2, ..., e_n)$ is an orthonormal frame and ε_i are the indicators of e_i, $\varepsilon_i = g(e_i, e_i) = \pm 1$. The Weyl conformal curvature tensor C is defined in the usual way,

$$C(X,Y) = R(X,Y) + \frac{1}{n-2}\left(\frac{r}{n-1} X \wedge Y - \widetilde{S}X \wedge Y - X \wedge \widetilde{S}Y\right), \tag{2}$$

where r (= trace $\widetilde{S}$) is the scalar curvature and for any X,Y, the operator $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \quad Z \in \mathfrak{X}(M).$$

Using (2) and (1), we find the following useful equality

$$\sum_i \varepsilon_i C(Je_i, JY)e_i = \frac{n}{n-2}\,\widetilde{S}Y + \frac{r^*}{n-2}\,JY - \frac{r}{(n-1)(n-2)}\,Y, \tag{3}$$

where r^* is the $*$-scalar curvature, which is defined as the trace of $J\widetilde{S}$. In the above, we have applied the identity $\sum_i \varepsilon_i g(Je_i, e_i) = 0$, which is a consequence of the traceless of J.

The holomorphically projective curvature tensor P is defined in the following way ([19], [15])

$$P(X,Y) = R(X,Y) - \frac{1}{n-2}(X \wedge_S Y - JX \wedge_S JY), \tag{4}$$

the operator $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y, \quad Z \in \mathfrak{X}(M).$$

We notice, for later use, that this tensor has the following properties

$$\begin{aligned} &P(X, Y, Z, W) = -P(Y, X, Z, W), \quad P(JX, JY, Z, W) = -P(X, Y, Z, W),\\ &\sum_i \varepsilon_i P(e_i, Y, Z, Je_i) = 0, \quad \sum_i \varepsilon_i P(X, Y, e_i, e_i) = 0, \end{aligned} \tag{5}$$

A Kähler-Norden manifold (M, J, g) is holomorphically projectively flat if and only if its holomorphically projective curvature tensor P vanishes identically (ibidem).

2. The Weyl conformal curvature

Theorem 1. *A Kähler-Norden manifold (M, J, g) is conformally flat if and only if it is holomorphically projectively flat and its scalar curvature vanishes.*

Proof. Let us assume that $C = 0$. Then, using (3), we get

$$\widetilde{S}Y = -\frac{r^*}{n}\,JY + \frac{r}{n(n-1)}\,Y.$$

Hence, taking the trace, we find $r = 0$, and consequently

$$\widetilde{S}Y = -\frac{r^*}{n} JY. \tag{6}$$

Then, from (2) by $r = 0$ and (6), we obtain

$$R(X,Y) = -\frac{r^*}{n(n-2)}(X \wedge JY + JX \wedge Y). \tag{7}$$

Applying (6) and (7) into the right hand side of (4) yields $P = 0$.

Conversely, suppose $P = 0$ and $r = 0$. In this case, the Ricci operator and the curvature tensor are as in the formulas (6) and (7) (see [15]). By applying these formulas into (2), we find $C = 0$, which completes the proof. □

Corollary 1. *The curvature tensor and the Ricci tensor of a conformally flat Kähler-Norden manifold* (M, J, g) *have the shapes* (6) *and* (7)*, respectively. Moreover, the* $*$*-scalar curvature is constant and the manifold is locally symmetric.*

Proof. Firstly, using a well known identity and $r = 0$, we obtain

$$\sum_i \varepsilon_i (\nabla_{e_i} S)(Y, e_i) = \frac{1}{2} dr(Y) = 0. \tag{8}$$

On the other hand, with the help of (6), we find

$$\sum_i \varepsilon_i (\nabla_{e_i} S)(Y, e_i) = -\frac{1}{n} dr^*(JY) = 0.$$

Hence, r^* is constant. Therefore, the local symmetry follows from (7). □

Theorem 2. *Any Kähler-Norden manifold* (M, J, g) *with parallel Weyl conformal curvature tensor is locally symmetric.*

Proof. Assume that the Weyl conformal curvature tensor of a Kähler-Norden manifold (M, J, g) is parallel. Covariant differentiation of (3) and our assumption yield

$$0 = n(\nabla_W S)(Y, Z) + dr^*(W) g(JY, Z) - \frac{1}{n-1} dr(W) g(Y, Z). \tag{9}$$

Contracting formula (9) with respect to the pair of arguments Y, Z (that is, taking $Y = Z = e_i$ into (9), multiplying by ε_i and summing up over i), one finds

$$dr(W) = 0. \tag{10}$$

Using the famous identity and (10), we obtain

$$\sum_i \varepsilon_i (\nabla_{e_i} S)(Y, e_i) = \frac{1}{2} dr(Y) = 0. \tag{11}$$

Equality (10) applied into (9) gives

$$(\nabla_W S)(Y, Z) = -\frac{1}{n} dr^*(W) g(JY, Z).$$

Taking JY instead of Y into the last relation, and transvecting the resulting equality with respect to Y, Z, one gets $dr^* = 0$, which implies also $\nabla S = 0$.

In view of (2), the covariant derivative ∇C can be expressed in the following form

$$(\nabla_W C)(X,Y) = (\nabla_W R)(X,Y) + \frac{1}{n-2}\left(\frac{1}{n-1}dr(W)X \wedge Y - X \wedge ((\nabla_W \widetilde{S})Y) - ((\nabla_W \widetilde{S})X) \wedge Y\right).$$

To have the proof complete it is now sufficient to apply $\nabla \widetilde{S} = 0$ together with $\nabla C = 0$ and $dr = 0$ into the last expression. □

In the context of the above theorem, it should be said that the symmetric Kähler-Norden manifolds are not yet classified, even locally. Examples of locally symmetric Kähler-Norden structures are given in [15].

In the rest of this section, we study Kähler-Norden manifolds satisfying the condition

$$\nabla C = \varphi \otimes C, \tag{12}$$

where φ is a 1-form.

A pseudo-Riemannian manifold is said to be of recurrent conformal curvature (cf. e.g. [13], [14]) if its conformally curvature Weyl tensor C is non identically zero and satisfies the condition (12), φ is then called the recurrence form of C.

Proposition 1. *If a Kähler-Norden manifold* (M, J, g) *satisfies the condition* (12)*, then its holomorphically projective curvature tensor* P *realizes*

$$\nabla P = \varphi \otimes P. \tag{13}$$

Proof. From (2) and (12), we obtain

$$\begin{aligned}(\nabla_W R)(X,Y) = \varphi(W)R(X,Y) - \frac{1}{n-2}\Bigg(&\frac{1}{n-1}(dr(W) - \varphi(W)r)X \wedge Y \\ &- X \wedge (\nabla_W \widetilde{S})Y + \varphi(W)X \wedge \widetilde{S}Y - (\nabla_W \widetilde{S})X \wedge Y + \varphi(W)\widetilde{S}X \wedge Y\Bigg).\end{aligned} \tag{14}$$

Formula (3) together with (12) let us find the condition

$$\begin{aligned}&n((\nabla_W S)(Y,Z) - \varphi(W)S(Y,Z)) \\ =\ &-(dr^*(W) - \varphi(W)r^*)g(JY,Z) + \frac{1}{n-1}(dr(W) - \varphi(W)r)g(Y,Z).\end{aligned} \tag{15}$$

Contracting (15) with respect to Y, Z, one finds

$$dr(W) - \varphi(W)r = 0.$$

Therefore, we can rewrite (15) in the following way

$$(\nabla_W S)(Y,Z) = \varphi(W)S(Y,Z) - \frac{1}{n}(dr^*(W) - \varphi(W)r^*)g(JY,Z). \tag{16}$$

When we substitute (16) into (14), we obtain

$$(\nabla_W R)(X,Y) = \varphi(W)R(X,Y) + \eta(X \wedge (JY) + (JX) \wedge Y), \tag{17}$$

with

$$\eta = \frac{1}{n(n-2)}(dr^*(W) - \varphi(W)r^*).$$

Differentiating covariantly (4) and using (16), (17) enable us to check that the holomorphically projective curvature tensor P satisfies the condition (13). □

Theorem 3. *Let (M, J, g) be a Kähler-Norden manifold of recurrent conformal curvature.*

(1) If $\dim M \geqslant 6$, then the manifold (M, J, g) is locally symmetric.

(2) If $\dim M = 4$, then the manifold (M, J, g) is locally symmetric or holomorphically projectively flat.

Proof. In virtue of Proposition 1, the tensor P satisfies the condition (13). As it is known, under such a condition, P vanishes everywhere or nowhere on M [17, Theorem 3.8], [18].

If P does not vanish at every point of the manifold, then (M, J, g) is of recurrent holomorphically projective curvature, and by Theorem 4 of [15] it is locally symmetric. Let us assume that P is identically zero, that is, the manifold (M, J, g) is holomorphically projectively flat. If $\dim M \geqslant 6$, then by Proposition 3 of [15], the manifold is locally symmetric. □

Theorem 4. *Let (M, J, g) be a holomorphically projectively flat Kähler-Norden manifold of dimension 4. If its scalar curvature does not vanish everywhere on M, then (M, J, g) is of recurrent conformal curvature.*

Proof. Let us assume that $P = 0$. By Proposition 3 of [15], the curvature tensor and the Ricci tensor have the shapes

$$\begin{aligned} R(X,Y) &= \frac{r}{n(n-2)}(X \wedge Y - JX \wedge JY) - \frac{r^*}{n(n-2)}(X \wedge JY + JX \wedge Y), \\ S(X,Y) &= \frac{r}{n}g(X,Y) - \frac{r^*}{n}g(JX,Y). \end{aligned}$$

Substituting these relations into (2), we find

$$C(X,Y) = \frac{r}{n(n-1)(n-2)}(X \wedge Y - (n-1)JX \wedge JY).$$

Therefore, the covariant derivative ∇C can be expressed in the following form

$$(\nabla_W C)(X,Y) = \frac{1}{n(n-1)(n-2)} dr(X \wedge Y - (n-1)JX \wedge JY).$$

Now it is easy to see that the covariant derivative of the non-zero tensor C satisfies the condition

$$\nabla C = \varphi \otimes C$$

with $\varphi = d \ln |r|$. This means that the manifold is of recurrent conformal curvature with φ as the recurrence form. $\square$

Remark. In the paper [15], we have constructed examples of 4-dimensional Kähler-Norden manifolds which are holomorphically projectively flat with non-constant scalar curvature as well as locally symmetric.

3. Pseudosymmetry curvature conditions

For an $(0,k)$-tensor $(k \geqslant 1)$ field T on a pseudo-Riemannian manifold (M,g), we define the $(0,k+2)$-tensor field $R \cdot T$ by the condition

$$\begin{aligned}(R\cdot T)(U,V,X_1,X_2,...,X_k) &= (R(U,V)\cdot T)(X_1,X_2,...,X_k)\\ &= -T(R(U,V)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,X_2,\ldots,X_{k-1},R(U,V)X_k).\end{aligned} \tag{18}$$

A pseudo-Riemannian manifold (M,g) is said to be semisymmetric [16] if $R\cdot R = 0$, Ricci-semisymmetric if $R\cdot S = 0$, Weyl-semisymmetric if $R\cdot C = 0$. Clearly, any semisymmetric manifold is Ricci-semisymmetric as well as Weyl-semisymmetric, and the converse statements do not hold in general [6], [5].

To initiate the definition of pseudosymmetry, we define also an $(0,k+2)$-tensor field $Q(g,T)$ associated with any $(0,k)$-tensor $(k \geqslant 1)$ field T on a pseudo-Riemannian manifold

$$\begin{aligned}Q(g,T)(U,V,X_1,X_2,...,X_k) &= ((U\wedge V)\cdot T)(X_1,X_2,...,X_k)\\ &= -T((U\wedge V)X_1,X_2,...,X_k)) - \cdots - T(X_1,X_2,...,X_{k-1},(U\wedge V)X_k).\end{aligned} \tag{19}$$

A pseudo-Riemannian manifold (M,g) is called pseudosymmetric [6] if there exists a function $L_R : M \to \mathbb{R}$ such that

$$R\cdot R = L_R\, Q(g,R). \tag{20}$$

Clearly, every semisymmetric manifold is also pseudosymmetric. The converse is not true in general [6]. Examples of semisymmetric Kähler-Norden metrics are found in [15]. In the class of Kähler-Norden metrics, pseudosymmetry reduces to semisymmetry. Indeed, we prove the following:

Theorem 5. *Every pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. Assume that a Kähler-Norden manifold (M,J,g) satisfies the condition (20). Using (1) and (18), we claim that

$$-(R\cdot R)(U,V,X,Y,Z,W) = (R\cdot R)(JU,JV,X,Y,Z,W).$$

Thus, by (20), we have

$$-L_R\, Q(g,R)(U,V,X,Y,Z,W) = L_R\, Q(g,R)(JU,JV,X,Y,Z,W).$$

Suppose that L_R is non-zero at a certain point $p \in M$. Then the above equality gives at the point p

$$-Q(g,R)(U,V,X,Y,Z,W) = Q(g,R)(JU,JV,X,Y,Z,W).$$

Contracting the last identity with respect to V, X, we obtain

$$-\sum_i \varepsilon_i Q(g,R)(U,e_i,e_i,Y,Z,W) = \sum_i \varepsilon_i Q(g,R)(JU,Je_i,e_i,Y,Z,W),$$

which, with the help of (19), can be rewritten in the following form

$$\begin{aligned} & nR(U,Y,Z,W) - R(U,Y,Z,W) + R(Y,U,Z,W) + R(Z,Y,U,W) \\ & + R(W,Y,Z,U) + g(U,Z)S(Y,W) - g(U,W)S(Y,Z) \\ = & -R(U,Y,Z,W) - R(JY,JU,Z,W) - R(JZ,Y,JU,W) - R(JW,Y,Z,JU) \\ & - g(JU,Z)\sum_i \varepsilon_i R(e_i,Y,W,Je_i) + g(JU,W)\sum_i \varepsilon_i R(e_i,Y,Z,Je_i). \end{aligned}$$

Hence, using (1) and the first Bianchi identity, we get

$$\begin{aligned} nR(U,Y,Z,W) \;=\; & g(JU,W)S(JY,Z) - g(JU,Z)S(JY,W) \\ & + g(U,W)S(Y,Z) - g(U,Z)S(Y,W). \end{aligned} \tag{21}$$

Contracting (21) with respect to Y, Z, we find

$$S(U,W) = \frac{1}{n}(rg(U,W) + r^*(JU,W)).$$

This leads to $R \cdot S = 0$. Using this fact and (21), we obtain $R \cdot R = 0$, that means the semisymmetry of our manifold. □

A pseudo-Riemannian manifold (M, g) is called Ricci-pseudosymmetric [6] if there exists a function $L_S : M \to \mathbb{R}$ such that

$$R \cdot S = L_S\, Q(g,S). \tag{22}$$

Clearly, every Ricci-semisymmetric manifold is also Ricci-pseudosymmetric. The converse is not true in general [6]. However, we shall prove that the Ricci-pseudosymmetry reduces to the Ricci-semisymmetry in the class of Kähler-Norden metrics.

Theorem 6. *Every Ricci-pseudosymmetric Kähler-Norden manifold is Ricci-semisymmetric.*

Proof. Let us assume then the equality (22) holds on M. Now, in the same way as in the proof of Theorem 5, we have

$$-L_S\, Q(g,S)(U,V,X,Y) = L_S\, Q(g,S)(JU,JV,X,Y). \tag{23}$$

Suppose that the function L_S is non-zero at a certain point of M. Therefore, (23) takes the form

$$-Q(g,S)(U,V,X,Y) = Q(g,S)(JU,JV,X,Y). \tag{24}$$

Contracting the last identity with respect to V, X, we get

$$-\sum_i \varepsilon_i Q(g,S)(U,e_i,e_i,Y) = \sum_i \varepsilon_i Q(g,S)(JU,Je_i,e_i,Y),$$

which in virtue of (19) and (1) can be written as

$$S(X,Y)=\frac{1}{n}(rg(X,Y)+r^{*}g(JX,Y)).$$

This gives immediately $R\cdot S=0$, which completes the proof. □

It is obvious that a pseudosymmetric manifold is Ricci-pseudosymmetric. And as it is already known (see [6]), the converse does not hold in the class of pseudo-Riemannian manifolds. The below example shows that the converse is not true even in the class of Kähler-Norden manifolds.

Example. Let $(x^1,...,x^{4m},x^{4m+1}=u,x^{4m+2}=v)$ be the Cartesian coordinates in the Cartesian space $\mathbb{R}^n$, with $n=4m+2\geqslant 10$. Let

$$K_1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\qquad K_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Define a Norden metric g on $\mathbb{R}^n$ by

$$[g_{ij}]=\begin{bmatrix}FI_m & I_m & -GI_m & 0 & 0\\ I_m & 0 & 0 & 0 & 0\\ -GI_m & 0 & -FI_m & -I_m & 0\\ 0 & 0 & -I_m & 0 & 0\\ 0 & 0 & 0 & 0 & K_1\end{bmatrix},$$

where I_m is the identity matrix of rank m and functions F, G are defined on $\mathbb{R}^n$, depend on u, v only and satisfy the Cauchy-Riemann equations

$$\frac{\partial F}{\partial u}=\frac{\partial G}{\partial v},\qquad \frac{\partial F}{\partial v}=-\frac{\partial G}{\partial u}.$$

Moreover, define a complex structure J on $\mathbb{R}^n$ by

$$[J^i_j]=\begin{bmatrix}0 & 0 & -I_m & 0 & 0\\ 0 & 0 & 0 & -I_m & 0\\ I_m & 0 & 0 & 0 & 0\\ 0 & I_m & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & K_2\end{bmatrix}.$$

One checks that (J,g) is a Kähler-Norden structure on $\mathbb{R}^n$. Moreover, it can be verified that the condition $R\cdot S=0$ is fulfilled, which means our manifold is Ricci-semisymmetric. On the other hand, $R\cdot R$ is non-zero, which means that the manifold is not semisymmetric. □

A pseudo-Riemannian manifold (M,g) is called Weyl-pseudosymmetric [6] if there exists a function $L_C : M\to\mathbb{R}$ such that

$$R\cdot C=L_C\,Q(g,C). \tag{25}$$

Clearly, every pseudosymmetric as well as Weyl-semisymmetric manifold is Weyl-pseudosymmetric. In dimensions $\geqslant 5$, for a pseudo-Riemannian manifold, it is proved in [8, Theorem 1] that the condition (25) reduces to (20) with $L_R = L_C$ on the set of points at which the tensor C does not vanish. This is not true in dimension 4 [7]. Below, we prove that a stronger result holds good in the class of Kähler-Norden metrics. Namely, we prove the following:

Theorem 7. *Every Weyl-pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. Let (M, J, g) be a Kähler-Norden manifold which is Weyl-pseudosymmetric. At first, we prove that the Weyl-pseudosymmetric reduces to the Weyl-semisymmetry.

To do it let suppose that $R \cdot C$ is non-zero at a certain point $p \in M$. By the Weyl-pseudosymmetry, in the same manner as in the proof of Theorem 5, we obtain

$$L_C\, Q(g,C)(JU,V,X,Y,Z,W) = L_C\, Q(g,C)(U,JV,X,Y,Z,W).$$

Since L_C is non-zero at p, it must be satisfied at p

$$Q(g,C)(JU,V,X,Y,Z,W) = Q(g,C)(U,JV,X,Y,Z,W).$$

Contracting the last identity with respect to V, X, and next using (19) and the first Bianchi identity for C, we obtain

$$\begin{aligned} nC(JU,Y,Z,W) - C(JY,U,Z,W) + C(JZ,Y,W,U) - C(JW,Y,Z,U) \\ = g(U,Z)\sum_i \varepsilon_i C(e_i,Y,W,Je_i) - g(U,W)\sum_i \varepsilon_i C(e_i,Y,Z,Je_i). \end{aligned} \tag{26}$$

Contracting the above relation with respect to U, Z, we find

$$\sum_i \varepsilon_i C(e_i,Y,W,Je_i) = 0,$$

which turns (26) into

$$nC(JU,Y,Z,W) - C(JY,U,Z,W) + C(JZ,Y,W,U) - C(JW,Y,Z,U) = 0. \tag{27}$$

We write this equation three times, with the vector fields U, Z, W cyclically permuted. Summing all three equations gives

$$C(JU,Y,Z,W) + C(JZ,Y,W,U) + C(JW,Y,U,Z) = 0,$$

which used in (27) leads to

$$(n-1)C(JU,Y,Z,W) - C(JY,U,Z,W) = 0. \tag{28}$$

If we substitute JU and JY instead of U and Y, respectively, into (28) and compare the obtained relation with (28), we deduce that $C = 0$ at point p. Consequently, $R \cdot C = 0$ at p. This is a contradiction.

Thus, our Kähler-Norden manifold (M, J, g) is Weyl-semisymmetric, that is, satisfies the condition $R \cdot C = 0$. Using this and (3) one can easily see that $R \cdot S = 0$. Consequently, we have $R \cdot R = 0$ on M. □

A pseudo-Riemannian manifold (M, g) will be called holomorphically projective-pseudosymmetric if there exists a function $L_P : M \mapsto \mathbb{R}$ such that

$$R \cdot P = L_P \, Q(g, P). \tag{29}$$

Theorem 8. *Every holomorphically projective-pseudosymmetric Kähler-Norden manifold is semisymmetric.*

Proof. At the first part of the proof, we show that the holomorphically projective-pseudosymmetry implies the holomorphically projective semisymmetry.

To do it, let us assume that (M, J, g) is a holomorphically projective-pseudosymmetric Kähler-Norden manifold and $R \cdot P$ is non-zero at a certain point $p \in M$. Using the holomorphically projective-pseudosymmetry, in the same way as in the proof of Theorem 5, we find

$$-L_P \, Q(g, P)(U, V, X, Y, Z, W) = L_P \, Q(g, P)(JU, JV, X, Y, Z, W).$$

Since L_P is non-zero at p, the following formula

$$-Q(g, P)(U, V, X, Y, Z, W) = Q(g, P)(JU, JV, X, Y, Z, W)$$

must be satisfied at p. Contracting the last identity with respect to V, X, we have

$$\begin{aligned}
&nP(U, Y, Z, W) - P(U, Y, Z, W) + P(Y, U, Z, W) + P(Z, Y, U, W) \\
&+ P(W, Y, Z, U) - g(U, Z) \sum_i \varepsilon_i P(e_i, Y, e_i, W) \\
&= - P(U, Y, Z, W) - P(JY, JU, Z, W) - P(JZ, Y, JU, W) - P(JW, Y, Z, JU) \\
&+ g(U, JZ) \sum_i \varepsilon_i P(e_i, Y, Je_i, W) + g(U, JW) \sum_i \varepsilon_i P(e_i, Y, Z, Je_i).
\end{aligned}$$

Hence, using (5), we obtain

$$\begin{aligned}
&nP(U, Y, Z, W) + P(Z, Y, U, W) + P(W, Y, Z, U) \\
&+ P(JZ, Y, JU, W) + P(JW, Y, Z, JU) \\
&= g(U, JZ) \sum_i \varepsilon_i P(e_i, Y, Je_i, W) + g(U, Z) \sum_i \varepsilon_i P(e_i, Y, e_i, W).
\end{aligned} \tag{30}$$

Contracting the above equality with respect to Z, U, we find

$$\sum_i \varepsilon_i P(JW, Y, e_i, Je_i) = 0. \tag{31}$$

Taking JU instead of U in (30), contracting the resulting equation with respect to Z, U, and making use of (31), we deduce

$$\sum_i \varepsilon_i P(e_i, Y, Je_i, W) = 0. \tag{32}$$

Moreover, contracting (30) with respect to Y, Z and using

$$\sum_i \varepsilon_i P(JW, e_i, e_i, JU) = \sum_i \varepsilon_i P(W, e_i, e_i, U),$$

we obtain

$$n\sum_i \varepsilon_i P(U,e_i,e_i,W) = -\sum_i \varepsilon_i P(W,e_i,e_i,U),$$

consequently,

$$\sum_i \varepsilon_i P(U,e_i,e_i,W) = 0. \tag{33}$$

Now, (30) with the help of (32) and (33) can be rewritten in the following form

$$\begin{aligned} nP(U,Y,Z,W) &= P(Y,Z,U,W) + P(Y,W,Z,U) \\ &\quad -P(JZ,Y,JU,W) - P(JW,Y,Z,JU). \end{aligned} \tag{34}$$

When we put JU, JY instead of U, Y into (34), respectively, and compare the obtained formula with (34), we conclude that $P = 0$ at point p. This gives $R\cdot P = 0$, which is a contradiction.

Thus, the manifold (M, J, g) is holomorphically projective semisymmetric. To have the proof complete it is sufficient to use Theorem 2 of [15], which states that holomorphically projective semisymmetry always implies the semisymmetry. □

Final remarks. The conditions of the semisymmetry and pseudosymmetry type for the Riemann, Ricci and Weyl curvature tensors of Kählerian and para-Kählerian manifolds were studied in the papers [2] - [4], [10] - [11], [12], and others. Some of them have inspired the author in her investigations.

Acknowledgment. The author is greatly indebted to Prof. Zbigniew Olszak for his help during the preparation of this paper.

References

[1] A. Borowiec, M. Francaviglia and I. Volovich, *Anti-Kählerian manifolds*, Diff. Geom. Appl. **12** (2000), 281-289.

[2] F. Defever, R. Deszcz and L. Verstraelen, *On semisymmetric para-Kähler manifolds*, Acta Math. Hungarica **74** (1997), 7-17.

[3] F. Defever, R. Deszcz and L. Verstraelen, *On pseudosymmetric para-Kähler manifolds*, Colloq. Math. **74** (1997), 253-260.

[4] J. Deprez, R. Deszcz and L. Verstraelen, *Pseudosymmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kählerian manifolds*, Ann. Fac. Sci. Toulouse **9** (1988), 183-192.

[5] A. Derdziński, *Examples de métriques de Kaehler et d'Einstein autoduales sur le plan complexe*, In: Géométrie riemannienne en dimension 4, Séminaire Arthur Besse 1978/79, Cedic/Fernand Nathan, Paris, 1981, 334-346.

[6] R. Deszcz, *On pseodsymmetric spaces*, Bull. Soc. Math. Belg. Sér. A **44** (1992), 1-34.

[7] R. Deszcz, *Examples of four-dimensional Riemannian manifolds satisfying some pseudosymmetry coditions*, Geometry and Topology of Submanifolds, **2**, (1990), 134-143, World Sci., Singapore.

[8] R. Deszcz and W. Grycak, *On manifolds satisfying some curvature condition*, Colloq. Math. **57** (1989), 89-92.

[9] G.T. Ganchev and A.V. Borisov, *Note on the almost complex manifolds with Norden metric*, Compt. Rend. Acad. Bulg. Sci. **39** (1986), 31-34.

[10] M. Hotloś, *On a certain class of Kählerian manifolds*, Demonstr. Math. **12**, (1979), 935-945.

[11] M. Hotloś, *On holomorphically pseudosymmetric Kählerian manifolds*, In: Geometry and Topology of Submanifolds, VII, pp. 139-142, World Sci. Publishing, River Edge, N.J., 1995.

[12] Z. Olszak, *Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor*, Simon Stevin **63** (1989), 295-303.

[13] W. Roter, *On the existence of conformally recurrent Ricci-recurrent spaces*, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astr. Phys., **24** (1976), 973–979.

[14] W. Roter, *On the existence of certain conformally recurrent metrics*, Colloq. Math. **51** (1987), 315–327.

[15] K. Słuka, *On Kähler manifolds with Norden metrics*, An. Ştiint. Univ. "Al.I.Cuza" Iaşi, Ser. Ia Mat., in print.

[16] Z. I. Szabó, *Structure theorems on Riemannian manifolds satisfying* $R(X,Y)R = 0$, I, *Local version*; II, *Global versions*, J. Diff. Geom. **17** (1982), 531-582; Geom. Dedicata **19** (1985), 65-108.

[17] Y. C. Wong, *Recurrent tensors on a linearly connected differentiable manifold*, Trans. Amer. Math. Soc. **99**, 1961, 325-334.

[18] Y. C. Wong, *Linear connexions with zero torsion and recurrent curvature*, Trans. Amer. Math. Soc. **102**, 1962, 471-506.

[19] K. Yano, *Differential geometry of complex and almost complex spaces*, Pergamon Press, New York, 1965.

Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego, 27, 50–370 Wrocław, Poland

E-mail address: `sluka@im.pwr.wroc.pl`