SPECTRA OF SUBMERSIONS *

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This talk presents a review of the results concerning the spectra of submersions obtained during the time by the author and others. Let $(M, g), (B, j)$ be two connected compact Riemannian manifolds without boundary and let $\pi : M \longrightarrow B$ be a submersion. We investigate the relations between the spectra of the Laplace-Beltrami operators acting on functions defined respectively on M, on B and on the fibers $F_x = \pi^{-1}(x),\ x \in B$. The problem is completely solved, via representation theory, when the submersion is Riemannian and the fibers are totally geodesic submanifolds of M. When the submersion is Riemannian and the fibers are minimal submanifolds of M, another technique gives comparisons between the spectrum of Δ_M and the one of Δ_B. Some recent results concern Riemannian submersions with fibers of basic mean curvature vector field and almost-Riemannian submersions. In particular, the first non-zero eigenvalue of Δ_M has a lower bound depending on the geometry of B and on the volume of the fibers.

1. Introduction

1.1. *Submersions*

The concept of submersion is, in a certain sense, the converse of the concept of immersion and it was introduced by B. O'Neill [13] in 1966. Submersions were studied by several mathematicians, among them I would remember A. Gray.

Let $(M, g), (B, j)$ be two compact boundaryless Riemannian manifolds. A surjective C^∞ mapping $\pi : M \longrightarrow B$ is a *submersion* if its differential $(d\pi)_y : T_yM \longrightarrow T_{\pi(y)}B$ is a surjective mapping of maximal rank $n = \dim B$ at any point $y \in M$. The fibers $F_x = \pi^{-1}(x),\ x \in B$, are regular p-dimensional submanifolds of M ($p = m - n$, with m=dim$M > n$), and they

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are diffeomorphic to a model fiber F. A vector $X_y \in T_yM$ is *vertical* if it is tangent at y to the fiber $F_{\pi^{-1}(\pi(y))}$. The subspace $\mathcal{V}_y \subset T_yM$ of vertical vectors is the *vertical space*; the *horizontal space* $\mathcal{H}_y$ is the orthogonal complement of $\mathcal{V}_y$ in T_yM with respect to the metric g:

$$\mathcal{H}_y = \mathcal{V}_y^{\perp}\,, \qquad T_yM = \mathcal{V}_y \oplus \mathcal{H}_y\,, \qquad g(\mathcal{V}_y, \mathcal{H}_y) = 0. \tag{1}$$

The space $\mathcal{H}_y$ is naturally isomorphic to $T_{\pi(y)}B$. The submersion $\pi : (M,g) \longrightarrow (B,j)$ is *Riemannian* if the restriction of its differential to horizontal vectors, $(d\pi)_y\,|_{\mathcal{H}_y}\colon \mathcal{H}_y \longrightarrow T_{\pi(y)}B$, is an isometry:

$$j\left((d\pi)_y(X), (d\pi)_y(X)\right) = g(X,X) \tag{2}$$

for any horizontal vector $X \in \mathcal{H}_y$ and any $y \in M$.

The simplest (and trivial) example is $M = B \times F$, the product of two Riemannian manifolds $(B,j), (F,g_F)$ endowed with the product metric, and where $\pi = \pi_1$, the projection on the first factor.

1.2. *Spectra*

The Laplace-Beltrami operator $\Delta(M,g)$, briefly Δ_M, acting on functions $f \in C^\infty(M)$ is defined by

$$\Delta_M f = \delta df = -\mathrm{div}\ \mathrm{grad} f \tag{3}$$

and it takes an important part in vibration theory (the wave equation is $\frac{\partial^2 f}{\partial t^2} + \Delta_M f = h$), in heat diffusion (the heat equation is $\frac{\partial f}{\partial t} + \Delta_M f = h$), and in other problems. The classical methods of analysis lead to solve the eigenvalues and eigenfunctions problem

$$\Delta_M f = \lambda f \tag{4}$$

with Dirichlet or Neumann boundary conditions when the manifold has a nonempty boundary.

The Spectral Geometry studies the relations between the geometry of the Riemannian manifold (M,g) and the spectrum of Δ_M, consisting in a infinite increasing unbounded sequence of eigenvalues, each of them with finite multiplicity. When the manifold is not compact, the spectrum of Δ_M contains a bounded discrete part and a continuous part. The explicit computation of the spectrum is in general very hard or impossible, but it is often useful to obtain estimates of the eigenvalues. To do this, an idea is to reduce the spectral problem from a big manifold to a smaller and simpler one (the best should be a 1-dimensional manifold), and to compare the spectrum of the big manifold with the spectrum of the small. Then, for a submersion $\pi : (M, g) \longrightarrow (B, j)$ we are interested to establish the links between the spectra of the Laplacians of the total space M, of the basis B and of the fibers F_x.

Example 1.1 (trivial) In the case of the product $M = B \times F$, $\pi = \pi_1$ projection on the first factor, it is well known that the eigenvalues of Δ_M are the sums of the ones of Δ_B and Δ_F, and that the eigenfunctions are the product of eigenfunctions of Δ_B with the ones of Δ_F.

In section 2, we consider a Riemannian submersion with totally geodesic fibers: in this case, all the fibers are isometric to a model fiber (F, g_F). The spectral problem was then completely solved, via representation theory, by G. Besson and me (see [4], 1990). Namely, we gave an explicit method to compute the eigenvalues with multiplicities and the eigenfunctions of Δ_M by the ones of Δ_F and the ones of the horizontal Laplacian (Theorems 2.1, 2.2).

When the submersion is Riemannian with minimal fibers, the technique used for totally geodesic fibers does not work because in the case in point the fibers are only diffeomorphic, and not isometric, to a model fiber. However, we can apply a general theorem to produce comparisons between the spectrum of Δ_M and the spectrum of Δ_B (section 3, Theorems 3.2, 3.3; see [5], 1994, and [6], 2000). These estimates are not completely satisfactory because in them does not appear the contribution of the fibers.

Surprisingly, better results hold under the weaker assumption that the fibers have basic mean curvature vector field. In this case we get spectral estimates for the eigenvalues of Δ_M, in particular a lower bound for the first non-zero eigenvalue, depending on the geometry of B and on the volume of the fibers (section 4, Theorems 4.2, 4.5; see [7], 2005).

At last, section 5 presents how to compare the spectrum of the total space of an almost Riemannian submersion with the spectrum of the total space of the naturally associated Riemannian submersion (Proposition 5.1; see [7]).

2. Riemannian submersions with totally geodesic fibers

The simplest case after products is a Riemannian submersion whose fibers are all *totally geodesic submanifolds* of M, what corresponds to the fact that the second fundamental form of the fibers vanishes:

$$II(X, Y) = (\nabla^M_X Y - \nabla^{F_x}_X Y)_y = 0 \tag{5}$$

for any vertical X, Y, for any $y \in F_x$ and for any $x \in B$; here ∇^M and ∇^{F_x} are the Levi-Civita connection of (M, g) and (F_x, g_x) resp. (g_x denotes the metric induced on F_x by g). In this situation, all the fibers (F_x, g_x), $x \in B$, are *isometric* to a model fiber (F, g_F) endowed with a reference metric g_F, see R. Hermann [11].

In 1982, L. Bérard Bergery and J. P. Bourguignon introduced [1] a decomposition of the Laplacian:

$$\Delta_M = \Delta_v + \Delta_h \tag{6}$$

as a sum of two operators, vertical and horizontal, corresponding to the natural decomposition of the metric g. As these two operators commute in the case in point, the eigenvalues of Δ_M are sums of eigenvalues of each of them, but not all the sums are admissible, the choice depending on the global geometry of the submersion.

The problem to determine which eigenvalues are in fact involved in the above sums was completely solved by G. Besson and myself in 1990, [4], via representation theory; we construct also an explicit Hilbertian basis of eigenfunctions of Δ_M.

The general theory says that for a submersion the fibration π is associated to the principal bundle $p : P \longrightarrow B$ of structural group G, the (compact) Lie group of the isometries of the model fiber (F, g_F), see Kobayashi-Nomizu [12], p. 54. The diagram of the fibration is

$$\begin{array}{ccc} P \times F & \xrightarrow{\ diag\ } & M = P \times_G F \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\ p\ ]{} & B \end{array}$$

where pr_1 is the projection on the first factor and $diag$ is the diagonal action of the group G on the product $P \times F$: then M is the quotient of $P \times F$ by $diag$, quotient denoted $P \times_G F$.

The idea was to search the eigenfunctions of the Laplacian $\Delta_{P \times F}$ which pass to the quotient. To do this, denote $\mathcal{R}$ the set of unitary irreducible complex representations of G: they are finite dimensional, since G is compact. As G acts by isometries on P and on F, we have the decompositions

$$L^2(P) = \oplus_{\rho \in \mathcal{R}} L^2_\rho(P)\,, \qquad L^2(F) = \oplus_{\rho \in \mathcal{R}} L^2_\rho(F).$$

For a fixed d-dimensional representation $\rho \in \mathcal{R}$, a *canonical pair* is a couple $V \subset L^2_\rho(P)$, $W \subset L^2_\rho(F)$ of irreducible non trivial G-invariant subspaces. Then by Schur's lemma there exists a unique, up to a complex number of modulus one, equivariant isometry between V and W. Now if $\Psi = (\psi_1, \ldots, \psi_d)$ is an orthonormal basis of V and $\Phi = (\varphi_1, \ldots, \varphi_d)$ is the orthonormal basis of W corresponding to Ψ, the scalar product

$$\langle \Psi, \Phi \rangle := \frac{1}{d} \sum_{i=1}^{d} \psi_i \cdot \overline{\varphi}_i \tag{7}$$

is a function on $P \times F$ invariant under the diagonal action of G, so it pass to the quotient M, and it does not depend on the choice of Ψ.

Theorem 2.1 *The family of functions $\langle \Psi, \Phi \rangle$ forms a Hilbertian basis of the space of the L^2-functions on $P \times F$ invariant under the action of G, i.e. of the space $L^2(M)$. This basis consists of eigenfunctions of Δ_M.*

As G acts by isometries, V and W are in fact included in some eigenspaces of Δ_P and of Δ_F resp., related to certain eigenvalues $\alpha+\beta \in \mathrm{Spec}\Delta_P$ with $\alpha \in \mathrm{Spec}\Delta_P^h$, $\beta \in \mathrm{Spec}\Delta_P^v$ and $\mu \in \mathrm{Spec}\Delta_F$ respectively.

Theorem 2.2 *The eigenvalue of Δ_M corresponding to the eigenfunction $\langle \Psi, \Phi \rangle$ is $\alpha + \mu$. The multiplicity of an eigenvalue $\nu \in \mathrm{Spec}\Delta_M$ is given by*

$$\mathit{mult}(\nu) = \sum_{\rho \in \mathcal{R}} \sum_{\alpha+\mu=\nu} m_\alpha(\rho) \cdot m_\mu(\overline{\rho})$$

where $m_\alpha(\rho)$ is the multiplicity of the representation ρ in the eigenspace of the horizontal Laplacian Δ_P^h related to the eigenvalue α and $m_\mu(\overline{\rho})$ is the multiplicity of the representation $\overline{\rho}$ conjugate to ρ.

Notice that not all the sums $\alpha + \mu$ are eigenvalues of Δ_M: if α corresponds to a representation ρ, μ *must* correspond to the conjugate representation $\overline{\rho}$. Notice also that the vertical Laplacian Δ_P^v does not take any part in computing the eigenvalues and that $\mathrm{Spec}\Delta_P^h$ is substantially $\mathrm{Spec}\Delta_B$.

3. Riemannian submersions with minimal fibers

The second step is to consider Riemannian submersions whose fibers F_x, $x \in B$, are *minimal submanifolds* of M. This is equivalent to the fact that the *mean curvature vector field* H of the fibers is identically equal to zero; recall that H is the trace of the vectorial secund fundamental form of the fibers, (5). In this case, all the fibers are only diffeomorphic to a model fiber F, but no more isometric.

Lemma 3.1 *If the fibers of the submersion are all minimal submanifolds, they have same volume:*

$$V(x) = Vol(F_x) = \int_{F_x} dv_{g_x} = constant$$

does not depend on $x \in B$.

It is always possible to split Δ_M in vertical and horizontal Laplacians, (6), but this technique does not produce results in this case. In order to obtain spectral estimates, we can apply a *general theorem of spectral comparison*, see [5]. Let T, T' be two self-adjoint semibounded operators acting respectively on Hilbert spaces H and H', with associated quadratic forms Q_T and $Q_{T'}$. We say that they satisfy *Kato's property* with respect to a mapping $\varpi : H' \longrightarrow H$ if and only if ϖ maps the domain of T' into the domain of T and

$$Q_T(\varpi f) \leq Q_{T'}(f)\,, \qquad \forall\ f \in \mathcal{D}(T') \tag{8}$$

(ϖ does not increase energy). The general theorem gives a comparison between SpecT' and Spec$T$ in the general frame of measure spaces. Here we consider directly a Riemannian submersion $\pi : M \longrightarrow B$ and the operators Δ_M, Δ_B acting on the spaces $L^2(M), L^2(B)$ respectively. As the fibers are minimal submanifolds of M, they obey Kato's inequality with respect to the mapping ϖ defined by

$$(\varpi f)(x) := \left(\int_{F_x} f(y)^2\, dv_{g_x}(y) \right)^{\frac{1}{2}}, \qquad f \in L^2(M) \text{ and } x \in B. \tag{9}$$

Theorem 3.2 *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with minimal fibers. Then, for any integer $N > 0$, one has*

(1) $\lambda_N(\Delta_M) \geq \dfrac{1}{8(p+1)^2} \lambda_{k+1}(\Delta_B)$;

(2) $\displaystyle\sum_{i=1}^{N} \lambda_i(\Delta_M) \geq \frac{1}{2} \sum_{j=1}^{k} \lambda_j(\Delta_B) + \frac{k}{8(p+1)} \lambda_{k+1}(\Delta_B)$

where p is the rank of the subspace spanned by the N first eigenfunction of Δ_M and where k is the integer part of $\frac{N}{p+1}$.

Another approach consists in showing that ϖ is a *symmetrization* in the sense of G. Besson [2]. As ϖ obeys Kato's inequality when the fibers are minimal, a generalized Beurling-Deny principle give the following domination theorem (see [6]):

Theorem 3.3 *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with minimal fibers. Then the resolvent operator $(\Delta_B + \lambda)^{-1}$ and the heat operator $e^{-t\Delta_B}$ dominate $(\Delta_M + \lambda)^{-1}$ and $e^{-t\Delta_M}$ respectively, for any positive t and λ.*

Recall that T' dominates T if $T(\varpi f) \geq \varpi(T'f)\,, \quad \forall\, f \in \mathcal{D}(T')$.

4. Riemannian submersions with fibers of basic mean curvature

A horizontal vector field $X \in \Gamma(TM)$ is *basic* if it is projectable by π, i.e. if its image by the differential $(d\pi)_y$ is the same vector $\overline{X}_x \in T_xB$ for all points y in the fiber $F_x = \pi^{-1}(x),\ x \in B$. In other words, X is basic if and only if it is the lift of a vector field $\overline{X} \in \Gamma(TB)$. The inner product of two basic vector fields X, Y is constant along any fiber F_x:

$$\langle X_y, Y_y\rangle = \langle (d\pi)_y(X), (d\pi)_y(Y)\rangle = \langle \overline{X}_x, \overline{Y}_x\rangle \quad \forall\ y \in F_x \tag{10}$$

(from now on, we shall write briefly $\langle\ ,\ \rangle$ to denote inner products). For any fixed $x \in B$, let us denote by g_x the restriction of the metric g to the fiber F_x, by v_{g_x} the induced canonical measure on F_x, and by $V(x)$ the corresponding volume of F_x, $V(x) = \int_{F_x} dv_{g_x}$. When X is the (basic) lift of $\overline{X} \in \Gamma(TB)$, then for any function $f \in C^\infty(M)$ one has (cf. G. Besson [3]):

$$\overline{X}\left(\int_{F_x} f(y) dv_{g_x}(y)\right) \int_{F_x} (Xf)_y dv_{g_x}(y) - \int_{F_x} f(y)\langle H_y, X_y\rangle dv_{g_x}(y)$$

where H_y is the *mean curvature vector* at $y \in F_x$ of the fiber F_x, i.e. the trace of the vectorial second fundamental form of F_x at y.

We shall assume in the sequel that H is basic, and denote $\overline{H}$ its projection.

Lemma 4.1 *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. Then the measure $\frac{v_{g_x}}{V(x)}$, $x \in B$, is invariant by the holonomy of the fibration.*

Let us define $\mathcal{E}_c, \mathcal{E}_0$ to be the subspaces of the Sobolev space $H^1(M)$ consisting of the functions $f \in H^1(M)$ which are constant on the fibers, respectively of zero average on the fibers:

$$\begin{aligned} \mathcal{E}_c &= \{f : M \longrightarrow \mathbb{R},\ f \in H^1(M)|\ f = u \circ \pi \,\text{with}\ u : B \longrightarrow \mathbb{R}\}, \\ \mathcal{E}_0 &= \{h : M \longrightarrow \mathbb{R},\ h \in H^1(M)|\ \int_{F_x} h(y) dv_{g_x}(y) = 0\}. \end{aligned} \tag{11}$$

Theorem 4.2 *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. Then:*

(1) the space $H^1(M)$ splits into the direct sum $H^1(M) = \mathcal{E}_c \oplus \mathcal{E}_0$;

(2) the decomposition in (1) is simultaneously L^2-orthogonal and q-orthogonal, where q is the quadratic form

$$q(f) := \int_M |\nabla f|_g^2 dv_g$$

(∇f denotes the gradient of f);

(3) the spaces $\mathcal{E}_c$ and $\mathcal{E}_0$ are stable under the action of the Laplace-Beltrami operator Δ_M,

$$\Delta_c := \Delta_M|_{\mathcal{E}_c} : \mathcal{E}_c \longrightarrow \mathcal{E}_c \quad \textit{and} \quad \Delta_0 := \Delta_M|_{\mathcal{E}_0} : \mathcal{E}_0 \longrightarrow \mathcal{E}_0,$$

hence

$$Spec(\Delta_M) = Spec(\Delta_c) \cup Spec(\Delta_0);$$

(4) the heat operator on M splits: $\exp^{-t\Delta_M} = \exp^{-t\Delta_c} \oplus \exp^{-t\Delta_0}$ (in the sense that, if $f = u \circ \pi + h \in \mathcal{E}_c \oplus \mathcal{E}_0$, then

$$\exp^{-t\Delta_M}(f) = \exp^{-t\Delta_c}(u \circ \pi) + \exp^{-t\Delta_0}(h),$$

and thus

$$Trace\left(\exp^{-t\Delta_M}\right) = Trace\left(\exp^{-t\Delta_c}\right) + Trace\left(\exp^{-t\Delta_0}\right).$$

For any fixed $x \in B$, denote $\lambda_1(F_x) = \lambda_1(F_x, g_x)$ the first non-zero eigenvalue of the Laplace-Beltrami operator Δ_{F_x} of the fiber F_x, and define

$$\Lambda := \inf_{x \in B} \lambda_1(F_x). \tag{12}$$

Corollary 4.3 *Any eigenvalue $\lambda_j \in Spec(\Delta_M)$ belonging to $Spec(\Delta_0)$ satisfies*

$$\lambda_j \geq \Lambda.$$

It follows that if λ_j is an eigenvalue of Δ_M such that $\lambda_j < \Lambda$, then $\lambda_j \in$Spec(Δ_c). Thus, in order to estimate "small" eigenvalues of Δ_M, it suffices to estimate the eigenvalues of Δ_c, where "small" means less than a lower bound of Λ.

In some cases, one can find such a lower bound. For instance, assume that the fibers F_x are diffeomorphic to a fiber-type F, endowed with a reference metric g_F, and that the metric on F_x is $g_x = (b(x))^2 g_F$ (typical example: manifolds of revolution). Then the min-max principle gives

$$\lambda_i(F_x, g_x) = \frac{1}{(b(x))^2}\lambda_i(F, g_F),$$

in particular $\lambda_1(F_x, g_x) = \frac{1}{(b(x))^2}\lambda_1(F, g_F) \geq \frac{1}{\sup_{x\in B}(b(x))^2}\lambda_1(F, g_F)$.

Example 4.1 Assume that F_x is a closed curve, i.e. F_x diffeomorphic to S^1, and that the metric on F_x is $g_x = (b(x))^2 d\theta^2$, where $d\theta^2$ is the canonical metric of S^1. As the length of F_x is $\ell(x) = \ell(F_x) = \int_0^{2\pi} b(x)d\theta = 2\pi b(x)$, one has $b(x) = \frac{\ell(x)}{2\pi}$ and thus

$$\lambda_1(F_x, g_x) = \frac{4\pi^2}{(\ell(x))^2} \geq \frac{4\pi^2}{\sup_{x\in B}(\ell(x))^2}$$

(recall that $\lambda_1(S^1, d\theta^2) = 1$).

Another situation in which it is possible to find a lower bound of Λ is when the fibers F_x have Ricci curvature bounded from below by $-(p-1)k^2$, where p is the dimension of the fibers, and diameter bounded from above by D: in this case one get

$$\lambda_1(F_x) \geq \Gamma(p, k, D)$$

where Γ is an explicit constant, see P. Li and S.T. Yau [10] and S. Gallot [8].

Proposition 4.4 *The mapping $\mathcal{E}_c \longrightarrow H^1(B)$ which maps the function $f = u \circ \pi$ onto the function u is bijective and maps:*

- *the L^2-norm $\|f\|^2_{L^2(M)} = \int_M f^2(y)dv_g(y)$ onto the quadratic form*

$$\|u\|_0^2 = \int_B V(x)u(x)^2 dv_j(x);$$

- *the quadratic form $q(f) = \int_M |\nabla f|_y^2 dv_g(y)$ onto the quadratic form*

$$q_0(u) = \int_B V(x)|\nabla u|_x^2 dv_j(x).$$

Let us call $R(u)$ the Rayleigh quotient of a function u with respect to the canonical L^2-norm and $R_0(u)$ the Rayleigh quotient of u of the quadratic form q_0 with respect to the L_0^2-norm:

$$R_0(u) = \frac{q_0(u)}{\|u\|_0^2} = \frac{\int_B V(x)|\nabla u|_x^2 dv_j(x)}{\int_B V(x)u(x)^2 dv_j(x)}.$$

Assume that the volume $V(x) = Vol(F_x)$ of the fibers is bounded when x ranges over B: $0 < V_0 \leq V(x) \leq V_1 < +\infty$. As $q(f) = q_0(u) \geq V_0 q(u)$ and $\|f\|^2_{L^2(M)} = \|u\|^2_0 \leq V_1 \|u\|^2_{L^2(B)}$, one has

$$R(f) = R_0(u) \geq \frac{V_0}{V_1} R(u) \quad \text{and} \quad R(f) = R_0(u) \leq \frac{V_1}{V_0} R(u).$$

Therefore, $\lambda_i(\Delta_c)$ is equal to the i-th eigenvalue of the diagonalization of q_0 with respect to the L^2_0-norm. Moreover, the min-max and max-min principle give:

$$\frac{V_0}{V_1} \lambda_i(B) \leq \lambda_i(\Delta_c) \leq \frac{V_1}{V_0} \lambda_i(B). \tag{13}$$

In a similar way, from $R(f) = R_0(u) \geq \frac{q_0(u)}{V_1 \|u\|^2_{L^2(B)}}$, it follows

$$\lambda_i(\Delta_c) \geq \frac{1}{V_1} \lambda_i(q_0) \tag{14}$$

where now q_0 is diagonalized with respect to the canonical L^2-norm and no more with respect to the L^2_0-norm. Sobolev and Hölder inequalities give

Theorem 4.5 *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. The first non-zero eigenvalue of Δ_M verifies:*

$$\lambda_1(\Delta_M) \geq \frac{\Gamma^{-1}}{V_1} \left(\int_B V(x)^{-\frac{n}{2}} dv_j(x) \right)^{-\frac{2}{n}}$$

where $\Gamma = \Gamma(n, k, D, V)$ is a positive constant depending on the geometry of B: n is the dimension, k is a lower bound of the Ricci curvature, $Ric \geq -(n-1)k$, D is the diameter, and V is the volume; $V(x)$ is the volume of the fiber F_x, and V_1 is its lower bound.

Notice that this estimate is optimal by its dependance on the power of $\frac{1}{V}$, cf. S. Gallot and D. Meyer [9].

Remark 4.1 When all the fibers of the submersions are *minimal submanifolds* of M, which means $H \equiv 0$, Lemma 3.1 shows that they have the same volume: $V(x) = constant$. Then the inequality of Theorem 4.5 takes the simplified form:

$$\lambda_1(\Delta_M) \geq C \tag{15}$$

where the constant $C = C(n, k, D, V)$ is equal to $\Gamma^{-1} V^{-\frac{2}{n}}$.

5. Approximating an almost Riemannian submersion by a Riemannian one.

Let $(M, g'), (B, j)$ be two compact boundaryless Riemannian manifolds and let $\pi : M \longrightarrow B$ be a submersion. We shall say that $\pi : (M, g') \longrightarrow (B, j)$ is an *almost-Riemannian submersion* when the restriction of g' to horizontal vectors is an almost isometry, i.e. there exist two real constants a and b (not depending on the point $y \in M$) such that

$$a^2 j\left((d\pi)_y(X), (d\pi)_y(X)\right) \leq g'(X, X) \leq b^2 j\left((d\pi)_y(X), (d\pi)_y(X)\right) \quad (16)$$

for any horizontal vector $X \in \mathcal{H}_y$.

Define on M the Riemannian metric g at any $y \in M$ by $g_y|_{\mathcal{V}_y} = g'_y|_{\mathcal{V}_y}$, by $g_y(X, V) = 0$ for any horizontal X and vertical V, and by $g_y(X, Y) = (\pi^* j_x)(X, Y) = j_x\left((d\pi)_y(X), (d\pi)_y(Y)\right)$ for any horizontal X and Y. In substance, the metric g preserves the orthogonality between vertical and horizontal vectors, it reduces to the old metric g' for vertical vectors, and it is the metric of the basis B for horizontal vectors. Then $\pi : (M, g) \longrightarrow (B, j)$ is a Riemannian submersion and

$$a^2 g(X, X) \leq g'(X, X) \leq b^2 g(X, X) \quad (17)$$

for any horizontal X.

Let us denote by $\{\lambda_i(M, g)\}_{i=0,1,2,\ldots}$ the spectrum of the Laplace-Beltrami operator $\Delta_{(M,g)}$ (each eigenvalue is repeated according to its multiplicity). The min-max and max-min principle give:

Proposition 5.1 *The eigenvalues of $\Delta_{(M,g)}$ and of $\Delta_{(M,g')}$ satisfy, for any $i = 0, 1, 2, \ldots$:*

$$\frac{b^m}{a^{m+2}} \lambda_i(M, g) \geq \lambda_i(M, g') \geq \frac{a^m}{b^{m+2}} \lambda_i(M, g)$$

where m =dim(M).

Proposition 5.1 implies for the traces of the heat kernels the following inequality:

$$Z_{(M,g')}(t) \leq Z_{(M,g)}\left(\frac{a^m}{b^{m+2}} t\right) \quad (18)$$

for any positive t, where $Z_{(M,g')}(t) = \sum_{i=0}^{+\infty} \exp -\lambda_i(M, g')t$.

We should also consider the case when π is an almost-Riemannian submersion only outside a subset A of zero capacity in B. It is not hard to see that cap(A)=0 implies cap($\pi^{-1}(A)$)=0. *A priori* the capacity of a subset

A' in M depends on the metric, but when g and g' are almost isometric according to (17) the capacity of A' with respect to g is equal to zero if and only if the capacity of A' with respect to g' is equal to zero. Then, we could settle and prove the analogous of Proposition 5.1 and of (18) for the Neumann problem on $M \setminus \pi^{-1}(A)$.

References

1. L.B. Bergery and J.P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois Jour. of Math., **26, n. 2** (1982), 181-200.
2. G. Besson, *On symmetrization*, Contemporary Mathematics, **51** (1986), 9-21.
3. G. Besson, *A Kato type inequality for Riemannian submersions with totally geodesic fibers*, Ann. Glob. Analysis and Geometry, **4 n. 3** (1986), 273-289.
4. G. Besson and M. Bordoni, *On the spectrum of Riemannian submersions with totally geodesic fibers.* Rend. Mat. Accad. Naz. Lincei, **Ser. IX 1** (1990), 335-340.
5. M. Bordoni, *Spectral estimates for Schrödinger and Dirac-type operators on Riemannian Manifolds.* Math. Ann. **298** (1994), 693-718.
6. M. Bordoni, *Comparing heat operators through local isometries or fibrations.* Bull. Soc. Math. France **128** (2000), 151-178.
7. M. Bordoni, *Spectral estimates for submersions with fibers of basic mean curvature.* Preprint Me.Mo.Mat. (2005).
8. S. Gallot, *Inégalités isopérimétriques et analytiques sur les variétés riemanniennes*, Astérisque **163-164** (1988), 33-91.
9. S. Gallot and D. Meyer, *D'un résultat hilbertien à un principe de comparaison entre spectre. Applications*, Ann. Sci. Éc. Norm. Supér., **IV Sér. 21** (1988), 561-591.
10. P. Li and S.T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Sympos. Pure Math. **36** (1980), 205-239.
11. R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle.* Proc. Amer. Math. Soc., **vol. II** (1960), 236-242.
12. S. Kobayashi and K. Nomizu, *Foundations of differental Geometry.* Vol. I, Wiley Interscience, New York - London 1963.
13. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459-469.