SOME ASPECTS OF NONCOMMUTATIVITY ON REAL, p-ADIC AND ADELIC SPACES *

BRANKO DRAGOVICH †

Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

E-mail: dragovich@phy.bg.ac.yu

ZORAN RAKIĆ

Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia

E-mail: zrakic@matf.bg.ac.yu

Classical and quantum mechanics for an extended Heisenberg algebra with canonical commutation relations for position and momentum coordinates are considered. In this approach additional noncommutativity is removed from the algebra by linear transformation of phase space coordinates and transmitted to the Hamiltonian (Lagrangian). This transformation does not change the quadratic form of Hamiltonian (Lagrangian) and Feynman's path integral maintains its well-known exact expression for quadratic systems. The compact matrix formalism is presented and can be easily employed in particular cases. Some p-adic and adelic aspects of noncommutativity are also considered.

1. Introduction

Standard n-dimensional quantum mechanics (QM) is based on the Heisenberg algebra

$$[\hat{x}_a, \hat{p}_b] = i\,\hbar\,\delta_{ab}, \quad [\hat{x}_a, \hat{x}_b] = 0, \quad [\hat{p}_a, \hat{p}_b] = 0, \quad a, b = 1, 2, \cdots, n, \qquad (1)$$

for Hermitian operators of position $\hat{x}_a$ and momentum $\hat{p}_a$ coordinates in the Hilbert space. In the recent years there has been an intensive research

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of noncommutative quantum theory with an algebra

$$[\hat{x}_a, \hat{p}_b] = i\,\hbar\,\delta_{ab}, \quad [\hat{x}_a, \hat{x}_b] = i\,\hbar\,\theta_{ab}, \quad [\hat{p}_a, \hat{p}_b] = 0, \quad a, b = 1, 2, \cdots, n, \tag{2}$$

where θ_{ab} are constant elements of a real antisymmetric ($\theta_{ab} = -\theta_{ab}$) $n \times n$-matrix Θ. The initial considerations of noncommutativity (NC) (2) go back to the 1930's (see, e.g. [1]) but the real excitement began in the 1998 when spatial NC of the form $[\hat{x}_a, \hat{x}_b] = i\,\hbar\,\theta_{ab}$ was observed in the low energy string theory with D-branes in a constant background B-field (see reviews [2], [3] and references therein).

The NC (2) leads to extended uncertainty, i.e.

$$\Delta x_a\,\Delta p_b \geq \frac{\hbar}{2}\,\delta_{ab}, \quad \Delta x_a\,\Delta x_b \geq \frac{\hbar}{2}\,|\theta_{ab}|, \tag{3}$$

which prevents from simultaneous accurate measuring not only x_a and p_a but also spatial coordinates x_a and x_b $\quad (a \neq b)$. To simplify exploration one often takes $\theta_{ab} = \theta\,\varepsilon_{ab}$, where $(\varepsilon_{ab}) = \mathcal{E}$ is the unit $n \times n$ antisymmetric matrix with $\varepsilon_{ab} = +1$ if $a < b$. Due to the uncertainty $\Delta x_a\,\Delta x_b \geq \frac{\hbar}{2}\,|\theta|, \quad (a \neq b)$, a spatial point is not a well defined concept and the space becomes fuzzy at distances of the order $\sqrt{\hbar\,|\theta|}$, which may be much larger than the Planck or string length.

The most attention in this subject has been paid to noncommutative field theory (for reviews, see e.g. [2] and [3]). Noncommutative quantum mechanics (NCQM) has been also actively explored. Namely, NCQM can be regarded as the corresponding one-particle nonrelativistic sector of noncommutative quantum field theory. It also provides study of NC on simple models and their potential experimental verification.

Models of NCQM have been mainly investigated using the Schrödinger equation. The path integral method has attracted less attention, however for systems with quadratic Lagrangians a systematic investigation started recently (see [4]-[8] and references therein).

We consider here n-dimensional NCQM which is based on the following algebra

$$[\hat{x_a}, \hat{p_b}] = i\,\hbar\,(\delta_{ab} - \frac{1}{4}\,\theta_{ac}\,\sigma_{cb})\,, \quad [\hat{x_a}, \hat{x_b}] = i\,\hbar\,\theta_{ab}\,, \quad [\hat{p_a}, \hat{p_b}] = i\,\hbar\,\sigma_{ab}\,, \tag{4}$$

where $(\theta_{ab}) = \Theta$ and $(\sigma_{ab}) = \Sigma$ are the antisymmetric matrices with constant elements. This kind of an extended noncommutativity maintains a symmetry between canonical variables and yields (2) in the limit $\sigma_{ab} \longrightarrow 0$. The algebra (4) allows simple reduction to the usual commutation relations

$$[\hat{q_a}, \hat{k_b}] = i\,\hbar\,\delta_{ab}, \qquad [\hat{q_a}, \hat{q_b}] = 0, \qquad [\hat{k_a}, \hat{k_b}] = 0, \tag{5}$$

using the following phase space linear transformation:

$$\hat{x_a} = \hat{q_a} - \frac{\theta_{ab}\,\hat{k_b}}{2}\,, \qquad \hat{p_a} = \hat{k_a} + \frac{\sigma_{ab}\,\hat{q_b}}{2}\,, \tag{6}$$

where summation over repeated indices is understood. It was shown recently [9] that NC (4) is suitable to study possible dynamical control of decoherence by applying perpendicular magnetic field to a charged particle in the plane. This property also gives possibility to observe NC.

In this paper we present compact general formalism of NCQM for quadratic Lagrangians (Hamiltonians) with the (4) form of the NC. The formalism developed here is suitable for both Schrödinger and Feynman approaches to quantum evolution. There are now many papers on some concrete models in NCQM. However, to our best knowledge, there is no article on the evaluation of general quadratic Lagrangians (Hamiltonians). Especially, Feynman's path integral method to the NC has been almost ignored. Note that quadratic Lagrangians contain an important class of physical models, and that some of them are rather simple and exactly solvable (a free particle, a particle in a constant field, a harmonic oscillator). The obtained relations between coefficients in commutative and noncommutative regimes give possibility to easily construct effective Hamiltonians and Lagrangians in the particular noncommutative cases.

Sec. 2 is devoted to noncommutativity on real space and contains: matrix formalism, some expressions for quadratic Lagrangians and Hamiltonians as well as relations between them, Schrödinger equation and Feynman path integral. Some p-adic and adelic aspects of noncommutativity are presented in Sec. 3. In the last section we give a few concluding remarks.

2. Noncommutativity on real phase space

Matrix formalism. Let us introduce the following (matrix) formalism. If we put $\hat{x}^T = (\hat{x}_1, \hat{x}_2, \cdots, \cdots, \hat{x}_n)$ and $\hat{p}^T = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n)$ (superscript ^{T} denotes transposition), then all commutators between $\hat{x}_a$ and $\hat{p}_b$ we can rewrite in the following way:

$$\text{for} \quad \hat{P} = \begin{bmatrix} \hat{p} \\ \hat{x} \end{bmatrix} \quad \text{we put} \quad [\hat{P}, \hat{P}] = \begin{pmatrix} [\hat{p},\hat{p}] & [\hat{p},\hat{x}] \\ [\hat{x},\hat{p}] & [\hat{x},\hat{x}] \end{pmatrix}, \tag{7}$$

where $[\hat{p},\hat{p}]$, $[\hat{p},\hat{x}]$, $[\hat{x},\hat{p}]$, $[\hat{x},\hat{x}]$ are $n\times n$ matrices with entries given by $\big([\hat{x},\hat{p}]\big)_{ab} = [\hat{x}_a,\hat{p}_b]$, and so on. Let us mention that $[\hat{p},\hat{x}] = -\big([\hat{x},\hat{p}]\big)^T$.
We want to rewrite in this spirit the change of coordinates. Namely, if we have another coordinates of the same type, $\hat{q}^T = (\hat{q}_1,\hat{q}_2,\cdots,\hat{q}_n)$ and $\hat{k}^T = (\hat{k}_1,\hat{k}_2,\cdots,\hat{k}_n)$, and if a linear connection is defined by

$$\hat{P} = \hat{\mathcal{A}}\,\hat{K}\,, \quad \text{i.e.} \quad \begin{bmatrix} \hat{p} \\ \hat{x} \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} \hat{k} \\ \hat{q} \end{bmatrix}, \tag{8}$$

we want to find dependance of $[\hat{P},\hat{P}]$ on $[\hat{K},\hat{K}]$. To do this, let us prove the following useful statement.

Lemma 2.1 *Let A be an arbitrary $n\times n$ matrix whose entries commute with the coordinates of $\hat{q}^T = (\hat{q}_1,\hat{q}_2,\cdots,\hat{q}_n)$ and $\hat{k}^T = (\hat{k}_1,\hat{k}_2,\cdots,\hat{k}_n)$. Then the following commutation relations hold*

(i1) $[A\hat{e},\hat{r}] = A\,[\hat{e},\hat{r}]\,,$ (i2) $[\hat{e},A\hat{r}] = [\hat{e},\hat{r}]\;A^T\,,$

where $\hat{e},\hat{r}\in\{\hat{q},\hat{k}\}$.

Proof. Let us prove (i2) for $\hat{e}=\hat{q}$ and $\hat{r}=\hat{k}$. We have,

$$\Big(\Big[\hat{q},A\hat{k}\Big]\Big)_{ab} = \Big[\hat{q}_a,A\hat{k}_b\Big] = \left[\hat{q}_a,\sum_{c=1}^{n} A_{bc}\hat{k}_c\right] = \sum_{c=1}^{n} A_{bc}\Big[\hat{q}_a,\hat{k}_c\Big]$$
$$= \sum_{c=1}^{n}\Big[\hat{q},\hat{k}\Big]_{ac} A^T_{cb} = \Big(\Big[\hat{q},\hat{k}\Big]\,A^T\Big)_{ab}\,.$$

The proof of all other relations is similar. ♢

Eqs. (6) and (5) can be rewritten in the compact form as

$$\hat{P} = \Xi\;\hat{K}, \qquad \Xi = \begin{pmatrix} I & \frac{1}{2}\,\Sigma \\ -\frac{1}{2}\,\Theta & I \end{pmatrix}, \qquad \hat{K} = \begin{pmatrix} \hat{k} \\ \hat{q} \end{pmatrix}, \tag{9}$$

$$[\hat{K},\hat{K}] = i\,\hbar \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{10}$$

and using Lemma 2.1, relations (5), and skew-symmetricity of Σ and Θ, we have

$$[\hat{P},\hat{P}] = [\Xi\;\hat{K},\Xi\;\hat{K}] = \begin{pmatrix} [\hat{k}+\frac{\Sigma}{2}\,\hat{q},\hat{k}+\frac{\Sigma}{2}\,\hat{q}] & [\hat{k}+\frac{\Sigma}{2}\,\hat{q},\hat{q}-\frac{\Theta}{2}\,\hat{k}] \\ [\hat{q}-\frac{\Theta}{2}\,\hat{k},\hat{k}+\frac{\Sigma}{2}\,\hat{q}] & [\hat{q}-\frac{\Theta}{2}\,\hat{k},\hat{q}-\frac{\Theta}{2}\,\hat{k}] \end{pmatrix}$$
$$= i\,\hbar \begin{pmatrix} \Sigma & \frac{1}{4}\,\Sigma\,\Theta - I \\ I - \frac{1}{4}\,\Theta\,\Sigma & \Theta \end{pmatrix}, \tag{11}$$

since

$$
\begin{aligned}
[\hat{k}+\frac{\Sigma}{2}\,\hat{q},\hat{k}+\frac{\Sigma}{2}\,\hat{q}] &= [\hat{k},\hat{k}]+[\frac{\Sigma}{2}\,\hat{q},\hat{k}]+[\hat{k},\frac{\Sigma}{2}\,\hat{q}]+[\frac{\Sigma}{2}\,\hat{q},\frac{\Sigma}{2}\,\hat{q}] \\
&= \frac{\Sigma}{2}[\hat{q},\hat{k}]+[\hat{k},\hat{q}]\frac{\Sigma^T}{2} = \frac{\Sigma}{2}[\hat{q},\hat{k}]+[\hat{q},\hat{k}]\frac{\Sigma}{2} = i\,\hbar\,\Sigma, \\
[\hat{q}-\frac{\Theta}{2}\,\hat{k},\hat{k}+\frac{\Sigma}{2}\,\hat{q}] &= [\hat{q},\hat{k}]-[\frac{\Theta}{2}\,\hat{k},\hat{k}]+[\hat{q},\frac{\Sigma}{2}\,\hat{q}]-[\frac{\Theta}{2}\,\hat{k},\frac{\Sigma}{2}\,\hat{q}] \\
&= [\hat{q},\hat{k}]-\frac{\Theta}{2}[\hat{k},\hat{q}]\frac{\Sigma^T}{2} = i\,\hbar\,\big(I-\frac{1}{4}\,\Theta\,\Sigma\big), \\
[\hat{q}-\frac{\Theta}{2}\,\hat{k},\hat{q}-\frac{\Theta}{2}\,\hat{k}] &= [\hat{q},\hat{q}]-[\frac{\Theta}{2}\,\hat{k},\hat{q}]-[\hat{q},\frac{\Theta}{2}\,\hat{k}]+[\frac{\Theta}{2}\,\hat{k},\frac{\Theta}{2}\,\hat{k}] \\
&= -\frac{\Theta}{2}[\hat{k},\hat{q}]-[\hat{q},\hat{k}]\frac{\Theta^T}{2} = \frac{\Theta}{2}[\hat{q},\hat{k}]+[\hat{q},\hat{k}]\frac{\Theta}{2} = i\,\hbar\,\Theta\,.
\end{aligned}
$$

Let us note that (11) contains formulas (4) rewritten in the above matrix formalism.

Quadratic Lagrangians and Hamiltonians. We start with general quadratic Lagrangian for an n-dimensional system with position coordinates, $x^T=(x_1,x_2,\cdots,x_n)$, which has the form:

$$
L(X,t)=\frac{1}{2}\,X^T\,M\,X+N^T\,X+\phi, \tag{12}
$$

where $2n\times 2n$ matrix M and $2n$-dimensional vectors X, N are defined as

$$
M=\begin{pmatrix}\alpha & \beta\\ \beta^T & \gamma\end{pmatrix}, \qquad X^T=(\dot{x}^T\,,\,x^T)\,, \qquad N^T=(\delta^T\,,\,\eta^T). \tag{13}
$$

where the coefficients of the $n\times n$ matrices $\alpha=((1+\delta_{ab})\,\alpha_{ab}(t))$, $\beta=(\beta_{ab}(t))$, $\gamma=((1+\delta_{ab})\,\gamma_{ab}(t))$, n-dimensional vectors $\delta=(\delta_a(t))$, $\eta=(\eta_a(t))$ and a scalar $\phi=\phi(t)$ are some analytic functions of the time t. Matrices α and γ are symmetric, α is nonsingular ($\det\alpha\neq 0$).

Using $p_a=\frac{\partial L}{\partial \dot{x}_a}$ one finds $\dot{x}=\alpha^{-1}\,(p-\beta\,x-\delta)$. Since the function $\dot{x}$ is linear in p and x, then by the Legendre transformation $H(p,x,t)=p^T\,\dot{x}-L(\dot{x},x,t)$ classical Hamiltonian is also quadratic, i.e.

$$
H(P,t)=\frac{1}{2}\,P^T\mathcal{M}\,P+\mathcal{N}^T\,P+F, \tag{14}
$$

where matrix $\mathcal{M}$ and vectors P, $\mathcal{N}$ are

$$
\mathcal{M}=\begin{pmatrix}A & B\\ B^T & C\end{pmatrix}, \qquad P^T=(p^T\,,\,x^T)\,, \qquad \mathcal{N}^T=(D^T\,,\,E^T)\,, \tag{15}
$$

and

$$\begin{aligned} A &= \alpha^{-1}, & B &= -\,\alpha^{-1}\,\beta, & C &= \beta^T\,\alpha^{-1}\,\beta - \gamma, \\ D &= -\,\alpha^{-1}\,\delta, & E &= \beta^T\,\alpha^{-1}\,\delta - \eta, & F &= \frac{1}{2}\,\delta^T\,\alpha^{-1}\delta\, - \phi\,. \end{aligned} \tag{16}$$

From the symmetry of matrices α and γ follows that the matrices $A = ((1+\delta_{ab})\,A_{ab}(t))$ and $C = ((1+\delta_{ab})\,C_{ab}(t))$ are also symmetric ($A^T = A,\ C^T = C$). The nonsingular ($\det\alpha \neq 0$) Lagrangian $L(\dot{x}, x, t)$ implies nonsingular ($\det A \neq 0$) Hamiltonian $H(p, x, t)$. Note that the inverse Legendre transformation, i.e. $H \longrightarrow L$, is given by the same relations (16).

One can show that

$$\mathcal{M} = \sum_{i=1}^{3} \Upsilon_i^T(M)\,M\,\Upsilon_i(M), \tag{17}$$

where

$$\begin{aligned} \Upsilon_1(M) &= \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & -I \end{pmatrix}, & \Upsilon_2(M) &= \begin{pmatrix} 0 & \alpha^{-1}\beta \\ 0 & 0 \end{pmatrix}, \\ \Upsilon_3(M) &= \begin{pmatrix} 0 & 0 \\ 0 & i\sqrt{2}\,I \end{pmatrix}, \end{aligned} \tag{18}$$

and I is $n \times n$ unit matrix. One has also $\mathcal{N} = Y(M)\,N$, where

$$Y(M) = \begin{pmatrix} -\,\alpha^{-1} & 0 \\ \beta^T\,\alpha^{-1} & -I \end{pmatrix} = -\Upsilon_1(M) + \Upsilon_2^T(M) + i\,\sqrt{2}\;\Upsilon_3(M)\,, \tag{19}$$

and $F = N^T\,Z(M)\,N - \phi$, where

$$Z(M) = \begin{pmatrix} \frac{1}{2}\,\alpha^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\,\Upsilon_1(M) - \frac{i}{2\sqrt{2}}\,\Upsilon_3(M)\,. \tag{20}$$

Using auxiliary matrices $\Upsilon_1(M), \Upsilon_2(M)$ and $\Upsilon_3(M)$ in the above way, Hamiltonian quantities $\mathcal{M}, \mathcal{N}$ and F are connected to the corresponding Lagrangian ones M, N and ϕ.

Since Hamiltonians depend on canonical variables, the transformation (9) leads to the transformation of Hamiltonian (14). By quantization the Hamiltonian (14) easily becomes

$$H(\hat{P}, t) = \frac{1}{2}\;\hat{P}^T\mathcal{M}\;\hat{P} + \mathcal{N}^T\;\hat{P} + F, \tag{21}$$

because (14) is already written in the Weyl symmetric form.

Performing linear transformation (9) in (21) we again obtain quadratic quantum Hamiltonian

$$\hat{H}_{\theta\sigma}(\hat{K},t) = \frac{1}{2}\,\hat{K}^T\,\mathcal{M}_{\theta\sigma}\,\hat{K} + \mathcal{N}_{\theta\sigma}^T\,\hat{K} + F_{\theta\sigma}, \tag{22}$$

where $2n \times 2n$ matrix $\mathcal{M}_{\theta\sigma}$ and $2n$-dimensional vectors $\mathcal{N}_{\theta\sigma}$, $\hat{K}$ are

$$\mathcal{M}_{\theta\sigma} = \begin{pmatrix} A_{\theta\sigma} & B_{\theta\sigma} \\ B_{\theta\sigma}^T & C_{\theta\sigma} \end{pmatrix}, \qquad \mathcal{N}_{\theta\sigma}^T = (D_{\theta\sigma}^T\,,\, E_{\theta\sigma}^T), \qquad \hat{K}^T = (\hat{k}^T\,,\, \hat{q}^T)\,, \tag{23}$$

and where

$$\begin{aligned} A_{\theta\sigma} &= A - \frac{1}{2}\,B\,\Theta + \frac{1}{2}\,\Theta\,B^T - \frac{1}{4}\,\Theta\,C\,\Theta, & D_{\theta\sigma} &= D + \frac{1}{2}\,\Theta\,E, \\ B_{\theta\sigma} &= B + \frac{1}{2}\,\Theta\,C + \frac{1}{2}\,A\,\Sigma + \frac{1}{4}\,\Theta\,B^T\,\Sigma, & E_{\theta\sigma} &= E - \frac{1}{2}\,\Sigma\,D, \\ C_{\theta\sigma} &= C - \frac{1}{2}\,\Sigma\,B + \frac{1}{2}\,B^T\,\Sigma - \frac{1}{4}\,\Sigma\,A\,\Sigma, & F_{\theta\sigma} &= F. \end{aligned} \tag{24}$$

Note that for the nonsingular Hamiltonian $H(\hat{p},\hat{x},t)$ and for sufficiently small θ_{ab} the Hamiltonian $H_{\theta\sigma}(\hat{k},\hat{q},t)$ is also nonsingular. It is worth noting that $A_{\theta\sigma}$ and $D_{\theta\sigma}$ do not depend on σ, as well as $C_{\theta\sigma}$ and $E_{\theta\sigma}$ do not depend on θ. Classical analogue of (22) maintains the same form.

From (14), (9) and (22) one can find connections between $\mathcal{M}_{\theta\sigma}$, $\mathcal{N}_{\theta\sigma}$, $F_{\theta\sigma}$ and $\mathcal{M}$, $\mathcal{N}$, F, which are given by the following relations:

$$\mathcal{M}_{\theta\sigma} = \Xi^T\,\mathcal{M}\;\Xi\,, \qquad \mathcal{N}_{\theta\sigma} = \Xi^T\,\mathcal{N}, \qquad F_{\theta\sigma} = F. \tag{25}$$

Using equations $\dot{q}_a = \frac{\partial H_{\theta\sigma}}{\partial k_a}$ which give $k = A_{\theta\sigma}^{-1}\,(\dot{q} - B_{\theta\sigma}\,q - D_{\theta\sigma})$, we can pass from the classical form of Hamiltonian (22) to the corresponding Lagrangian by relation $L_{\theta\sigma}(\dot{q},q,t) = k^T\dot{q} - H_{\theta\sigma}(k,q,t)$. Note that coordinates q_a and x_a coincide when $\theta = \sigma = 0$. Performing necessary computations we obtain

$$L_{\theta\sigma}(Q,t) = \frac{1}{2}\,Q^T\,M_{\theta\sigma}\,Q + N_{\theta\sigma}^T\,Q + \phi_{\theta\sigma}, \tag{26}$$

where

$$M_{\theta\sigma} = \begin{pmatrix} \alpha_{\theta\sigma} & \beta_{\theta\sigma} \\ \beta_{\theta\sigma}^T & \gamma_{\theta\sigma} \end{pmatrix}, \quad N_{\theta\sigma}^T = (\delta_{\theta\sigma}^T\,,\, \eta_{\theta\sigma}^T)\,, \quad Q^T = (\dot{q}^T\,,\, q^T)\,. \tag{27}$$

Then the connections between $M_{\theta\sigma}$, $N_{\theta\sigma}$, $\phi_{\theta\sigma}$ and M, N, ϕ are given by the following relations:

$$\begin{aligned} M_{\theta\sigma} &= \sum_{i,j=1}^{3} \Xi_{ij}^T\;M\;\Xi_{ij}, & \Xi_{ij} &= \Upsilon_i(M)\;\Xi\;\Upsilon_j(\mathcal{M}_{\theta\sigma}), \\ N_{\theta\sigma} &= Y(\mathcal{M}_{\theta\sigma})\,\Xi^T\,Y(M)\,N, & \phi_{\theta\sigma} &= \mathcal{N}_{\theta\sigma}^T\,Z(\mathcal{M}_{\theta\sigma})\,\mathcal{N}_{\theta\sigma} - F\,. \end{aligned} \tag{28}$$

In more detail, the connection between coefficients of the Lagrangians $L_{\theta\sigma}$ and L is given by the relations:

$$
\begin{aligned}
\alpha_{\theta\sigma} &= \left[\,\alpha^{-1} - \tfrac{1}{2}\,(\Theta\,\beta^T\,\alpha^{-1} - \alpha^{-1}\,\beta\,\Theta) - \tfrac{1}{4}\,\Theta\,(\beta^T\,\alpha^{-1}\,\beta - \gamma)\,\Theta\,\right]^{-1}, \\
\beta_{\theta\sigma} &= \alpha_{\theta\sigma}\left(\alpha^{-1}\,\beta - \tfrac{1}{2}\,(\alpha^{-1}\,\Sigma - \Theta\,\gamma + \Theta\,\beta^T\,\alpha^{-1}\,\beta) + \tfrac{1}{4}\,\Theta\,\beta^T\,\alpha^{-1}\,\Sigma\right), \\
\gamma_{\theta\sigma} &= \gamma + \beta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\beta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\beta + \tfrac{1}{4}\,\Sigma\,\alpha^{-1}\,\Sigma \\
&\quad - \tfrac{1}{2}\,(\Sigma\,\alpha^{-1}\,\beta - \beta^T\,\alpha^{-1}\,\Sigma)\,, \qquad\qquad (29)\\
\delta_{\theta\sigma} &= \alpha_{\theta\sigma}\left(\alpha^{-1}\,\delta + \tfrac{1}{2}\,\left(\Theta\,\eta - \Theta\,\beta^T\,\alpha^{-1}\,\delta\right)\right), \\
\eta_{\theta\sigma} &= \eta + \beta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\delta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\delta - \tfrac{1}{2}\,\Sigma\,\alpha^{-1}\,\delta\,, \\
\phi_{\theta\sigma} &= \phi + \tfrac{1}{2}\,\delta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\delta_{\theta\sigma} - \tfrac{1}{2}\,\delta^T\,\alpha^{-1}\,\delta\,.
\end{aligned}
$$

Note that $\alpha_{\theta\sigma}$, $\delta_{\theta\sigma}$ and $\phi_{\theta\sigma}$ do not depend on σ.

Noncommutative Schrödinger equation and path integral. The corresponding Schrödinger equation in this NCQM is

$$
i\,\hbar\frac{\partial\Psi(q,t)}{\partial t} = H_{\theta\sigma}(\hat{k},q,t)\,\Psi(q,t)\,, \qquad (30)
$$

where $\hat{k}_a = -\,i\,\hbar\frac{\partial}{\partial q_a}$, $a = 1,2,\cdots,n$ and $H_{\theta\sigma}(\hat{k},q,t)$ is given by (22). Investigations of dynamical evolution have been mainly performed using the Schrödinger equation and this aspect of NCQM is much more developed than the noncommutative Feynman path integral. For this reason and importance of the path integral method, we will give now also a description of this approach.

To compute a path integral, which is a basic instrument in Feynman's approach to quantum mechanics, one can start from its Hamiltonian formulation on the phase space. However, when Hamiltonian is a quadratic polynomial with respect to momentum k (see, e.g. [5]) such path integral on a phase space can be reduced to the Lagrangian path integral on configuration space. Hence, for the Hamiltonian (22) we have derived the corresponding Lagrangian (26).

The standard Feynman path integral [10] is

$$
\mathcal{K}(x'',t'';x',t') = \int_{x'}^{x''} \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L(\dot{q},q,t)\,dt\right)\mathcal{D}q\,, \qquad (31)
$$

where $\mathcal{K}(x'',t'';x',t')$ is the kernel of the unitary evolution operator $U(t)$ and $x'' = q(t'')$, $x' = q(t')$ are end points. In ordinary quantum mechanics

(OQM), Feynman's path integral for quadratic Lagrangians can be evaluated analytically and its exact expression has the form [11]

$$\mathcal{K}(x'',t'';x',t') = \frac{1}{(ih)^{\frac{n}{2}}}\sqrt{\det\left(-\frac{\partial^2\bar{S}}{\partial x''_a\partial x'_b}\right)}\exp\left(\frac{2\pi i}{h}\,\bar{S}(x'',t'';x',t')\right), \quad (32)$$

where $\bar{S}(x'',t'';x',t')$ is the action for the classical trajectory. According to (4), (5) and (6), NCQM related to the quantum phase space ($\hat{p}$, $\hat{x}$) can be regarded as an OQM on the standard phase space ($\hat{k}$, $\hat{q}$) and one can apply usual path integral formalism.

A systematic path integral approach to NCQM with quadratic Lagrangians (Hamiltonians) has been investigated during the last few years in [4]- [8]. In [4] and [5], general connections between quadratic Lagrangians and Hamiltonians for standard and $\theta \neq 0$, $\sigma = 0$ NC are established, and this formalism was applied to a particle in the two-dimensional noncommutative plane with a constant field and to the noncommutative harmonic oscillator. Papers [6] - [8] present generalization of articles [4] and [5] towards noncommutativity (4). This formalism was illustrated by a charged particle in a noncommutative plane with electric and perpendicular magnetic field.

If we know Lagrangian (12) and algebra (4) we can obtain the corresponding effective Lagrangian (26) suitable for the path integral in NCQM. Exploiting the Euler-Lagrange equations

$$\frac{\partial L_{\theta\sigma}}{\partial q_a} - \frac{d}{dt}\frac{\partial L_{\theta\sigma}}{\partial \dot{q}_a} = 0\,, \quad a = 1,2,\cdots,n\,,$$

one can obtain the classical trajectory $q_a = q_a(t)$ connecting end points $x' = q(t')$ and $x'' = q(t'')$, and the corresponding action is

$$\bar{S}_{\theta\sigma}(x'',t'';x',t') = \int_{t'}^{t''} L_{\theta\sigma}(\dot{q},q,t)\,dt\,.$$

Path integral in NCQM is a direct analogue of (32) and its exact form expressed through quadratic action $\bar{S}_{\theta\sigma}(x'',t'';x',t')$ is

$$\mathcal{K}_{\theta\sigma}(x'',t'';x',t') = \frac{1}{(ih)^{\frac{n}{2}}}\sqrt{\det\left(-\frac{\partial^2\bar{S}_{\theta\sigma}}{\partial x''_a\,\partial x'_b}\right)}\exp\left(\frac{2\,\pi\,i}{h}\,\bar{S}_{\theta\sigma}(x'',t'';x',t')\right).$$

3. Noncommutativity on p-adic and adelic spaces

We want to explore now some possible p-adic and adelic generalizations of the above noncommutativity on real phase space. Let us first recall some elementary properties of p-adic numbers and adeles.

p-Adic numbers and adeles. When we are going to consider basic properties of p-adic numbers it is instructive to start with the field $\mathbb{Q}$ of rational numbers, which is the simplest field of numbers with characteristic 0. $\mathbb{Q}$ also contains all results of physical measurements. Any non-zero rational number can be expanded into two different ways of infinite series:

$$\pm\, 10^n \sum_{k=0}^{-\infty} a_k\, 10^k\,, \quad a_k \in \{0,1,2,\cdots,9\}\,, \tag{33}$$

$$p^\nu \sum_{k=0}^{+\infty} b_k\, p^k\,, \quad b_k \in \{0,1,\cdots,p-1\}\,, \tag{34}$$

where p is a prime number, and $n, \nu \in \mathbb{Z}$. These expansions have the usual repetition of digits depending on rational number but different for (33) and (34).

The series (33) and (34) are convergent with respect to the usual absolute value $|\cdot|_\infty$ and p-adic norm (p-adic absolute value) $|\cdot|_p$. Allowing all possibilities for digits, as well as for integers n and ν, by (33) and (34) one can represent any real and p-adic number, respectively. According to the Ostrowski theorem, the field $\mathbb{R}$ of real numbers and the field $\mathbb{Q}_p$ of p-adic numbers exhaust all possible completions of $\mathbb{Q}$. Consequently $\mathbb{Q}$ is a dense subfield in $\mathbb{R}$ as well as in $\mathbb{Q}_p$. These local fields have many distinct geometric and algebraic properties. Geometry of p-adic numbers is the non-Archimedean (ultrametric) one.

There are mainly two kinds of analysis on $\mathbb{Q}_p$, which are mathematically well developed and employed in applications. They are related to two different mappings: $\mathbb{Q}_p \to \mathbb{Q}_p$ and $\mathbb{Q}_p \to \mathbb{C}$. Some elementary p-adic valued functions are defined by the same series as in the real case, but the region of convergence is rather different. For instance, $\exp_p x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ converges in $\mathbb{Q}_p$ if $|x|_p \leq |2p|_p$. Derivatives of p-adic valued functions are also defined as in the real case, but using p-adic norm instead of the absolute value.

Very important usual complex-valued p-adic functions are: (i) an additive character

$$\chi_p(x) = \exp 2\pi i \{x\}_p\,, \tag{35}$$

where

$$\{x\}_p = \begin{cases} p^{-m}(a_0 + a_1\, p + \cdots + a_{m-1}\, p^{m-1})\,, & m \geq 1\,, \\ 0\,, & \nu \geq 0\,, \end{cases} \tag{36}$$

is the fractional part of x presented in the canonical form (34); (*ii*) a multiplicative character

$$\pi_s(x) = |x|_p^s\,, \quad s \in \mathbb{C}\,; \tag{37}$$

and (*iii*) locally constant functions with compact support, whose simple example is

$$\Omega(|x|_p) = \begin{cases} 1,\ |x|_p \leq 1, \\ 0,\ |x|_p > 1. \end{cases} \tag{38}$$

An adele x is an infinite sequence

$$x = (x_\infty,\, x_2,\, x_3,\, \cdots, x_p,\, \cdots)\,, \tag{39}$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $\mathcal{P}$ of primes p we have $x_p \in \mathbb{Z}_p = \{y \in \mathbb{Q}_p\ :\ |y|_p \leq 1\}$. Addition and multiplication of adeles is componentwise. The ring of all adeles can be presented as

$$\mathbb{A} = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}), \qquad \mathbb{A}(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p\,, \tag{40}$$

where $\mathbb{Z}_p$ is the ring of p-adic integers. $\mathbb{A}$ is locally compact topological space with well defined Haar measure. There are mainly two kinds of analysis over $\mathbb{A}$, which generalize those on $\mathbb{R}$ and $\mathbb{Q}_p$.

On p-adic and adelic noncommutative analogs. Since 1987, p-adic numbers and adeles have been successfully employed in many topics of modern mathematical physics (for a review, see e.g. [12]). In particular, p-adic and adelic string theory (as a review, see [13]), quantum mechanics (see [14] as a recent review) and quantum cosmology (see [15] as a recent review) have been investigated. For much more information on p-adic numbers, adeles and their analysis one can see [16] and [17].

It is well known that combining quantum mechanics and relativity one concludes existence of a spatial uncertainty Δx which reads

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm. \tag{41}$$

The uncertainty (41) may be regarded as a reason to consider simultaneously noncommutative and p-adic aspects of spatial coordinates at the

Planck scale. Henceforth we are interesting here in p-adic analogs of the above noncommutativity considerations on real space. Adelic approach enables to treat real and all p-adic aspects of a quantum system simultaneously and as essential parts of a more complete description. Adelic quantum mechanics was formulated [18] and successfully applied to some simple and exactly solvable models. Here we consider also adelic approach to noncommutativity.

Note that instead of (1) one can use an equivalent quantization based on relations ($h = 1$)

$$\chi_\infty(-\alpha_a \hat{x}_a)\, \chi_\infty(-\beta_b \hat{p}_b) = \chi_\infty(\alpha_a \beta_b\, \delta_{ab})\, \chi_\infty(-\beta_b \hat{p}_b)\, \chi_\infty(-\alpha_a \hat{x}_a), \quad (42)$$

$$\chi_\infty(-\alpha_a \hat{x}_a)\, \chi_\infty(-\alpha_b \hat{x}_b) = \chi_\infty(-\alpha_b \hat{x}_b)\, \chi_\infty(-\alpha_a \hat{x}_a), \quad (43)$$

$$\chi_\infty(-\beta_a \hat{p}_a)\, \chi_\infty(-\beta_b \hat{p}_b) = \chi_\infty(-\beta_b \hat{p}_b)\, \chi_\infty(-\beta_a \hat{p}_a), \quad (44)$$

where $\chi_\infty(u) = \exp(-2\pi i u)$ is real additive character and (α, β) is a point of classical phase space.

Quantization of expressions which contain products of x_i and p_j is not unique. According to the Weyl quantization any function $f(p, x)$, of classical canonical variables p and x, which has the Fourier transform $\tilde{f}(\alpha, \beta)$ becomes a self-adjoint operator in $L_2(\mathbb{R}^n)$ in the following way:

$$\hat{f}(\hat{p}, \hat{x}) = \int \chi_\infty(-\alpha \hat{x} - \beta \hat{p}) \tilde{f}(\alpha, \beta)\, d^n\alpha\, d^n\beta. \quad (45)$$

It is significant that quantum mechanics on a real space can be generalized to p-adic spaces for any prime number p. However there is not a unique way to perform generalization. As a result there are two main approaches: with complex-valued and p-adic valued elements of the Hilbert space on $\mathbb{Q}_p^n$. For approach with p-adic valued wave functions see [19]. p-Adic quantum mechanics with complex-valued wave functions is more suitable for connection with ordinary quantum mechanics, and in the sequel we will refer only to this kind of p-adic quantum mechanics (as a review, see [14]).

Since wave functions are complex-valued, one cannot construct a direct analog of the Schrödinger equation. The Weyl approach to quantization is suitable in p-adic quantum mechanics (see e.g. [14]).

Let $\hat{x}$ and $\hat{k}$ be some operators of position x and momentum k, respectively. Let us define operators $\chi_v(\alpha \hat{x})$ and $\chi_v(\beta \hat{k})$ by formulas

$$\chi_v(\alpha \hat{x})\, \chi_v(ax) = \chi_v(\alpha x)\, \chi_v(ax) = \chi_v((a + \alpha)x)\,, \quad (46)$$

$$\chi_v(\beta\hat{k})\,\chi_v(bk) = \chi_v(\beta k)\,\chi_v(bk) = \chi_v((b+\beta))\,, \tag{47}$$

where index v denotes real ($v = \infty$) and any p-adic case, $v = \infty, 2, \cdots, p, \cdots$, taking into account all non-trivial and inequivalent valuations on $\mathbb{Q}$. It is obvious that these operators also act on a function $\psi_v(x) \in L_2(\mathbb{Q}_v^n)$, which has the Fourier transform $\tilde{\psi}_v(k)$, in the following way:

$$\chi_v(-\alpha\hat{x})\,\psi_v(x) = \chi_v(-\alpha\hat{x}) \int \chi_v(-kx)\,\tilde{\psi}_v(k)\,d^nk = \chi_v(-\alpha x)\,\psi_v(x), \tag{48}$$

$$\chi_v(-\beta\hat{k})\,\psi_v(x) = \int \chi_v(-\beta k)\,\chi_v(-kx)\,\tilde{\psi}(k)_v\,d^nk = \psi_v(x+\beta), \tag{49}$$

where integration in p-adic case is with respect to the Haar measure dk with the properties: $d(k+a) = dk$, $d(ak) = |a|_p\,dk$ and $\int_{|k|_p \le 1} dk = 1$. Now relations (42), (43), (44) can be straightforwardly generalized, including p-adic cases, by replacing formally index ∞ by v . Thus we have

$$\chi_v(-\alpha_a\hat{x}_a)\,\chi_v(-\beta_b\hat{k}_b) = \chi_v(\alpha_a\beta_b\,\delta_{ab})\,\chi_v(-\beta_b\hat{k}_b)\,\chi_v(-\alpha_a\hat{x}_a), \tag{50}$$

$$\chi_v(-\alpha_a\hat{x}_a)\,\chi_v(-\alpha_b\hat{x}_b) = \chi_v(-\alpha_b\hat{x}_b)\,\chi_v(-\alpha_a\hat{x}_a), \tag{51}$$

$$\chi_v(-\beta_a\hat{k}_a)\,\chi_v(-\beta_b\hat{k}_b) = \chi_v(-\beta_b\hat{k}_b)\,\chi_v(-\beta_a\hat{k}_a). \tag{52}$$

It is worth noting that equation (49) suggests to introduce

$$\left\{\frac{\beta\hat{k}}{h}\right\}_p^m \psi_p(x) = \int \left\{\frac{\beta k}{h}\right\}_p^m \chi_p(-k\,x)\,\tilde{\psi}_p(k)\,d^nk \tag{53}$$

which may be regarded as a new kind of the p-adic pseudo-differential operator (for Vladimirov's pseudo-differential operator, see [12]). Also equation (50) suggests a p-adic analog of the Heisenberg algebra in the form

$$\left\{\frac{\alpha_a\hat{x}_a}{h}\right\}_p \left\{\frac{\beta_b\hat{k}_b}{h}\right\}_p - \left\{\frac{\beta_b\hat{k}_b}{h}\right\}_p \left\{\frac{\alpha_a\hat{x}_a}{h}\right\}_p = -\frac{i}{2\pi}\,\delta_{ab}\left\{\frac{\alpha_a\beta_b}{h}\right\}_p, \tag{54}$$

where h is the Planck constant. According to (54), p-adic noncommutativity depends on $\left\{\frac{\alpha_a\beta_b}{h}\right\}_p$ which is a rational number related to the size of phase space in units of h. When $\frac{\alpha_a\beta_b}{h} \in \mathbb{Z}$ then $\left\{\frac{\alpha_a\beta_b}{h}\right\}_p = 0$ and system is p-adically commutative.

From (35) one can derive uncertainty relation [20]

$$\Delta\left\{\frac{\alpha_a x_a}{h}\right\}_p \Delta\left\{\frac{\beta_b k_b}{h}\right\}_p \ge \frac{\delta_{ab}}{4\pi}\left\{\frac{\alpha_a\beta_b}{h}\right\}_p, \tag{55}$$

which is p-adic analog of the first inequality in (3).

Taking product of (50) over all valuations v we have

$$\prod_v \chi_v(-\alpha_a \hat{x}_a)\,\chi_v(-\beta_b \hat{k}_b) = \prod_v \chi_v(-\beta_b \hat{k}_b)\,\chi_v(-\alpha_a \hat{x}_a)\,, \quad \frac{\alpha_a\,\beta_b}{h} \in \mathbb{Q}\,, \tag{56}$$

since

$$\prod_v \chi_v(\alpha_a\,\beta_b\,\delta_{ab}) = 1\,, \quad \frac{\alpha_a\,\beta_b}{h} \in \mathbb{Q}\,. \tag{57}$$

It follows that in an adelic quantum system with the same rational value of $\frac{\alpha_a \beta_b}{h}$ in real and all p-adic counterparts one has commutativity between canonical operators $\hat{x}_a$ and $\hat{k}_a$.

p-Adic version of (4) can be obtained from (50) - (52) by adding the corresponding prefactors on the RHS. Adelic product will be again commutative for rational values of parameters $\alpha_a\,, \beta_b\,, \theta_{ab}$ and σ_{ab}.

p-Adic and adelic path integrals have been investigated and for quadratic Lagrangians an analog of (32) was obtained with number field invariant form (see [21] and references therein). For some other considerations of p-adic and adelic noncommutativity including the Moyal product in the context of scalar field theory one can see [22] and [23].

4. Concluding remarks

At the first glance one can conclude that the phase space transformation (6) is not appropriate because it is not a canonical one. However this transformation should not be the canonical one since initial problem is given not only by Hamiltonian (14) but also with relations (4). Using transformations (6), Hamiltonian (14) with commutation relations (4) is equivalent to Hamiltonian (22) with conventional relations (5).

Let us mention that taking $\sigma = 0,\ \theta = 0$ in the above formulas we recover expressions for the Lagrangian $L(X,t)$, classical action $\bar{S}(x'', T; x', 0)$ and probability amplitude $\mathcal{K}(x'', T; x', 0)$ of the ordinary commutative case.

A similar path integral approach with $\sigma = 0$ has been considered in the context of the Aharonov-Bohm effect, the Casimir effect, a quantum system in a rotating frame, and the Hall effect (references on these and some other related subjects can be found in [4] - [8]). Our investigation contains all quantum-mechanical systems with quadratic Hamiltonians (21) (Lagrangians (12)) on noncommutative phase space given by relations (4).

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