MARGARITA MATHEMATICA EN MEMORIA DE
JOSÉ JAVIER (CHICHO) GUADALUPE HERNÁNDEZ
(Luis Español y Juan L. Varona, editores),
Servicio de Publicaciones, Universidad de La Rioja,
Logroño, Spain, 2001.

FIBRE BUNDLES OVER ORBITS OF STATES

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A la memoria de Chicho

Abstract. We review topologic properties of orbits of states of von Neumann algebras, starting with unitary orbits, and proceeding with more general sets of states, namely vector states with symbol a spheric vector in a Hilbert C^*-module of the algebra. This is done by considering natural bundles over these sets, which enable one to relate their topologic properties to those of the unitary groups of von Neumann algebras related to the original algebra and the state involved. These views are applied to the topologic study of states, partial isometries and projections of the hyperfinite II_1 factor.

1. Introduction

In this paper we treat results contained in work done previously in [4], [5], [6] and [1], and try to give a unified exposition of them. Some of the proofs are only outlined. The main objects of this study are a von Neumann algebra, i.e. a ring of bounded operators acting on a Hilbert space H, which is closed under the strong operator topology, and a state of the algebra, that is, a positive functional of norm one. Typical states are obtained by means of unit vectors in the Hilbert space: if $f \in H$, with $\|f\| = 1$, then $\omega_f(a) = (af, f)$ is a positive functional of norm 1 (for a an operator in the von Neumann algebra). These are called vector states. We shall consider more general types of vectors states, with symbols in a right Hilbert C^*-module, rather than a Hilbert space.

Tools from homotopy theory have been used in operator algebras for quite some time. Starting with N. H. Kuiper's theorem [18], establishing the contractibility of the unitary group of an infinite dimensional Hilbert space, following with further generalizations, to properly infinite von Neumann algebras ([10], [8], [11]). Araki, M. Smith and L. Smith considered the case when the von Neumann algebra is finite, and in [8] showed for example that the π_1 group of the unitary group of a II_1 factor is isomorphic to the additive group $\mathbb{R}$. These results were later extended by Schröder in [26]. Also some results appeared computing the homotopy type of the unitary groups of certain classes of C^*-algebras ([15], [30]). The topology considered for the unitary groups of the von Neumann algebras in these papers is the one induced by the norm of the algebra. Only a few years ago Popa and Takesaki [24] studied the

2000 *Mathematics Subject Classification.* 46L30, 46L05, 46L10.
Key words and phrases. State space, C^*-module.

homotopy theory of the unitary and automorphism groups of a factor in the weak topologies of the algebra.

We shall establish here certain natural bundles, and use them to obtain topologic information about our sets of states. Let us describe which are these sets.

First we shall consider unitary orbits. Let $\mathcal{B}$ be a von Neumann algebra, we denote by $U_{\mathcal{B}}$ the group of unitary operators of $\mathcal{B}$, or shortly, the unitary group of $\mathcal{B}$. If φ is a state of $\mathcal{B}$, and $u \in U_{\mathcal{B}}$, then φ_u given by $\varphi_u(a) = \varphi(u^*au)$ is another state of $\mathcal{B}$. This gives an action of the group $U_{\mathcal{B}}$ on the set of states. For simplicity let us restrict to faithful states, i.e. states with the property that $\varphi(a^*a) = 0$ implies $a = 0$, or equivalently states with support equal to the identity (in general, the support will be a projection of the algebra). Let us denote by $\mathcal{U}_\varphi = \{\varphi_u : u \in U_{\mathcal{B}}\}$ the orbit of φ under this action, the unitary orbit of φ. The natural map over this set $\mathcal{U}_\varphi$ is

$$U_{\mathcal{B}} \to \mathcal{U}_\varphi, \quad u \mapsto \varphi_u.$$

The fibre over φ is the set of unitaries v satisfying that $\varphi(v^*av) = \varphi(a)$ for all $a \in \mathcal{B}$. Or equivalently $\varphi(va) = \varphi(av)$ for all $a \in \mathcal{B}$. The set of operators $b \in \mathcal{B}$ verifying that $\varphi(ba) = \varphi(ab)$ is a von Neumann algebra, usually called the the centralizer algebra of φ, and denoted by $\mathcal{B}^\varphi$. Then the fibre of this map over φ is $U_{\mathcal{B}^\varphi}$ the unitary group of $\mathcal{B}^\varphi$. So there is a natural bijection between $\mathcal{U}_\varphi$ and the quotient $U_{\mathcal{B}}/U_{\mathcal{B}^\varphi}$, by means of $\varphi_u \mapsto [u]$, where $[u]$ denotes the class of u in this quotient. We shall endow $\mathcal{U}_\varphi$ with the topology induced by this bijection, that is, we identify these sets, where the homogeneous space $U_{\mathcal{B}}/U_{\mathcal{B}^\varphi}$ is considered with the quotient topology of the usual norm topology of $\mathcal{B}$. The first fact is that with this topology, the map $U_{\mathcal{B}} \to \mathcal{U}_\varphi$ is a fibre bundle. This bundle will be studied in section 2 of this paper. The main result about it is that though the unitary group $U_{\mathcal{B}}$ has non trivial homotopy groups, $\mathcal{U}_\varphi$ is simply —but in general not doubly— connected.

A right Hilbert C^*-module over $\mathcal{B}$ is a right $\mathcal{B}$-module X with a $\mathcal{B}$-valued inner product $\langle\ ,\ \rangle$, which is additive in both variables, and satisfying the following axioms:

$$\langle x, x\rangle \text{ is a positive operator of } \mathcal{B},$$

$$\langle x, x\rangle = 0 \text{ implies } x = 0,$$

$$\langle x, y.a\rangle = \langle x, y\rangle a,$$

and

$$\langle x, y\rangle = \langle y, x\rangle^*.$$

Moreover, these axioms imply that $\|x\|_X = \|\langle x, x\rangle\|^{1/2}$ is a norm for X. We make the assumption that X is complete with this norm. Since we are dealing with von Neumann algebras, which are closed under a topology weaker than the norm topology, we shall eventually further require that X behaves well with respect to weak topologies. Namely we shall require that X is selfdual, which briefly means that $\mathcal{B}$-valued, $\mathcal{B}$-module forms of X are of the form $x \mapsto \langle y, x\rangle$ for appropriate $y \in X$.

There is an algebra of operators associated to such a module X, the set of operators $t : X \to X$ which are adjointable for the inner product, i.e. there exists another operator $s : X \to X$ such that $\langle tx, y\rangle = \langle x, sy\rangle$. Remarkably, adjointable operators are automatically $\mathcal{B}$-linear and bounded, and the set of all adjointable operators, denoted by $\mathcal{L}_{\mathcal{B}}(X)$, is a C^*-algebra. If moreover X is selfdual then $\mathcal{L}_{\mathcal{B}}(X)$ is a von Neumann algebra. These are standard facts on C^*-modules, and can be found in the original paper [21] by W. Paschke. Further references on this subject are [22], [25] and [19].

A vector $x \in X$ will be called spherical if $\langle x, x\rangle = 1$, and we shall denote by $S_1(X)$ the unit sphere of X, or the set of all spherical vectors. More generally if p is a projection in $\mathcal{B}$, $S_p(X)$ denotes the set of $x \in X$ such that $\langle x, x\rangle = p$. Geometric and topologic properties of these spheres and p-spheres were studied in [2] and [3]. Their homotopy groups can be computed in some cases, though not always, because they include as particular cases the classical finite dimensional spheres. However, if X is selfdual, the π_1-group can be computed in terms of the type decomposition of $\mathcal{B}$ [2].

If $x \in S_p(X)$ and φ is a state with support p, then one obtains a state of $\mathcal{L}_{\mathcal{B}}(X)$, called φ_x, by means of

$$\varphi_x(t) = \varphi(\langle x, tx\rangle), \quad t \in \mathcal{L}_{\mathcal{B}}(X).$$

If $X = \mathcal{B}$ (with the inner product given by $\langle x, y\rangle = x^*y$), then $\mathcal{L}_{\mathcal{B}}(X)$ identifies with $\mathcal{B}$. A unitary operator $u \in U_{\mathcal{B}}$ is a spherical vector of this X, and clearly the notation φ_u is consistent with the previous definition of this symbol. In other words, this notion of vector state generalizes the unitary action considered above.

We shall denote by $\mathcal{O}_\varphi$ the set of all vector states, with φ fixed and x varying in $S_p(X)$, where p is the support projection of φ. The natural map over this set is

$$S_p(X) \to \mathcal{O}_\varphi, \quad x \mapsto \varphi_x,$$

which generalizes the previous map. Again, we endow $\mathcal{O}_\varphi$ with the quotient topology induced by this map ($S_p(X)$ considered with the norm topology of X). This map is considered in section 3. It is shown that the topology above is given by the following metric

$$d_\varphi(\varphi_{x_0}, \varphi_{y_0}) = \inf\{\|x - y\| : x, y \text{ such that } \varphi_x = \varphi_{x_0},\ \varphi_y = \varphi_{y_0}\}.$$

Again, with these topologies this map is a fibre bundle, with fibre equal to the unitary group of the centralizer of the state φ restricted to the reduced algebra $p\mathcal{B}p$.

In section 4 we let φ vary over the set of all states with support p (with p fixed). The set thus obtained is shown to be the set of all states of $\mathcal{L}_{\mathcal{B}}(X)$ with *equivalent* supports. The natural map here is

$$(x, \varphi) \mapsto \varphi_x,$$

for $x \in \mathcal{S}_p(X)$ and φ a state of $\mathcal{B}$ with support p. If these two sets are considered with the norm topology, the quotient topology induced on the set of modular vector states φ_x is given by a metric d, given by $d(\Phi, \Psi) = \|\Phi - \Psi\| + \|\operatorname{supp}(\Phi) - \operatorname{supp}(\Psi)\|$. It is shown that the map above is a fibre bundle.

Apparently, this metric gives a topology which is much stronger that the norm topology (of the dual of $\mathcal{L}_{\mathcal{B}}(X)$). If one is interested in the set of states φ_x in the norm topology, one is forced to consider a weaker topology for the sphere $\mathcal{S}_p(X)$. This is done in sections 5 and 6. A well known faithful representation ([25], [21]) of

$\mathcal{L}_{\mathcal{B}}(X)$ enables one to rewrite the map $(x, \varphi) \mapsto \varphi_x$ as a map $f \mapsto \omega_f$ for a set of vectors f in this representation. The weak topology in $\mathcal{S}_p(X)$ is harder to handle, but imposing conditions on the algebra $\mathcal{B}$ one obtains that this map is again a fibre bundle.

These facts are applied in section 6, to prove that the set of states of (for example) the hyperfinite II_1 factor, with support equivalent to a fixed projection, has trivial homotopy groups of all orders. It is also shown that the set of partial isometries of this factor, with initial space fixed, in the ultraweak topology, has trivial homotopy groups of all orders. Finally, it is shown, that unitary orbits of states of this algebra are simply connected in the norm topology as well.

We include an application of these results in section 7. First we prove a statement which in our opinion is of interest in itself, and follows as an easy consequence of a result in section 5: the map which consists of taking the support projection of a state is continuous, when restricted to states of a finite von Neumann algebra with a priori equivalent supports, in the norm topology, with range in the set of projections of the algebra, regarded with the strong operator topology. Then it is shown that the support map in this setting defines a strong deformation retract. Therefore applying the result of section 6, it follows that the set of projections of the class of algebras considered there, has trivial homotopy groups for all orders $n \geq 1$ (this set is not connected).

If $\mathcal{A}$ is a von Neumann algebra and $q \in \mathcal{A}$ is a projection, $\Sigma_q(\mathcal{A})$ denotes the set of normal states of $\mathcal{A}$ with support equal to q, and $P\Sigma_q(\mathcal{A})$ the set of normal states with support equivalent to q.

2. Unitary orbits of faithful states

Throughout this section φ will denote a faithful normal (i.e. ultraweakly continuous) state of a von Neumann algebra $\mathcal{B}$. As remarked above, if u is a unitary element of $\mathcal{B}$, then φ_u given by $\varphi_u(a) = \varphi(u^*au)$, is also a faithful state. We denote $\mathcal{U}_\varphi$ the unitary orbit of φ, i.e. $\mathcal{U}_\varphi = \{\varphi_u : u \in U_{\mathcal{B}}\}$. The set of unitaries of this action which leave φ fixed is the unitary group of the centralizer $\mathcal{B}^\varphi$ of φ, $\mathcal{B}^\varphi = \{b \in \mathcal{B} : \varphi(ab) = \varphi(ba)$ for all $\mathcal{A} \in \mathcal{B}\}$. Thus the orbit $\mathcal{U}_\varphi$ identifies with the homogeneous space $U_{\mathcal{B}}/U_{\mathcal{B}^\varphi}$. We will consider on $\mathcal{U}_\varphi$ the topology induced by this identification, where both $U_{\mathcal{B}}$ and $U_{\mathcal{B}^\varphi}$ are considered with the norm topology of $\mathcal{B}$. In other words, we endow $\mathcal{U}_\varphi$ with the quotient topology given by the map

$$\pi_\varphi : U_{\mathcal{B}} \to \mathcal{U}_\varphi, \quad \pi_\varphi(u) = \varphi_u.$$

Now, as $\mathcal{U}_\varphi$ is a set of bounded functionals of $\mathcal{B}$, there is another natural topology on it, namely the norm topology of the dual $\mathcal{B}^*$.

There is yet a third norm-induced topology on $\mathcal{U}_\varphi$. Recall that a conditional expectation between C^*-algebras $\mathcal{A} \subset \mathcal{B}$ is a norm 1 projection $E : \mathcal{B} \to \mathcal{A}$, which automatically preserves adjoints, positive operators, and is $\mathcal{A}$-linear. E is said to be faithful if $E(b^*b) = 0$ implies $b = 0$, and normal when it is continuous for the ultraweak topology. By the modular theory of states in von Neumann algebras, given a faithful normal state such as φ, there exists a unique faithful and normal conditional expectation $E_\varphi : \mathcal{B} \to \mathcal{B}^\varphi$ which is φ-invariant, $\varphi \circ E_\varphi = \varphi$. Using E_φ

one can define a new norm in $\mathcal{B}$, which is the $\mathcal{B}^\varphi - C^*$-Hilbert module norm, given by the inner product

$$\langle b, c\rangle = E_\varphi(b^*c).$$

This modular norm is therefore $\|b\|_{E_\varphi} = \|E_\varphi(b^*b)\|^{1/2}$. The usual norm and this latter norm are equivalent in $\mathcal{B}$ if and only if the index of the expectation E_φ is finite. An expectation $E : \mathcal{B} \to \mathcal{A}$ is said to be of finite index ([16], [23]) if there exists a positive number κ such that $E - \kappa I$ is a positive mapping in $\mathcal{B}$. It is a strong condition, particularly for expectations onto state centralizers such as E_φ. It forces that the algebra $\mathcal{B}$ must be finite, and if it is a factor, then the state φ must be of the form $\varphi(b) = \tau(ha)$, where τ is the unique trace of the finite factor and h is a positive operator with finite spectrum (see [4]).

On the other hand, both norms clearly coincide in $\mathcal{B}^\varphi$. We are interested in the topologies they induce in the quotient $U_\mathcal{B}/U_{\mathcal{B}^\varphi}$. The following results clarify the relationship between these three topologies: *norm of the dual*, *usual norm quotient* and *modular norm quotient*.

Lemma 2.1. *Let $E : \mathcal{B} \to \mathcal{A} \subset \mathcal{B}$ be a faithful conditional expectation of infinite index. Then the norm of $\mathcal{B}$ and the norm $\| \ \|_E$ induced by E define topologies in $U_\mathcal{B}/U_\mathcal{A}$ which are not equivalent.*

Proof. Since the index of E is infinite ([9], [14]), there exist elements $a_n \in \mathcal{B}$ with $0 \le a_n \le 1$, $\|a_n\| = 1$ and $E(a_n) \to 0$ as n tends to infinity. It is straightforward to verify that the distance $d(a_n, \mathcal{A}) = \inf\{\|a_n - b\| : b \in \mathcal{A}\}$ does not tend to zero with n. Let $u_n \in U_\mathcal{B}$ be unitaries such that $1 - a_n = \frac{u_n + u_n^*}{2}$. Then

$$\|u_n - 1\|_E^2 = \|2 - E(u_n + u_n^*)\| = 2\|E(a_n)\| \to 0.$$

Therefore the sequence of the classes of the elements u_n tends to the class of 1 in the modular topology. We claim that $[u_n]$ does not tend to $[1]$ in the usual topology (induced by the norm of $\mathcal{B}$). Suppose not. Then there exist unitaries $v_n \in U_\mathcal{A}$ such that $u_n v_n \to 1$. Then

$$\|u_n - v_n^*\|^2 = \|(u_n - v_n^*)(u_n^* - v_n)\| = \|2 - u_n v_n - v_n^* u_n^*\| \to 0.$$

This implies that $d(u_n, \mathcal{A}) \to 0$, and therefore $d(a_n, \mathcal{A}) \to 0$, a contradiction. □

Proposition 2.2. *The usual norm quotient and the modular norm quotient topologies coincide in $\mathcal{U}_\varphi$ if and only if the index of E_φ is finite.*

The following inequalities show the order that prevails between the three topologies:

Proposition 2.3. *Let u and w be unitaries in $\mathcal{B}$, then*

$$\|\varphi_u - \varphi_w\| \le 2\|u - w\|_E \le 2\|u - w\|. \tag{1}$$

Proof. The second inequality is obvious, because E_φ is contractive. In order to prove the first note that for any $x \in \mathcal{B}$,

$$|\varphi(u^*xu) - \varphi(w^*xw)| \le |\varphi(u^*x(u - w))| + |\varphi((u^* - w^*)xw)|.$$

Note that if v is unitary, by the Cauchy-Schwarz inequality we have that $|\varphi(zv)| \leq \varphi(zz^*)^{1/2}$ and $|\varphi(v^*z)| \leq \varphi(z^*z)^{1/2}$. Applying these inequalities we obtain

$$|\varphi(u^*x(u-w))| \leq \varphi((u^*-w^*)x^*x(u-v))^{1/2} = \varphi \circ E_\varphi((u^*-w^*)x^*x(u-v))^{1/2},$$

and

$$|\varphi((u^*-w^*)xw)| \leq \varphi \circ E_\varphi((u^*-w^*)xx^*(u-w))^{1/2}.$$

Note that $(u^*-w^*)x^*x(u-v) \leq \|x\|^2(u^*-w^*)(u-v)$, and analogously for the other term. Thus we obtain

$$\begin{aligned} |\varphi(u^*xu) - \varphi(w^*xw)| &\leq 2\|x\| \, \varphi \circ E_\varphi((u^*-w^*)(u-v))^{1/2} \\ &\leq 2\|x\| \, \|E_\varphi((u^*-w^*)(u-v))\|^{1/2}. \end{aligned}$$

□

These inequalities also show that the inclusion $\mathcal{U}_\varphi \hookrightarrow \mathcal{B}^*$ is continuous, when $\mathcal{U}_\varphi$ is considered both with the usual norm quotient or the modular norm quotient topologies. We will return to the dual norm topology in section 6.

For the remaining of the section we shall consider the features of these two topologies separately. For the usual norm quotient topology, perhaps the most remarkable fact is that $\mathcal{U}_\varphi$ is simply connected. Let us establish this fact. To do so our main tool will be the map

$$\pi_\varphi : U_\mathcal{B} \to \mathcal{U}_\varphi, \quad \pi_\varphi(u) = \varphi_u.$$

First we check that it is a fibre bundle. The following fact is perhaps well known, the reference we know for it is [7].

Proposition 2.4. *Let $A \subset B$ be complex Banach algebras with the same unit, such that A is complemented in B. Denote by G_A , G_B the groups of invertible elements of A and B. Then the quotient map*

$$G_B \to G_B/G_A$$

has continuous local cross sections.

In our setting, $\mathcal{B}^\varphi$ is complemented in $\mathcal{B}$, because we have the projection E_φ. Starting with continuous local cross sections for quotient of invertible groups it is not difficult to obtain unitary cross sections for the quotient of unitary groups: it suffices to restrict to the unitary quotient, and to compose the cross section on the invertible group of $\mathcal{B}$ with the map which consists in taking the unitary part in the polar decomposition (the unitary on the left hand side), which is continuous on the group of invertibles.

It follows that the homogeneous space $U_\mathcal{B}/U_{\mathcal{B}^\varphi}$ has continuous local cross sections, and therefore π_φ is a fibre bundle. Once this fact is clear, we use the tail of the homotopy exact sequence of this bundle to prove that $\pi_1(\mathcal{U}_\varphi)$ is trivial. That $\pi_0(\mathcal{U}_\varphi)$ is trivial follows from the fact that the unitary group of a von Neumann algebra is connected. One has

$$\ldots \pi_1(U_{\mathcal{B}^\varphi}, 1) \to \pi_1(U_\mathcal{B}, 1) \stackrel{\pi_\varphi^*}{\to} \pi_1(\mathcal{U}_\varphi, \varphi) \to 0.$$

A von Neumann algebra has a type decomposition, one can find projections p_f,p_i in the centre of $\mathcal{B}$ such that $p_f + p_i = 1$, $p_f\mathcal{B}$ is a finite von Neumann algebra (with unit p_f) and $p_i\mathcal{B}$ is properly infinite (with unit p_i). These projections factor the algebras, $\mathcal{B} = p_f\mathcal{B} \oplus p_i\mathcal{B}$, the states, $\psi = \psi_f + \psi_i$ where $\psi_f(x) = \psi(p_f x)$ and $\psi_i(x) = \psi(p_i x)$, and the centralizer algebras $\mathcal{B}^\varphi = (p_f\mathcal{B})^{\varphi_f} \oplus (p_i\mathcal{B})^{\varphi_i}$. In other words, this projections enable one to consider the properly infinite and the finite case separately. One can further decompose the algebra, for our purposes it will suffice to proceed with the finite part, which splits into the type I part and the type II_1 part. The type I part further decomposes in the the type I_n parts, $1 \leq n < \infty$. Let us state the result, with an outline of the proof.

Theorem 2.5. *Let φ be a faithful and normal state on a von Neumann algebra $\mathcal{B}$. Then the unitary orbit $\mathcal{U}_\varphi$ with the norm quotient topology is simply connected.*

Proof. By the above remark, we may proceed by cases.

(1) If $\mathcal{B}$ is properly infinite, it was proved by Breuer in [10] that $U_\mathcal{B}$ is contractible in the norm topology. It follows that $\pi_1(\mathcal{U}_\varphi, \varphi) = 0$.

(2) If $\mathcal{B}$ is of type II_1, then $\mathcal{B}^\varphi$ is finite, but may have type I and/or type II parts. To deal with this situation, we need the following lemma, which can be found in [3]. It is based on the fact [13] that if $p \in \mathcal{B}$ is a projection, the map

$$U_\mathcal{B} \to \{upu^* : u \in U_\mathcal{B}\}, \quad u \mapsto upu^*$$

is a fibre bundle, with fibre equal to the unitary group of the commutant $\{p\}' \cap \mathcal{B}$.

Lemma 2.6. *Let $\mathcal{B}$ be a von Neumann algebra and p a projection. Then the unitary orbit $\{upu^* : u \in U_\mathcal{B}\}$ of p is simply connected.*

Note that the unitary group of the commutant $\{upu^* : u \in U_\mathcal{B}\}$ can be identified with the product $U_{p\mathcal{B}p} \times U_{(1-p)\mathcal{B}(1-p)}$. In our case, we have projections p_I and p_{II} in the centre of $\mathcal{B}^\varphi$ (which may be bigger than the centre of $\mathcal{B}$) with $p_I + p_{II} = 1$, $p_I\mathcal{B}^\varphi$ of type I and $p_{II}\mathcal{B}^\varphi$ of type II. Therefore

$$U_\mathcal{B} / \left(U_{p_I\mathcal{B}p_I} \times U_{p_{II}\mathcal{B}p_{II}}\right)$$

is simply connected. The inclusion $U_{\mathcal{B}^\varphi} \subset U_\mathcal{B}$ can be factorized

$$U_{\mathcal{B}^\varphi} = U_{p_I\mathcal{B}^\varphi} \times U_{P_{II}\mathcal{B}^\varphi} \subset U_{p_I\mathcal{B}p_I} \times U_{p_{II}\mathcal{B}p_{II}} \subset U_\mathcal{B}.$$

In the inclusion $U_{p_I\mathcal{B}^\varphi} \subset U_{p_I\mathcal{B}p_I}$, $p_I\mathcal{B}^\varphi$ is of type I and $p_I\mathcal{B}p_I$ is of type II_1. Analogously, the inclusion $U_{p_{II}\mathcal{B}^\varphi} \subset U_{p_{II}\mathcal{B}p_{II}}$ involves type II_1 algebras. Therefore it suffices to prove the result when $\mathcal{B}^\varphi$ is either of type II_1 or of type I.

(a) If both $\mathcal{B}$ and $\mathcal{B}^\varphi$ are of type II_1, their π_1 groups are isomorphic, as additive groups, to the sets of selfadjoint elements of their centres (see [15], [26]). Moreover, it can be shown using the arguments of these papers cited, that the morphism $i^* : \pi_1(U_{\mathcal{B}^\varphi}, 1) \to \pi_1(U_\mathcal{B}, 1)$ induced by the inclusion map $i : U_{\mathcal{B}^\varphi} \hookrightarrow U_\mathcal{B}$ at the π_1 level, under that identification, becomes the restriction of the center valued trace τ of $\mathcal{B}$ to $\mathcal{Z}(\mathcal{B}^\varphi)$,

$$\tau|_{\mathcal{Z}(\mathcal{B}^\varphi)} : \mathcal{Z}(\mathcal{B}^\varphi) \to \mathcal{Z}(\mathcal{B}).$$

Here $\mathcal{Z}(\mathcal{A})$ denotes the centre of $\mathcal{A}$. This morphism is clearly onto, because $\mathcal{Z}(\mathcal{B}^\varphi)$ contains $\mathcal{Z}(\mathcal{B})$. It follows that $\pi_1(\mathcal{U}_\varphi, \varphi) = 0$.

(b) If $\mathcal{B}$ is of type II_1 and $\mathcal{B}^\varphi$ is of type I (and finite), let p_n be the projections in the centre of $\mathcal{B}^\varphi$ decomposing it in its I_n types, $n < \infty$. Since $\mathcal{B}^\varphi$ is of type I ([26], [15]), $\pi_1(U_{\mathcal{B}^\varphi}, 1)$ identifies with the additive group of selfadjoint elements in the centre of $\mathcal{B}^\varphi$ which have their spectrum contained in $\mathbb{Z}$. Here the inclusion map $i : U_{\mathcal{B}^\varphi} \hookrightarrow U_{\mathcal{B}}$ again induces the morphism i^* at the π_1 level which identifies with the restriction of the center valued trace τ of $\mathcal{B}$, to the set of selfadjoint elements in the centre of $\mathcal{B}^\varphi$ with integer spectrum. We must also show here that this morphism is onto. Pick $c \in \mathcal{Z}(\mathcal{B})$, and put $c_n = cp_n$. Suppose that for each n we can find a projection q_n in the centre of $p_n\mathcal{B}^\varphi$ (equal to $p_n\mathcal{Z}(\mathcal{B}^\varphi)$), such that $\tau(q_n) = c_n$. Then the element $r = \sum_n q_n$ would be a selfadjoint element in the centre of $\mathcal{B}^\varphi$ with integer spectrum, satisfying $\tau(r) = c$. This in turn would mean that i^* is onto, and therefore $\pi_1(\mathcal{U}_\varphi, \varphi)$ would be trivial. This remark implies that it suffices to prove our statement when $\mathcal{B}^\varphi$ is of type I_n. Let us make this assumption, and let e be an abelian projection in $\mathcal{B}^\varphi$ with $\tau(e) = 1/n$. Again pick $0 \le c \le 1$ in $\mathcal{Z}(\mathcal{B})$. Now $e\mathcal{B}e$ is of type II_1, and the restriction of φ to $e\mathcal{B}e$ has centralizer equal to the commutative algebra $e\mathcal{B}^\varphi e$. Suppose now that we have proven our result for the case when $\mathcal{B}^\varphi$ is commutative. Then there would exist a projection $q \in \mathcal{Z}(e\mathcal{B}^\varphi e) = e\mathcal{Z}(\mathcal{B}^\varphi)$ such that

$$\tau(q) = ec,$$

where here τ denotes the center valued trace of $e\mathcal{B}e$. Taking trace in the above inequality yields $(1/n)\tau(q) = (1/n)c$, and the statement follows. Therefore it suffices to prove the result in the case when $\mathcal{B}^\varphi$ is commutative. Since it is the centralizer of a state, it must be maximal commutative inside $\mathcal{B}$, and it is generated by a single positive operator h, essentially satisfying $\varphi = \tau(h)$. Here a straightforward spectral theoretic argument shows our result (see [5] for the details).

(3) Finally, it remains to check the case when $\mathcal{B}$ is of type I and finite. A similar argument as above enables one to reduce to the case when $\mathcal{B}$ is of type I_n. But in this case the result is apparent, elements in $\pi_1(U_{\mathcal{B}}, 1)$ are of finite sums $\sum_i m_i p_i$ with m_i integers and p_i mutually orthogonal projections in $\mathcal{Z}(\mathcal{B})$. Since $\mathcal{Z}(\mathcal{B}) \subset \mathcal{Z}(\mathcal{B}^\varphi)$, the mentioned restriction of the centre valued trace is surjective.

$\square$

Let us now consider the modular norm quotient topology in $\mathcal{U}_\varphi$, i.e. the topology on the quotient $U_{\mathcal{B}}/U_{\mathcal{B}^\varphi}$ induced by the modular norm $\|a\|_{E_\varphi} = \|E_\varphi(a^*a)\|^{1/2}$ on $\mathcal{B}$.

We shall make use of the Jones basic extension (see for example [9]) of the conditional expectation E_φ. In our case this means the following. Let H_φ be the GNS Hilbert space of φ, i.e. the completion of the pre-Hilbert space $\mathcal{B}$ with the scalar product $\langle a,b\rangle_\varphi = \langle a,b\rangle = \varphi(b^*a)$. It is easy to see that the linear map $E_\varphi : \mathcal{B} \to \mathcal{B}^\varphi \subset \mathcal{B}$ is bounded in the norm of H_φ, and therefore extends to a selfadjoint projection in $B(H_\varphi)$, denoted by e_φ (usually called the Jones projection of E_φ), whose range is the closure of $\mathcal{B}^\varphi$ in H_φ. Denote by $\mathcal{B}_1$ the von Neumann subalgebra of $B(H_\varphi)$ generated by $\mathcal{B}$ and e_φ. Among the properties of this construction, we shall need the following:

(1) $e_\varphi a e_\varphi = E_\varphi(a)e_\varphi$, $a \in \mathcal{B}$. In particular, e_φ commutes with $\mathcal{B}^\varphi$.
(2) $\mathcal{B} \cap \{e_\varphi\}' = \mathcal{B}^\varphi$.
(3) The map $x \mapsto xe_\varphi$ is a *-isomorphism between $\mathcal{B}^\varphi$ and $\mathcal{B}^\varphi e_\varphi$.

The first pleasant fact about this topology is that it enables one to represent the space $\mathcal{U}_\varphi$ as a set of operators in $\mathcal{B}_1$. Consider the following map:

$$\mathcal{U}_\varphi \to U_{\mathcal{B}}(e_\varphi) = \{ue_\varphi u^* : u \in \mathcal{B}\}, \quad \varphi_u \mapsto ue_\varphi u^*.$$

Strictly speaking, $U_{\mathcal{B}}(e_\varphi)$ is not the unitary orbit of a projection, because the projection e_φ does not belong to $\mathcal{B}$ (with the exception of the trivial case when $\mathcal{B} = \mathcal{B}^\varphi$, i.e. φ is a trace and $\mathcal{U}_\varphi$ reduces to a point). First note that this map is well defined: if $ue_\varphi u^* = we_\varphi w^*$ for $w, u \in U_{\mathcal{B}}$, then w^*u commutes with e_φ, which by the second property cited above implies that $w^*u \in \mathcal{B}^\varphi$, which means that $\varphi_u = \varphi_w$.

This map is continuous, if $U_{\mathcal{B}}(e_\varphi) \subset \mathcal{B}_1 \subset B(H_\varphi)$ is considered with the norm topology [4]. Moreover, it is a homeomorphism. Indeed, if $ue_\varphi u^*$ is close (in norm) to e_φ, then also $u^*e_\varphi u$ is close to e_φ. Using the properties of the basic extension, this implies that both $E_\varphi(u)E_\varphi(u^*)$ and $E_\varphi(u^*)E_\varphi(u)$ are close to 1, and therefore $E_\varphi(u^*)$ is invertible. Let $\mu(g)$ be the continuous map consisting of taking the unitary part of the invertible element $g \in \mathcal{B}$, $g = \mu(g)|g|$ (explicitly, $\mu(g) = g(g^*g)^{-1/2}$). Then $\mu(E_\varphi(u^*))$ is a unitary in $\mathcal{B}^\varphi$, and $u\mu(E_\varphi(u^*))$ is close to 1 in the norm $\|\ \|_{E\varphi}$,

$$\|u\mu(E_\varphi(u^*)) - 1\|^2_{E_\varphi} = \|2 - E_\varphi(u)\mu(E_\varphi(u^*)) - \mu(E_\varphi(u))E_\varphi(u)\|.$$

Note that

$$E_\varphi(u)\mu(E_\varphi(u^*)) = E_\varphi(u)E_\varphi(u^*)[E_\varphi(u)E_\varphi(u^*)]^{-1/2} = (E_\varphi(u)E_\varphi(u^*))^{1/2},$$

which is close to 1 because $E_\varphi(u)E_\varphi(u^*)$ is close to 1. The other term inside the norm is dealt in a similar way. This implies not only that the map $\varphi_u \mapsto ue_\varphi u^*$ is a homeomorphism, but also that the assignment $ue_\varphi u^* \mapsto u\mu(E_\varphi(u^*))$, which is continuous and well defined on a neighbourhood of e_φ in $\mathcal{B}_1$, defines a continuous local cross section for

$$U_{\mathcal{B}} \to \mathcal{U}_\varphi \simeq U_{\mathcal{B}}(e_\varphi), \quad u \mapsto \varphi_u \sim ue_\varphi u^*$$

when $U_{\mathcal{B}}$ is considered with the modular norm $\|\ \|_{E_\varphi}$ and $\mathcal{U}_\varphi$ with the quotient of this topology.

However, by [5] $U_{\mathcal{B}} \subset \mathcal{B}_1$ is a submanifold, or equivalently, the map above has local cross sections which are continuous in the *norm* topology of $U_{\mathcal{B}}$, if and only if the index of E_φ is finite.

It is easy to see that $\mathcal{U}_\varphi$ is closed (in $\mathcal{B}^*$) when regarded with the usual norm quotient topology. This may not be true in the modular norm quotient topology. In the closure of $\mathcal{U}_\varphi$ with this topology, there come up states of the form φ_x, $\varphi_x(a) = \langle x, xa\rangle$, where $\langle\ ,\ \rangle$ denotes the $\mathcal{B}^\varphi$-valued inner product of the completion of the pre-Hilbert C^*-module $\mathcal{B}$. Namely, the elements x are limits of unitaries in $\mathcal{B}$, in the modular norm $\|\ \|_{E_\varphi}$. These elements are spherical elements: if $u_n \to x$ in the norm $\|\ \|_{E_\varphi}$, then

$$1 = E_\varphi(u_n u_n^*) = \langle u_n, u_n\rangle \to \langle x, x\rangle.$$

This motivates the generalization considered in the next section.

3. Orbits of states under spherical elements

Let $\mathcal{B}$ be a von Neumann algebra, X a right C^*-module over $\mathcal{B}$ which is selfdual, and $\mathcal{L}_\mathcal{B}(X)$ the von Neumann algebra of adjointable operators of X. All states considered will supposed to be normal. If $p \in \mathcal{B}$ is a projection, denote by $\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$ the p-sphere of X. We shall study the states of $\mathcal{L}_\mathcal{B}(X)$ which are vector states in the modular sense. That is, for a state φ of $\mathcal{B}$ and a vector $x \in \mathcal{S}_p(X)$, we consider the state φ_x with density x, given by

$$\varphi_x(t) = \varphi(\langle x, t(x)\rangle), \quad t \in \mathcal{L}_\mathcal{B}(X).$$

If $x, y \in X$, let $\theta_{x,y} \in \mathcal{L}_\mathcal{B}(X)$ be the "rank one" operator given by $\theta_{x,y}(z) = x\langle y, z\rangle$. If $\langle x, x\rangle = p$ then the operator $\theta_{x,x} = e_x$ is a selfadjoint projection, and all projections arising in this manner, from vectors in $\mathcal{S}_p(X)$, are mutually (Murray-von Neumann) equivalent. It turns out that these *modular vector states* as we shall subsequently call them, are precisely the states of $\mathcal{L}_\mathcal{B}(X)$ with support of rank one, i.e. equal to one of these projections e_x.

In this section we will consider the following generalization of the unitary orbit of φ:

$$\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}$$

for φ a fixed state in $\mathcal{B}$, with support projection $\mathrm{supp}(\varphi) = p$. We denote by $\Sigma_p(\mathcal{B})$ the set of states of $\mathcal{B}$ with support p.

Let us state some elementary facts about modular vector states ([5]):

Proposition 3.1. *Let $\psi, \varphi \in \Sigma_p(\mathcal{B})$, $x, y \in \mathcal{S}_p(X)$. Then*

(a) *$\varphi_x = \psi_x$ if and only if $\varphi = \psi$.*

(b) *$\varphi_x = \psi_y$ if and only if $\psi = \varphi \circ Ad(u)$, with $y = xu$ and $u \in U_{p\mathcal{B}p}$.*

(c) *$\varphi_x = \varphi_y$ if and only if $y = xv$, for v a unitary element in $\mathcal{B}_p^\varphi$.*

Proof. Let us start with (a): $\varphi(b) = \varphi_x(\theta_{xb,x}) = \psi_x(\theta_{xb,x}) = \psi(b)$.

To prove (b), suppose that $\varphi_x = \psi_y$. Then they have the same support, i.e. $e_x = e_y$, which implies that there exists a unitary element $u \in U_{p\mathcal{B}p}$ such that $y = xu$ (see [3]). Then

$$\varphi_x(t) = \psi_y(t) = \psi(\langle xu, t(xu)\rangle) = \psi(u^*\langle x, t(x)\rangle u) = [\psi \circ Ad(u^*)]_x(t).$$

Using part (a), this implies that $\varphi = \psi \circ Ad(u^*)$, or $\psi = \varphi \circ Ad(u)$.

To prove (c), use (b), and note that the unitary element $u \in U_{p\mathcal{B}p}$ satisfies $\varphi = \varphi \circ Ad(u)$, i.e. $u \in \mathcal{B}_p^\varphi$. □

Our main tool here will be the natural map

$$\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi,\ \sigma(x) = \varphi_x.$$

Let us consider the following natural metric in $\mathcal{O}_\varphi$:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x' - y'\| : x', y' \in \mathcal{S}_p(X), \varphi_{x'} = \varphi_x, \varphi_{y'} = \varphi_y\}$$

It is clear that this metric induces the same topology as the quotient topology given by the map σ, also, that in view of 3.1 it can be computed as follows:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x - yv\| : v \text{ unitary in } \mathcal{B}_p^\varphi\}.$$

First note that this is indeed a metric. For instance, if $d_\varphi(\varphi_x, \varphi_y) = 0$, then there exist unitaries v_n in $\mathcal{B}_p^\varphi$ such that $\|x - yv_n\| \to 0$, i.e. $yv_n \to x$ in $\mathcal{S}_p(X)$. In particular yv_n is a Cauchy sequence, and therefore v_n is a Cauchy sequence, converging to a unitary v in $\mathcal{B}_p^\varphi$. Then $x = yv$ and $\varphi_x = \varphi_y$. The other properties follow similarly.

With this metric, $\mathcal{O}_\varphi$ is homeomorphic to the quotient $\mathcal{S}_p(X)/U_{\mathcal{B}_p^\varphi}$. The following result implies that the inclusion $\mathcal{O}_\varphi \subset \mathcal{B}^*$ is continuous.

Lemma 3.2. *If $x, y \in \mathcal{S}_p(X)$, then $\|\varphi_x - \varphi_y\| \le 2\|x - y\|$. In particular*

$$\|\varphi_x - \varphi_y\| \le 2d_\varphi(\varphi_x, \varphi_y)$$

where the norm $\|\ \|$ of the functionals denotes the usual norm of the conjugate space $\mathcal{L}_\mathcal{B}(X)^*$.

Proof. If $t \in \mathcal{L}_\mathcal{B}(X)$, then $|\varphi_x(t) - \varphi_y(t)| \le |\varphi(\langle x, t(x - y)\rangle| + |\varphi(\langle x - y, ty\rangle)|$. Now by the Cauchy-Schwarz inequality $\|\langle x, t(x - y)\rangle\| \le \|t\|\ \|x - y\|$, and $\|\langle x - y, ty\rangle\| \le \|x - y\|\ \|t\|$. Then $\|\varphi_x(t) - \varphi_y(t)\| \le 2\|t\|\ \|x - y\|$, and the result follows. □

Recall that for a normal state φ with support p there exists a conditional expectation $E_\varphi : p\mathcal{B}p \to \mathcal{B}_p^\varphi$.

Theorem 3.3. *The map $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$, $\sigma(x) = \varphi_x$ is a locally trivial fibre bundle. The fibre of this bundle is the unitary group $U_{\mathcal{B}_p^\varphi}$ of $\mathcal{B}_p^\varphi$.*

We give an outline of the proof. It suffices to construct continuous local cross sections for σ at every point φ_{x_0}, $x_0 \in \mathcal{S}_p(X)$. Suppose that $d_\varphi(\varphi_x, \varphi_{x_0}) < r < 1$, and let us adjust r. There exists a unitary operator $v \in U_{\mathcal{B}^\varphi}$ such that $\|xv - x_0\| < r < 1$. In particular,

$$\|p - \langle xv, x_0\rangle\| = \|\langle x_0, x_0\rangle - \langle xv, x_0\rangle\| = \|\langle x_0 - xv, x_0\rangle \le \|x_0 - xv\| < 1$$

and therefore $\langle xv, x_0\rangle$ is invertible in the algebra $p\mathcal{B}p$ (with unit p). Therefore one can find r such that also $E_\varphi(\langle xv, x_0\rangle)$ is invertible. Let us put

$$\eta_{x_0}(\varphi_x) = x\mu(E_\varphi(\langle xv, x_0\rangle)),$$

defined on the ball $\{\varphi_x : d_\varphi(\varphi_x, \varphi_{x_0}) < r\}$, where as before, μ denotes the unitary part in the polar decomposition. Then all it remains is to verify that this map η_{x_0} does the job: it is well defined, continuous, and is a cross section for σ.

We shall need the following fact, which is straightforward to verify.

Lemma 3.4. *Suppose that one has the following commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & X \\ & \searrow^{\pi_2} & \Big\downarrow p \\ & & Y, \end{array}$$

where E, X, Y are topological spaces, π_1, π_2 are fibrations and p is continuous and surjective. Then p is also a fibration.

Denote by $\mathcal{E} = \mathcal{E}(\mathcal{L}_{\mathcal{B}}(X))$ the set of projections of $\mathcal{L}_{\mathcal{B}}(X)$. In general, the space of projections of a von Neumann algebra is a differentiable submanifold of the algebra, whose components are the unitary orbits of single projections [13]. Let $\mathcal{E}_e \subset \mathcal{E}$ denote the set of projections which are Murray-von Neumann equivalent to $e \in \mathcal{E}$. It is clear that $\mathcal{E}_e$, being a union of connected components of $\mathcal{E}$, is also a submanifold of $\mathcal{L}_{\mathcal{B}}(X)$. There is another natural map associated to $\mathcal{O}_\varphi$,

$$\mathcal{O}_\varphi \to \mathcal{E}_e, \quad \varphi_x \mapsto e_x,$$

where e is any projection of the form e_{x_0} for some $x_0 \in \mathcal{S}_p(X)$ (they are all equivalent). Since $e_x = \mathrm{supp}(\varphi_x)$, we shall call this map supp. In general, taking support of positive functionals does not define a continuous map. However it is continuous in this context, i.e. restricted to the set $\mathcal{O}_\varphi$ with the metric d_φ. Indeed, as seen before, convergence of $\varphi_{x_n} \to \varphi_x$ in this metric implies the existence of unitaries v_n of $\mathcal{B}_p^\varphi \subset p\mathcal{B}p$ such that $x_n v_n \to x$ in $\mathcal{S}_p(X)$. This implies that $e_{x_n v_n} = e_{x_n} \to e_x$. Moreover, one has

Theorem 3.5. *The map* supp $: \mathcal{O}_\varphi \to \mathcal{E}_e$ *is a fibration with fibre $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. One has the following commutative diagram of fibre bundles*

$$\begin{array}{ccc} \mathcal{S}_p(X) & \xrightarrow{\rho} & \mathcal{O}_\varphi \\ & \searrow^{\sigma} & \Big\downarrow \mathrm{supp} \\ & & \mathcal{E}_e. \end{array}$$

This is a consequence of 3.4, and the fact that $\mathcal{S}_p(X) \to \mathcal{E}_e$ is a fibre bundle [3].

One can use the homotopy exact sequences of these bundles to relate the homotopy groups of $\mathcal{O}_\varphi$, $\mathcal{S}_p(X)$, $\mathcal{E}_e$, $U_{p\mathcal{B}p}$, $U_{\mathcal{B}_p^\varphi}$ and $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. There are many results concerning the homotopy groups of the unitary group of a von Neumann algebra, the survey by Schröder [27] is an excellent reference to these. The homotopy groups of $\mathcal{S}_p(X)$ where considered in [2], [3]. Finally, the set $\mathcal{U}_\varphi$ was considered in the previous section. The sequences are:

$$\dots \pi_n(U_{\mathcal{B}_p^\varphi}, p) \to \pi_n(\mathcal{S}_p(X), x_0) \overset{\sigma_*}{\to} \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \to \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p) \to \dots$$

where x_0 is a fixed element in $\mathcal{S}_p(X)$, and

$$\dots \pi_n(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \overset{\mathrm{supp}_*}{\to} \pi_n(\mathcal{E}, e_{x_0}) \to \pi_{n-1}(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \dots$$

with φ a fixed state in $\Sigma_p(\mathcal{B})$.

In [3] it was shown that if $\mathcal{B}$ is finite von Neumann algebra, then $\mathcal{S}_p(X)$ is connected. It follows that if $\mathcal{B}$ is finite, then $\mathcal{O}_\varphi$ is connected as well. Let us cite some conclusions which follow from direct observation of the above sequences:

(1)
$$\pi_1(\mathcal{O}_\varphi, \varphi_x) \simeq \pi_1(\mathcal{E}_e, e_x).$$
If moreover Xp is selfdual (as a $p\mathcal{B}p$-module), then $\pi_1(\mathcal{O}_\varphi, \varphi_x) = 0$. For the first assertion we use the fact proved in the previous section, that $\mathcal{U}_\varphi = U_\mathcal{B}/U_{\mathcal{B}^\varphi}$ is simply connected. For the second, we use [3] that unitary orbits of projections of a von Neumann algebra are simply connected.

(2) If Xp is selfdual, then for any $x_0 \in \mathcal{S}_p(X)$ fixed and any closed continuous path $x(t) \in \mathcal{S}_p(X)$, with $x(0) = x(1) = x_0$, there exists a path of unitaries $v(t)$ in $\mathcal{B}_p^\varphi$, with $v(0) = v(1) = p$, such that $x(t)$ is homotopic to $x_0v(t)$. This is because the inclusion map $i : U_{\mathcal{B}_p^\varphi} \hookrightarrow \mathcal{S}_p(X)$ given by $v \mapsto x_0v$ is onto at the π_1 level.

(3) Suppose that Xp is selfdual and $p\mathcal{B}p$ is properly infinite, then for $n \geq 1$
$$\pi_n(\mathcal{O}_\varphi, \varphi_x) \simeq \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p).$$

(4) The same conclusion follows if Xp is selfdual, $p\mathcal{B}p$ is of type II_1 and $\mathcal{L}_\mathcal{B}(X)$ is properly infinite. This is the case if for example $p\mathcal{B}p$ is a II_1 factor and Xp is not finitely generated.

These last two follow from the fact that if one has either of the two conditions, then $\mathcal{S}_p(X)$ is contractible ([3]). A consequence from these is that (in both situations) $\pi_1(\mathcal{O}_\varphi)$ is trivial. But $\pi_2(\mathcal{O}_\varphi)$ may not, because $\mathcal{B}_p^\varphi$ is a finite von Neumann algebra ([26], [15]), which can have non trivial π_1 group.

4. Modular vector states

The set we consider in this section is the union of the orbits $\mathcal{O}_\varphi$, with φ ranging in the set $\Sigma_p(\mathcal{B})$ of normal states with support p, and p fixed. It was remarked before that these states are characterized as states of $\mathcal{L}_\mathcal{B}(X)$ with support *equivalent* to e_x, for any $x \in \mathcal{S}_p(X)$. Recall that if $\mathcal{A}$ is a von Neumann algebra and $q \in \mathcal{A}$ is a projection, $P\Sigma_q(\mathcal{A})$ denotes the set of normal states with support equivalent to q. Our set is then $P\Sigma_e(\mathcal{L}_\mathcal{B}(X))$, with $e = e_x$ as above. We continue in the fashion of relating our sets with other spaces already studied. The natural map to study here is
$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to P\Sigma_e(\mathcal{L}_\mathcal{B}(X)), \quad (x, \varphi) \mapsto \varphi_x.$$
Let us endow $P\Sigma_e(\mathcal{L}_\mathcal{B}(X))$ with the quotient topology given by this map, where $\mathcal{S}_p(X)$ is considered with the norm topology of X, and $\Sigma_p(\mathcal{B})$ with the norm topology of $\mathcal{B}^*$. We shall find a metric which induces this topology. First note that the unitary group $U_{p\mathcal{B}p}$ acts both on $\mathcal{S}_p(X)$ (via the right action of the module X) and on $\Sigma_p(\mathcal{B})$ (by inner conjugation, $u.\varphi = \varphi_u$, the action introduced in section 1). Consider the diagonal action of $U_{p\mathcal{B}p}$ on the product of both spaces. It is easy to see, using 3.4, that the set $P\Sigma_e(\mathcal{L}_\mathcal{B}(X))$ is the quotient of $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$ by this diagonal action.

Proposition 4.1. *The metric d in $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ given by*

$$d(\Phi,\Psi)=\|\Phi-\Psi\|+\|\operatorname{supp}(\Phi)-\operatorname{supp}(\Psi)\|$$

induces the same topology as the quotient topology described above.

We omit the proof, which can be found in [6]. If $\mathcal{L}_{\mathcal{B}}(X)$ is not finite dimensional, this metric is stronger than the norm metric. It is not hard to find examples. On the other hand, if $\mathcal{L}_{\mathcal{B}}(X)$ is finite dimensional, it can be proved that taking the support is continuous when one restricts to states with equivalent support. We shall return to this question of continuity of the support under certain conditions.

Note that the inclusion $(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)),d)\subset\mathcal{L}_{\mathcal{B}}(X)^*$ is continuous.

At this point it will be convenient to give a name to the map $(x,\varphi)\mapsto\varphi_x$.

Theorem 4.2. *The map $\wp_1:\mathcal{S}_p(X)\times\Sigma_p(\mathcal{B})\to P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$, $\wp_1(x,\varphi)=\varphi_x$ is a principal fibre bundle with fibre $U_{p\mathcal{B}p}$.*

We give an outline of the proof. We shall use here the projective bundle studied in [3],

$$\rho:\mathcal{S}_p(X)\to\mathcal{E}_e,\quad \rho(x)=e_x$$

which is a principal fibre bundle also with fibre $U_{p\mathcal{B}p}$. To prove our statement it suffices to exhibit a local cross section around a generic base point φ_x. We claim that there is a neighborhood of φ_x such that elements ψ_y in this neighborhood satisfy that $\langle y,x\rangle$ is invertible. Indeed, if $d(\varphi_x,\psi_y)<r$, then $\|e_x-e_y\|<r$. If we choose r small enough so that e_y lies in the ball around e_x in which a local cross section of $\rho(x)=e_x$ is defined, then there exists a unitary u in $p\mathcal{B}p$ such that $\|x-yu\|<1$. Note that

$$\|p-\langle yu,x\rangle\|=\|\langle x-yu,x\rangle\|\le\|x-yu\|<1.$$

Then $\langle yu,x\rangle=u^*\langle y,x\rangle$ is invertible in $p\mathcal{B}p$, and therefore also $\langle y,x\rangle$. In this neighborhood put

$$s(\psi_y)=(y\mu(\langle y,x\rangle),\psi\circ Ad(\mu(\langle y,x\rangle)),$$

where μ denotes the unitary part in the polar decomposition of invertible elements in $p\mathcal{B}p$ as before, and $Ad(v)(x)=vxv^*$. The proof finishes by showing that s is well defined, is a local cross section and is continuous.

Now we have seen that

$$P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))\simeq\mathcal{S}_p(X)\times\Sigma_p(\mathcal{B})/(x,\varphi)\sim(xu,\varphi_u).$$

There arise two more natural maps, namely

$$\wp_2:\mathcal{S}_p(X)\times\Sigma_p(\mathcal{B})\ /\ (x,\varphi)\sim(xu,\varphi_u)\to\mathcal{S}_p(X)/x\sim xu\,,\ \wp_2([(x,\varphi)])=[x]$$

with fibre $\Sigma_p(\mathcal{B})$, and

$$\wp_3:\mathcal{S}_p(X)\times\Sigma_p(\mathcal{B})\ /\ (x,\varphi)\sim(xu,\varphi_u)\to\Sigma_p(\mathcal{B})/\varphi\sim\varphi_u\,,\ \wp_3([(x,\varphi)])=[\varphi]$$

with fibre $\mathcal{S}_p(X)$.

We will see that $\wp_2$ is a fibre bundle, but that $\wp_3$, which is far more interesting, is not. To see this, consider the case when $X=\mathcal{B}$ is a finite algebra, and $p=1$. Here $\mathcal{L}_{\mathcal{B}}(\mathcal{B})=\mathcal{B}$ and $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ consists of the states of $\mathcal{B}$ with support equivalent to 1 (note that $x\in\mathcal{S}_1(X)$ verifies $x^*x=1$, i.e. $x\in U_{\mathcal{B}}$, and $e_x=1$). That is, $P\Sigma_e(\mathcal{B})$ is

the set of faithful states of $\mathcal{B}$ ($= \Sigma_1(\mathcal{B})$ in our notation). It follows that $\wp_3$ is just the quotient map
$$\Sigma_1(\mathcal{B}) \to \Sigma_1(\mathcal{B})/U_{\mathcal{B}}.$$
Take $\mathcal{B} = M_n(\mathbb{C})$ ($n < \infty$). Then the quotient map above is not a weak fibration. Indeed, both sets $\Sigma_1(M_n(\mathbb{C}))$ and $\Sigma_1(M_n(\mathbb{C}))/U_{M_n(\mathbb{C})}$ are convex metric spaces. The latter can be identified, using the density matrices, as the n-tuples of eigenvalues $(\lambda_1, ..., \lambda_n)$ arranged in decreasing order and normalized such that $\sum \lambda_k = 1$, with the ℓ_1 distance. If this quotient map were a weak fibration, then the fibre would have trivial homotopy groups of all orders. This is clearly not the case, since the fibre is the unitary group $U(n)$ of $M_n(\mathbb{C})$.

We focus on the other map $\wp_2$. First note that the quotient $\mathcal{S}_p(X)/x \sim xu$ is homeomorphic to $\mathcal{E}_e$ (recall the bundle ρ). The map can be written in the following fashion
$$\wp_2 : P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)) \to \mathcal{E}_e, \quad \wp_2(\varphi_x) = e_x.$$
Recall that $\mathrm{supp}(\varphi_x) = e_x$, so that this map could also have been named supp. The next result shows that taking support, under the current circumstances, is a fibration:

Theorem 4.3. *The map $\wp_2 : P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)) \to \mathcal{E}_e$, given by $\wp_2(\varphi_x) = e_x$ is a fibration with fibre $\Sigma_p(\mathcal{B})$.*

Consider the diagram

$$\begin{array}{ccc} \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) & \xrightarrow{\wp_1} & P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)) \\ & \searrow \pi & \downarrow \wp_2 \\ & & \mathcal{E}_e, \end{array}$$

where π is given by $p(x, \varphi) = e_x$. Clearly π is a fibre bundle, because it is the composition of the projective bundle $x \mapsto e_x$ with the projection $(x, \varphi) \mapsto x$. The map $\wp_1$ was shown to be a fibration. It follows from 3.4 that $\wp_2$ is a fibration.

As in the previous section, let us write down the homotopy exact sequences of these fibrations:
$$\begin{aligned} \ldots \pi_n\left(U_{p\mathcal{B}p}, p\right) \to \pi_n(\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}), (x_0, \varphi)) &\overset{(\wp_1)_*}{\to} \pi_n(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), \varphi_{x_0}) \\ \to \pi_{n-1}(U_{p\mathcal{B}p}, p) \to \ldots \end{aligned}$$
and
$$\ldots \pi_n(\Sigma_p(\mathcal{B}), \varphi) \to \pi_n(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), \varphi_{x_0}) \overset{(\wp_2)_*}{\to} \pi_n(\mathcal{E}_e, e) \to \pi_{n-1}(\Sigma_p(\mathcal{B}), \varphi) \ldots$$

First note that $\Sigma_p(\mathcal{B})$ is convex, therefore $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$ has the same homotopy type as $\mathcal{S}_p(X)$, and
$$\pi_*(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)) = \pi_*(\mathcal{E}_e).$$
Also note that if $p\mathcal{B}p$ is finite (i.e. p is finite in $\mathcal{B}$) then $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ is connected.

Let us state now two consequences, which again follow from direct inspection of the above sequences:

(1) If $p\mathcal{B}p$ is properly infinite, then for $n \geq 0$

$$\pi_n(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), \varphi_x) = 0.$$

It would be interesting to know if in such circumstances $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ is contractible.

(2) In general, one has

$$\pi_1(P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), \varphi_x) = 0.$$

Finally let us remark that the natural inclusion $(\mathcal{O}_\varphi, d_\varphi) \to (P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), d)$ is continuous (recall that taking support is continuous in $\mathcal{O}_\varphi$). However the identity mapping $(\mathcal{O}_\varphi, d_\varphi) \to (\mathcal{O}_\varphi, d)$ is not (in general) a homeomorphism ([6]).

5. Purification of $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$

One may argue not without reason that the metric d induces on $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ a topology which is weird, or at least too strong. However, note that this topology is forced on us if we consider the sets $\mathcal{S}_p(X)$ and $\Sigma_p(\mathcal{B})$ with their norm topologies. If one is interested in the set $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$ with, for example, its usual norm topology, as a subset of the dual space of $\mathcal{L}_{\mathcal{B}}(X)$, then one must weaken the topologies on $\mathcal{S}_p(X)$ or $\Sigma_p(\mathcal{B})$. We choose to do the first thing: we will consider (again) $\mathcal{B}$ a von Neumann algebra and X a selfdual module, which is then a conjugate space [21]. We shall endow $\mathcal{S}_p(X) \subset X$ with the relative w^*-topology.

First we shall recall a faithful representation ρ of $\mathcal{L}_{\mathcal{B}}(X)$ as operators in a Hilbert space $\mathcal{H}$, on which all the modular vector states φ_x will become genuine vector states, that is, on which one can find unit vectors $f \in \mathcal{H}$ which implement φ_x. This representation was introduced and studied in the seminal papers by M. Rieffel [25] and W. Paschke [21] on Hilbert C^*-modules over non commutative operator algebras.

Let us pick what is called a standard representation for $\mathcal{B}$ on a Hilbert space H. A standard representation has many remarkable properties. Among them, there is a cone $\mathcal{P} \subset H$, called the standard positive cone, such that every normal positive functional of $\mathcal{B}$ is implemented by a unique vector in this cone.

Consider the algebraic tensor product $X \otimes H$, and on this vector space consider the positive semidefinite form given by $[x \otimes \xi, y \otimes \eta] = (\xi, \langle x, y\rangle \eta)$. Denote by $Z = \{z \in X \otimes H : [z, z] = 0\}$, and let $\mathcal{H}$ be the Hilbert space obtained as the completion of the pre-Hilbert space $X \otimes H/Z$ with the positive definite form induced by $[\ ,\ ]$ on the quotient. The representation $\rho : \mathcal{L}_{\mathcal{B}}(X) \to B(\mathcal{H})$ is given by $\rho(t)(x \otimes \xi + Z) = t(x) \otimes \xi + Z$.

Let us state without proof the basic properties which make this representation useful to our study. Let us denote

$$\mathcal{A}(X) = \{x \otimes \xi + Z : x \in \mathcal{S}_p(X), \xi \text{ implements a state in } \Sigma_p(\mathcal{B})\}.$$

Proposition 5.1. *Let $\mathcal{B} \subset B(H)$ be a finite algebra in standard form, and $\mathcal{P} \subset H$ the positive standard cone.*

(1) *Let $\xi \in \mathcal{P}$ be the unique unit vector implementing φ ($\varphi(a) = \omega_\xi(a) = (a\xi, \xi)$), then the state φ_x is implemented by the vector $x \otimes \xi + Z$, namely*

$$\varphi_x(t) = \omega_{x\otimes\xi+Z}(\rho(t)) = [\rho(t)(x \otimes \xi), x \otimes \xi], \quad t \in \mathcal{L}_{\mathcal{B}}(X).$$

(2) *Let $x, y \in \mathcal{S}_p(X)$ and $\xi, \eta \in \mathcal{P}$, where ξ and η implement states in $\mathcal{B}$ with support p. Then the elementary tensors $x \otimes \xi$ and $y \otimes \eta$ induce the same element in $\mathcal{A}(X)$ (i.e. $x \otimes \xi - y \otimes \eta \in Z$) only if $x = y$ and $\xi = \eta$. In other words, there is a bijection*

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \leftrightarrow \mathcal{A}(X), \quad (x, \varphi) \to x \otimes \xi + Z.$$

The proofs follow straightforward from the definitions and the basic properties of the standard representation.

In order to simplify the exposition, we shall restrict to the case $p = 1$. This in fact will mean no restriction, because $\mathcal{S}_p(X)$ is the unit sphere of the $p\mathcal{B}p$-module Xp. But with this simplification, unit vectors implement states with support $p = 1$ if and only if they are cyclic and separating for $\mathcal{B}$ (the algebra $p\mathcal{B}p$ would have appeared anyway).

This proposition enables one to replace the map

$$\wp_1 : \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \to P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)), \quad \wp_1(x, \varphi) = \varphi_x$$

with the map

$$\omega : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}, \quad \omega(x \otimes \xi + Z) = \omega_{x\otimes\xi+Z},$$

where $\omega_{x\otimes\xi+Z}$ is the vector state induced by $x \otimes \xi + Z \in \mathcal{H}$, restricted to the von Neumann algebra $\rho(\mathcal{L}_{\mathcal{B}}(X))$, and $\Omega_{\mathcal{A}(X)}$ is the set of all ω_f for $f \in \mathcal{A}(X)$. In what follows we simplify the notation: $\rho(\mathcal{L}_{\mathcal{B}}(X))$ will be identified with $\mathcal{L}_{\mathcal{B}}(X)$, and the vectors $x \otimes \xi + Z$ will be denoted by $x \otimes \xi$. There is no ambiguity with this respect, because as shown above, there is only one such representative $x \otimes \xi$ in each coset $x \otimes \xi + Z$.

This standpoint enables one to study the set $P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X)) \sim \Omega_{\mathcal{A}(X)}$ with the norm topology of the dual of $\mathcal{L}_{\mathcal{B}}(X)$, while still having the map $\wp_1 \sim \omega$ to be a fibration, however this latter under strong assumptions on the algebra $\mathcal{B}$. It is for us an interesting problem if the map ω can be a fibration for a broader family of algebras.

The set $\mathcal{A}(X) \subset \mathcal{H}$ comes equipped with the Hilbert space norm topology. Much of the rest of this section is devoted to establish that:

(1) When such election is done, the bijection $\mathcal{A}(X) \leftrightarrow \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$ induces on this set the product topology of the w^*-topology of $\mathcal{S}_1(X) \subset X$ times the norm topology of $\Sigma_1(\mathcal{B}) \subset \mathcal{B}^*$.
(2) The quotient topology induced in $\Omega_{\mathcal{A}(X)}$ by the Hilbert space norm in $\mathcal{A}(X)$ and the map ω is the norm topology of $\mathcal{L}_{\mathcal{B}}(X)^*$.

Certain facts have to be established in the way. Recall that the other main feature of the standard representation of a von Neumann algebra, other than the positive cone $\mathcal{P}$, is the antiunitary operator $J : H \to H$ leaving the cone $\mathcal{P}$ fixed. Let us first examine the fibre of ω (which is a copy of $U_{\mathcal{B}}$) in this context.

Proposition 5.2. *Given a fixed element $x \otimes \xi \in \mathcal{A}(X)$, the fibre $\omega^{-1}(\omega_{x\otimes\xi})$ is the set $\{xu \otimes u^* Ju^* J\xi : u \in U_{\mathcal{B}}\}$ which is in one to one correspondence with $U_{\mathcal{B}}$. The relative topology induced on $U_{\mathcal{B}}$ by this bijection is the strong operator topology.*

The proof is straightforward.

Since $\mathcal{A}(X) \sim \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$, the sphere $\mathcal{S}_1(X)$ and the set $\Sigma_1(\mathcal{B})$ of faithful states of $\mathcal{B}$ lie inside $\mathcal{A}(X)$. Let us make explicit these inclusions, and their induced topologies. Pick a fixed element $x_0 \in \mathcal{S}_1(X)$ and $\xi_0 \in \mathcal{P}$ unit, cyclic and separating, inducing the state φ_0. The following maps are one to one:

$$\mathcal{S}_1(X) \to \{x \otimes \xi_0 : x \in \mathcal{S}_1(X)\} \subset \mathcal{A}(X), \quad x \mapsto x \otimes \xi_0,$$

and

$$\Sigma_1(\mathcal{B}) \to \{x_0 \otimes \xi : \xi \in \mathcal{P} \text{ unit, cyclic and separating}\},$$
$$\varphi \mapsto x_0 \otimes \xi,$$

where ξ is the vector in the cone associated to φ.

The first bijection endows $\mathcal{S}_1(X)$ with the relative topology induced from $\mathcal{H}$, which is given by the following: a net x_α converges to x if and only if $\varphi_0(\langle x_\alpha - x, x_\alpha - x\rangle) \to 0$, if and only if $|x_\alpha - x| \to 0$ in the strong operator topology of $\mathcal{B} \subset B(H)$. The sphere $\mathcal{S}_1(X) \subset X$ is closed in this topology.

The second bijection is a homeomorphism when $\Sigma_1(\mathcal{B})$ is regarded with the norm topology and $\{x_0 \otimes \xi : \xi \in \mathcal{P}$ unit, cyclic and separating$\} \subset \mathcal{H}$ is regarded with the Hilbert space norm of $\mathcal{H}$.

Remark 5.3. Since X is selfdual, it is a conjugate space [21]. The result above shows that the topology of $\mathcal{S}_1(X)$ induced by the Hilbert space norm of $\mathcal{H}$ coincides with the w^* topology of $X \supset \mathcal{S}_1(X)$. Indeed, it was shown in [21] that a net $x_\alpha \to x$ in the w^* topology if and only if $\varphi(\langle x_\alpha, y\rangle) \to \varphi(\langle x, y\rangle)$ for all $y \in X$, $\varphi \in \mathcal{B}_*^+$. This clearly implies that $\varphi(\langle x_\alpha - x, x_\alpha - x\rangle) \to 0$, which is the topology considered above (here the fact $\langle x, x\rangle = \langle x_\alpha, x_\alpha\rangle = 1$ is crucial). Conversely

$$\varphi(\langle x_\alpha - x, y\rangle) \leq \varphi(\langle x_\alpha - x, x_\alpha - x\rangle)^{1/2} \varphi(\langle y, y\rangle)^{1/2}$$

yields the other implication.

We have examined the topologies induced on $\mathcal{S}_1(X)$ and $\Sigma_1(\mathcal{B})$ by the described inclusions on $\mathcal{A}(X)$. We have seen also that $\mathcal{A}(X) \sim \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$. These facts alone however do not imply that $\mathcal{A}(X)$ is homeomorphic to $\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$ in the product topology (of the w^* topology and the norm topology respectively). The next result states that this is the case.

Theorem 5.4. *The bijection*

$$\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \to \mathcal{A}(X), \quad (x, \varphi) \mapsto x \otimes \xi$$

is a homeomorphism when $\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$ is endowed with the product topology of the w^* topology of $\mathcal{S}_1(X)$ and the norm topology of $\Sigma_1(\mathcal{B})$.

We omit the proof (see [6]).

Note that since $\Sigma_1(\mathcal{B})$ is convex, this implies that the set $\mathcal{A}(X) \subset \mathcal{H}$ is homotopically equivalent to the sphere $\mathcal{S}_1(X)$ in the w^* topology.

Now we focus on the set of states $\Omega_{\mathcal{A}(X)}$ ($\sim P\Sigma_e(\mathcal{L}_{\mathcal{B}}(X))$) and the map ω. $\Omega_{\mathcal{A}(X)} \subset \mathcal{L}_{\mathcal{B}}(X)^*$ can be endowed with two topologies, the norm topology of the dual space $\mathcal{L}_{\mathcal{B}}(X)^*$ and the quotient norm given by ω. As said before, this two coincide, a fact which will be useful when trying to find fibration properties for this map. In general, a map of the form $f \mapsto \omega_f|_{\mathcal{A}}$ defined on a certain set of vectors f of a Hilbert space on which $\mathcal{A}$ acts will define a (quotient) topology stronger than the topology given by the norm of the functionals. Recall Bures metric [12] for states, which is a metric giving a topology equivalent to the norm topology and, roughly speaking, is defined as the infimum of the distances between vectors inducing the states measured, taken over all possible representations on which the states measured are vector states.

Theorem 5.5. *The quotient and the norm topology coincide in $\Omega_{\mathcal{A}(X)}$.*

The proof is elementary but rather long (see [6]).

In order to see if this map is a fibration, we shall look for local cross sections. A powerful tool to state the existence of cross sections is Michael's theory of continuous selections [20]. A remarkable example of the use of this theory in the context of operator algebras is the paper by S. Popa and M. Takesaki [24], which will be widely used in this paper. To invoke Michael's theorem one must check first that the set function $\omega_{z\otimes\xi} \mapsto \omega^{-1}(\omega_{z\otimes\xi})$ which assigns to each point in the base space the fibre over it, is *lower semicontinuous* [20].

In our context lower semicontinuity means that for any $r > 0$, and $x \otimes \xi \in \mathcal{A}(X)$ the set $\{\omega_{y\otimes\eta} : \|y \otimes Ju^*\eta - x \otimes \xi\| < r \text{ for some } u \in U_{\mathcal{B}}\}$ is open in $\Omega_{\mathcal{A}(X)}$. In other words, for a state $\omega_{y\otimes\eta}$ close to $\omega_{x\otimes\xi}$ one should find an element $y \otimes Ju^*\eta$ in the fibre of $\omega_{y\otimes\eta}$ at distance less than r to the fibre of $\omega_{x\otimes\xi}$. The theorem above states that this is granted for our map ω. Indeed, two states in $\Omega_{\mathcal{A}(X)}$ are close in this quotient topology if and only if there are elements in their fibres which are close in $\mathcal{A}(X)$.

The next result uses the proof of the crucial lemma 3 of the paper by S. Popa and M. Takesaki [24]. They consider separable von Neumann factors admitting a one parameter group of automorphism which scales the trace. This means that there is a one parameter group $\{\theta_s : s \in \mathbb{R}\}$ of automorphisms of $\mathcal{B} \otimes B(K)$ (K a separable Hilbert space) such that $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$, with τ a faithful normal trace of $\mathcal{B} \otimes B(K)$. This condition on $\mathcal{B}$ is strong, but there are remarkable examples fulfilling it, most notably the hyperfinite II_1-factor $\mathcal{R}_0$.

Theorem 5.6. *If $\mathcal{B}$ is a separable factor of type II_1 such that the tensor product $\mathcal{B} \otimes B(K)$ (K a separable Hilbert space) admits a one parameter automorphism group, then the map*

$$\omega : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}, \quad \omega(x \otimes \xi) = \omega_{x\otimes\xi}$$

admits a (global) continuous cross section when $\Omega_{\mathcal{A}(X)}$ is endowed with the norm topology of $\mathcal{L}_{\mathcal{B}}(X)^*$.

Proof. In this case, since $\mathcal{B}$ is finite, $U_{\mathcal{B}}$ is complete in the strong (= strong*) operator topology [29]. Moreover, Popa and Takesaki proved in [24] that it admits a geodesic structure in the sense of Michael [20]. It has been already remarked that the set function $\omega_{x\otimes\xi} \mapsto \{xu \otimes u^*Ju^*J\xi : u \in U_{\mathcal{B}}\}$ is lower semicontinuous in the norm topology. Therefore theorem 5.4 of [20] applies, and ω has a continuous cross section. □

Corollary 5.7. *If $\mathcal{B}$ is a II_1 factor satisfying the conditions of 5.6, then for all $n \geq 0$, $x \in \mathcal{S}_1(X)$, $\varphi = \omega_\xi \in \Sigma_1(\mathcal{B})$,*

$$\pi_n(\Omega_{\mathcal{A}(X)}, \omega_{x\otimes\xi}) = \pi_n(\mathcal{S}_1(X), x),$$

where $\Omega_{\mathcal{A}(X)}$ is considered with the norm topology, and $\mathcal{S}_1(X)$ with the w^* topology.

Proof. In [24] it was proven that the unitary group $U_{\mathcal{B}}$ of such a factor is contractible in the ultra strong operator topology, and therefore also in the strong operator topology. The statement follows using the above result, recalling that the fibre of the fibration $\vec{\wp_1}$ is $U_{\mathcal{B}}$ with this topology. □

6. States of the hyperfinite II_1 factor

We will apply the results of the previous section to obtain our main result, namely, that the set $P\Sigma_p(\mathcal{R}) \subset \mathcal{R}^*$ of states of $\mathcal{R}$, a factor satisfying the hypothesis of 5.6, having support equivalent to a given projection p, considered with the norm topology, has trivial homotopy groups of all orders.

To do so, we must first construct the appropriate module X. If $\mathcal{R}$ is a von Neumann factor satisfying 5.6, and $p \in \mathcal{R}$ is a proper projection, put $X = \mathcal{R}p$ and $\mathcal{B} = p\mathcal{R}p$. $\mathcal{B}$ is also factor which verifies the hypothesis of 5.6. Note that $\langle X, X\rangle = \text{span}\{px^*yp : x, y \in \mathcal{R}p\} = p\mathcal{R}p = \mathcal{B}$ in this case. Therefore by 2.2 of [22], $\{\theta_{x,y} : x, y \in X\}$ spans an ultraweakly dense two sided ideal of $\mathcal{L}_{\mathcal{B}}(X)$. On the other hand, it is clear that $\mathcal{R} \subset \mathcal{L}_{\mathcal{B}}(X)$ as left multipliers, and also that $\theta_{x,y} \in \mathcal{R}$, for $x, y \in X = \mathcal{R}p$. Indeed, $\theta_{x,y}(z) = x\langle y, z\rangle = xpy^*z$, i.e. left multiplication by $xpy^* \in \mathcal{R}$. Therefore $\mathcal{L}_{\mathcal{B}}(X) = \mathcal{R}$. In particular, if $x \in \mathcal{S}_1(X)$, $e_x = \theta_{x,x} = xpx^*$ which is equivalent to $px^*xp = \langle x, x\rangle = p$ in $\mathcal{L}_{\mathcal{B}}(X)$. The set $P\Sigma_e(\mathcal{B}) = \Omega_{\mathcal{A}(X)}$ equals then the set of states of $\mathcal{R}$ with support (unitarily) equivalent to p. Note that this set is (arcwise) connected in the norm topology. Indeed, if $\mathcal{B}$ is finite, $\mathcal{S}_1(X)$ is connected. Using the map $\wp_1$ of section 4, it follows that any two points in $P\Sigma_e(\mathcal{B})$ can be joined with a path (in $P\Sigma_e(\mathcal{B})$) continuous in the d-topology, and therefore also in the norm topology.

Applying 5.6 in this situation implies the following:

Lemma 6.1. *Let $\mathcal{R}$ be a factor as in 5.6, and $p \in \mathcal{R}$ an arbitrary projection. The set of states of $\mathcal{R}$ with support equivalent to p considered with the norm topology has the same homotopy groups as the set*

$$\mathcal{S}_p(\mathcal{R}) = \{v \in \mathcal{R} : v^*v = p\} \subset \mathcal{R}$$

of partial isometries of $\mathcal{R}$ with initial space p, regarded with the (relative) ultraweak topology.

Proof. In this case $\mathcal{S}_1(X)$ clearly equals $\mathcal{S}_p(\mathcal{R})$ above, and the topology is the w^* (i.e.) ultraweak topology of $\mathcal{R}$. If $p = 0$ the statement is trivial. If $p = 1$ it follows from the strong operator contractibility of $U_{\mathcal{R}}$ for such $\mathcal{R}$ proved in [24]. The case of a proper projection follows from 5.6 and the above remark. □

If $p = 0, 1$, then $\mathcal{S}_p(\mathcal{R})$ is contractible (if $p = 1$, $\mathcal{S}_p(\mathcal{R}) = U_{\mathcal{R}}$). A natural question would be if $\mathcal{S}_p(\mathcal{R})$ is contractible for proper $p \in \mathcal{R}$.

We need the following elementary fact:

Lemma 6.2. *Let $\mathcal{M} \subset B(H)$ be a finite von Neumann algebra, and let $a_n \in \mathcal{M}$ such that $\|a_n\| \leq 1$ and $a_n^* a_n$ tends to 1 in the strong operator topology. Then there exist unitaries u_n in $\mathcal{M}$ such that $u_n - a_n$ converges strongly to zero.*

Proof. Consider the polar decomposition $a_n = u_n|a_n|$, where u_n can be chosen unitaries because $\mathcal{M}$ is finite. Note that $|a_n| \to 1$ strongly. Indeed, since $\|a_n\| \leq 1$, $a_n^* a_n \leq (a_n^* a_n)^{1/2}$. Therefore, for any unit vector $\xi \in H$, $1 \geq (|a_n|\xi, \xi) \geq (a_n^* a_n \xi, \xi) \to 1$. Therefore

$$\|(a_n - u_n)\xi\|^2 = \|u_n(|a_n| - 1)\xi\|^2 \leq \||a_n|\xi - \xi\|^2 = 1 + (a_n^* a_n \xi, \xi) - 2(|a_n|\xi, \xi),$$

which tends to zero. □

In [3] it was proven that for a fixed $x_0 \in \mathcal{S}_1(X)$ the map $\pi_{x_0} : U_{\mathcal{L}_{\mathcal{B}}(X)} \to \mathcal{S}_1(X)$ given by $\pi_{x_0}(U) = U(x_0)$ is onto when $\mathcal{B}$ is finite. In that paper it was considered with the norm topologies. Here we shall regard it with the weak topologies and in the particular case at hand, namely $X = \mathcal{R}p$ and $\mathcal{B} = p\mathcal{R}p$ with $\mathcal{R}$ as above. Then, choosing $x_0 = p \in \mathcal{S}_1(X) = \mathcal{S}_p(\mathcal{R})$, the mapping π_p is

$$\pi_p : U_{\mathcal{R}} \to \mathcal{S}_p(\mathcal{R}), \quad \pi_p(u) = up.$$

Theorem 6.3. *If $\mathcal{R}$ is a factor satisfying the hypothesis of 5.6, then the map π_p above is a trivial principal bundle, when $U_{\mathcal{R}}$ is regarded with the strong operator topology and $\mathcal{S}_p(\mathcal{R})$ is regarded with the ultraweak topology. The fibre is (homeomorphic to) the unitary group of $q\mathcal{R}q$, where $q = 1 - p$, again with the strong operator topology.*

Proof. The key of the argument is again Lemma 3 of [24]. In that result it is shown that the homogeneous space $U_{\mathcal{R}}/U_{\mathcal{M}}$ admits a global continuous cross section, where $\mathcal{M} \subset \mathcal{R}$ are factors satisfying the hypothesis of 5.6, and their unitary groups are endowed with the strong operator topology. In our situation, the fibre of π_p (over p) is the set $\{u \in U_{\mathcal{R}} : up = p\} = \{qwq + p : qwq \in U_{q\mathcal{R}q}\} = U_{q\mathcal{R}q} \times \{p\}$. The fibre is not the unitary group of a subfactor with the same unit, nevertheless the argument carries on anyway. Therefore in order to prove our result it suffices to show that in $\mathcal{S}_p(\mathcal{R})$ the ultraweak topology (equal to the weak operator topology) coincides with the quotient topology induced by the map π_p. In other words, that the bijection

$$U_{\mathcal{R}}/U_{q\mathcal{R}q} \times \{p\} \to \mathcal{S}_p(\mathcal{R}), \quad [u] \mapsto up$$

is a homeomorphism in the mentioned topologies. It is clearly continuous. It suffices to check continuity of the inverse at the point p. Suppose that u_α is a net of unitaries in $U_{\mathcal{R}}$ such that $u_\alpha p$ converges weakly to p. Then we claim that there are unitaries

$qw_\alpha q$ in $q\mathcal{R}q$ such that $qw_\alpha q + p - u_\alpha$ converges strongly to zero, which would end the proof. This amounts to saying that there exists unitaries $qw_\alpha q$ verifying that

$$\mathrm{Re}((qw_\alpha q + p)\xi, u_\alpha \xi) \to \|\xi\|^2$$

for all $\xi \in H$. Now since $u_\alpha p \to p$, one has $u_\alpha p\xi \to p\xi$, the former limit is equivalent to the following

$$\mathrm{Re}(qw_\alpha q\xi, u_\alpha q\xi) \to \|q\xi\|^2.$$

Again, $u_\alpha p \to p$ strongly (and the fact that $\mathcal{R}$ is finite), imply that $qu_\alpha p$, $pu_\alpha q$, $qu_\alpha^* p$ and $pu_\alpha^* q$ all converge to zero strongly. Using that u_α are unitaries, these facts imply that $qu_\alpha^* qu_\alpha q \to q$ strongly. Using the lemma above, for the algebra $\mathcal{M} = q\mathcal{R}q$, and $a_\alpha = qu_\alpha q$, it follows that there exist unitaries $qw_\alpha q$ in $q\mathcal{R}q$ such that $qw_\alpha q - qu_\alpha q$ converges to zero strongly. Since $pu_\alpha q$ also tends to zero, it follows that

$$qw_\alpha q - u_\alpha q = qw_\alpha q - qu_\alpha q - pu_\alpha q \to 0$$

strongly. Clearly this last limit proves our claim. $\square$

Our main result then follows easily

Theorem 6.4. *Let $\mathcal{R}$ be a factor satisfying the hypothesis of 5.6, and let p be a projection in $\mathcal{R}$. Then both $\mathcal{S}_p(\mathcal{R})$ with the ultraweak topology, and the set $P\Sigma_p(\mathcal{B})$ of normal states of $\mathcal{R}$ with support equivalent to p with the norm topology, have trivial homotopy groups of all orders.*

Proof. By the above theorem, $\mathcal{S}_p(\mathcal{R})$ has trivial homotopy groups, since it is the base space of a fibration with contractible space and contractible fibre. The same consequence holds for the set of normal states with support equivalent to p, using 6.1. $\square$

We can restrict the fibration ω to obtain information about the unitary orbit $\mathcal{U}_\varphi$ of section 2, but this time with the norm topology.

Remark 6.5. Let $\mathcal{B}$ be a factor satisfying the condition of 5.6. Consider now the restriction of the fibration $\omega : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}$ to the subset $\{\omega_{x\otimes\xi_0} : x \in \mathcal{S}_1(X)\} \subset \Omega_{\mathcal{A}(X)}$, for a fixed unit, cyclic and separating vector ξ_0 i.e.

$$\{x \otimes \xi_0 : x \in \mathcal{S}_1(X)\} \simeq \mathcal{S}_1(X) \to \{\omega_{x\otimes\xi_0} : x \in \mathcal{S}_1(X)\}, \quad x \otimes \xi_0 \mapsto \omega_{x\otimes\xi_0},$$

which is again a fibration with the relative topologies. Note that the latter set is in one to one correspondence with $\mathcal{O}_\varphi$ of section 3, where $\varphi = \omega_{\xi_0}$. Therefore one recovers the map $\sigma : \mathcal{S}_1(X) \to \mathcal{O}_\varphi$, $\sigma(x) = \varphi_x = \omega_{x\otimes\xi_0}$ of section 2, now considered with the w^* topology for $\mathcal{S}_1(X)$ and the norm topology for $\mathcal{O}_\varphi$. It follows that this map is a fibration, with fibre equal to $U_{\mathcal{R}^\varphi}$ with the strong operator topology.

One can consider this fibration σ in the particular case $X = \mathcal{B} = \mathcal{R}$, for $\mathcal{R}$ as above, to obtain the following:

Corollary 6.6. *Let φ be a faithful normal state of a factor $\mathcal{R}$ as in 5.6. Then the map*

$$\pi_\varphi : U_{\mathcal{R}} \to \mathcal{U}_\varphi = \{\varphi_u : u \in U_{\mathcal{R}}\}, \quad \pi_\varphi(u) = \varphi_u$$

is a fibration when the unitary group $U_{\mathcal{R}}$ is considered with the strong operator topology and the unitary orbit $\mathcal{U}_\varphi$ of φ is considered with the norm topology. The fibre is the unitary group $U_{\mathcal{R}^\varphi}$ of the centralizer of φ also with the strong operator topology. Moreover, for $n \geq 0$ one has

$$\pi_{n+1}(\mathcal{U}_\varphi, \varphi) = \pi_n(U_{\mathcal{R}^\varphi}, 1),$$

where $U_{\mathcal{R}^\varphi}$ is regarded with the strong operator topology.

Proof. It was noted in section 2 that when $X = \mathcal{B}$ is a finite von Neumann algebra, then $\mathcal{S}_1(X)$ is $U_{\mathcal{B}}$ and $\mathcal{O}_\varphi$ is the unitary orbit of φ. $\mathcal{S}_1(X) = U_{\mathcal{B}}$ is endowed with the ultraweak topology, which coincides in $U_{\mathcal{B}}$ with the strong operator topology. The rest of the corollary follows using that in this case $\pi_\varphi = \sigma$ is (the restriction) of a fibration, and again [24] that for such factors $\mathcal{R}$ the unitary group is contractible in the strong operator topology. □

When $n = 0$, since $U_{\mathcal{R}^\varphi}$ is connected, one obtains that $\mathcal{U}_\varphi$ is simply connected in the norm topology as well.

7. Continuity of the support

Finally, let us address again the question of continuity of the support. As remarked before, taking support of a positive functional does not define a continuous map, no matter how weak the topologies involved, even in the finite dimensional setting. However one can check that if the algebra is finite dimensional, taking support is continuous if one restricts to the set of functionals which have a priori equivalent supports.

We shall obtain, as a consequence of theorem 5.5, that if $\mathcal{B}$ is finite, then the map

$$P\Sigma_p(\mathcal{B}) \to \mathcal{E}_p(\mathcal{B}), \quad \psi \mapsto \mathrm{supp}(\psi)$$

is continuous when $P\Sigma_p(\mathcal{B})$ is considered with the norm topology and $\mathcal{E}_p$ with the strong operator topology.

Put $\mathcal{A} = p\mathcal{B}p$ and $X = \mathcal{B}p$, where we make the assumption that $\mathcal{B}$ is finite. A state $\psi \in P\Sigma_p(\mathcal{B})$ is of the form $\tilde{\psi}_v$ for $v \in \mathcal{S}_1(X) = \mathcal{S}_p(\mathcal{B})$ and $\tilde{\psi} \in \Sigma_p(\mathcal{A})$. That is $\psi(x) = \tilde{\psi}(v^*xv)$ for an appropriate partial isometry v of $\mathcal{B}$ with initial space p and final space $\mathrm{supp}(\psi)$. Suppose that $\varphi_n \to \varphi$ in norm, and let $v_n, v \in \mathcal{S}_1(X)$ and $\tilde{\varphi_n}, \tilde{\varphi} \in \Sigma_p(\mathcal{A})$ such that $\varphi_n = \tilde{\varphi_n}_{v_n}$ and $\varphi = \tilde{\varphi}_v$. Theorem 5.5 implies the existence of unitaries $u_n \in U_{\mathcal{A}}$ such that $\tilde{\varphi_n}_{u_n} \to \tilde{\varphi}$ in norm and $v_n u_n \to v$ in the ultraweak topology of $\mathcal{B}$, and therefore also in the strong operator topology. Note that the support of $\varphi_n = \tilde{\varphi_n}_{v_n}$ is the final projection $v_n v_n^*$, and analogously $\mathrm{supp}(\varphi) = vv^*$. Since $\mathcal{B}$ is finite, $v_n u_n \to v$ implies $u_n^* v_n^* \to v^*$. The product is strong operator continuous when the operators involved have their norms uniformly bounded. Therefore

$$\mathrm{supp}(\varphi_n) = v_n v_n^* = v_n u_n (v_n u_n)^* \to vv^* = \mathrm{supp}(\varphi)$$

in the strong operator topology. We have proved the following:

Corollary 7.1. *Let $\mathcal{B}$ be a finite von Neumann algebra. Then the map*

$$\mathrm{supp} : P\Sigma_p(\mathcal{B}) \to \mathcal{E}_p$$

is continuous when $P\Sigma_p(\mathcal{B})$ is considered with the norm topology and $\mathcal{E}_p$ with the strong operator topology.

Fix a faithful and normal tracial state τ on $\mathcal{B}$. For $q \in \mathcal{E}$, denote by τ_q the state given by $\tau_q(x) = \tau(qx)/\tau(q)$. Note that if $q \in \mathcal{E}_p$ then $\tau(q) = \tau(p)$. Let $\mathcal{T}_p = \{\tau_q : q \in \mathcal{E}_p\}$.

Theorem 7.2. *The map $\mathcal{E}_p \to P\Sigma_p(\mathcal{B})$, $q \mapsto \tau_q$ is a continuous cross section for the support map, when $\mathcal{E}_p$ is considered with the strong operator topology, and $P\Sigma_p(\mathcal{B})$ with the norm topology. The set $\mathcal{E}_p$ is homeomorphic to the image $\mathcal{T}_p$ of this section (with the norm topology). Moreover, $\mathcal{T}_p$ is a strong deformation retract of $P\Sigma_p(\mathcal{B})$.*

Proof. It is clear that the support of τ_q is q, therefore this map is a cross section. Let us see that it is continuous. Let p_n converge strongly to q in $\mathcal{E}_p$. Suppose $\mathcal{B}$ represented in a Hilbert space H in such a way that τ is given by a (tracial) vector ν. For instance, take $H = L^2(\mathcal{B}, \tau)$. Then

$$|\tau(p_n a) - \tau(qa)| = |\langle (p_n - q)a\nu, \nu\rangle| = |\langle a\nu, (p_n - q)\nu\rangle| \leq \|a\| \, \|(p_n - q)\nu\|.$$

Dividing by the common trace of all these projections, one obtains that $\tau_{p_n} \to \tau_q$ in norm. It remains to prove the converse, that if $\tau_{p_n} \to \tau_q$ in norm for $p_n, q \in \mathcal{E}_p$, then $p_n \to q$ strongly. Since $H = L^2(\mathcal{B}, \tau)$ is a standard form for $\mathcal{B}$, convergence in norm of the positive functionals τ_{p_n} means convergence in H of their densities $p_n\nu$. It follows that $p_n\nu \to q\nu$. Let a' be an element in the commutant of $\mathcal{B}$. Then $a'p_n\nu = p_n a'\nu \to qa'\nu$. Therefore p_n converges to q in a dense subset of H, namely $\{a'\nu : a' \in \mathcal{B}'\}$. Since p_n, q have norm 1, it follows that $p_n \to q$ strongly.

Consider the continuous map $F_t(\Phi)$ given by

$$F_t(\Phi) = t\,\tau_{\mathrm{supp}(\Phi)} + (1-t)\Phi,$$

for $\Phi \in P\Sigma_p(\mathcal{A})$ and $t \in [0,1]$. Then $F_0 = Id$, F_1 is a retraction onto the image of the cross section of supp (that is, essentially $F_1 = \mathrm{supp}$), and for all $t \in [0,1]$, $F_t(\tau_q) = \tau_q$. □

An immediate corollary of this fact is the following:

Corollary 7.3. *Let $\mathcal{R}$ be a factor satisfying the hypothesis of 5.6, then the set $\mathcal{E}_p$ of projections of $\mathcal{R}$ which are equivalent to p, has trivial homotopy groups of all orders $n \geq 0$, in the strong operator topology.*

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