9

Incomplete Domain Decomposition LU Factorizations

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1 Introduction

The incomplete domain decomposition LU factorizations for the solution of systems of linear equations arising from the discretization of two-dimensional non selfadjoint PDEs are introduced. The construction of the factorizations is presented for positive definite M-matrices. The theoretical discussion is for two subdomains. Multidomain numerical illustrations are also included.

Consider a decomposition of the computational domain Ω into two overlapping subregions, arbitrarily ordered Ω_1 and Ω_2. The original method due to Schwarz [Sch70] consisted of alternating the solution on each subdomain until convergence was achieved. Domain decomposition methods have evolved this idea to the construction of preconditionings. Consider LU factorizations on each subdomain. Using these factorizations, a symmetrized domain decomposition preconditioning solves in domain Ω_1, then solves in domain Ω_2, and finally corrects in domain Ω_1, see [BW86]. The cost per iteration is the cost of 3 LU solves.

The method proposed here has the feel of an LU factorization: forward elimination followed by back substitution. First using the subdomains LU factorizations, forward eliminate in domain Ω_1, carry that information to domain Ω_2 forward eliminate there. Then, the back substitution is completed in the reverse order: first, in domain Ω_2 and then in the original domain Ω_1. The cost per iteration is equivalent to the cost of 2 LU solves. Thus, the cost per iteration of the incomplete domain decomposition LU factorizations is approximately 2/3 of the cost of traditional domain decomposition factorizations.

Just as the original idea of Schwarz and the multiplicative domain decomposition methods have the feel of a Gauss-Seidel iteration on the subdomains, the factorizations proposed here have the feel of a block symmetric Gauss-Seidel. This should make the factorizations proposed here somewhat more robust than traditional domain decomposition factorizations. This is born out in the application to time dependent problems where the step size is adaptively changed for the accuracy of the solution, [Kom96]. Incomplete domain decomposition LU factorizations are able to solve the

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linear systems for larger time steps.

The combination of less cost per iteration and robustness makes this factorization an attractive preconditioner. The incomplete domain decomposition LU factorizations can be extended to multiple subdomains, [DK97]. Furthermore, the multiple subdomains factorization is parallelizable through the use of coloring.

In Section 2, the domain Ω is decomposed into overlapping subdomains and the incomplete domain decomposition factorization is derived. A brief analysis of the factorization is presented in Section 3. The factorization is related to a regular splitting of an expanded matrix, whose dimensions exceed the original matrix according to the amount of overlap. Section 4 reports the results of some numerical experiments illustrating the potential of the factorization.

2 Incomplete Domain Decomposition *LU* Factorizations

The presentation centers in the solution of the linear system

$$Ax = b \tag{1}$$

arising from the finite difference discretization of two-dimensional PDEs on a rectangular domain Ω. A is an $n \times n$ nonsingular matrix, b is a given n-dimensional vector, and x is the n-dimensional unknown vector. The matrix A is assumed to be a positive definite M-matrix. The linear system will be solved using preconditioned conjugate gradient type methods [SSF95, VdV92].

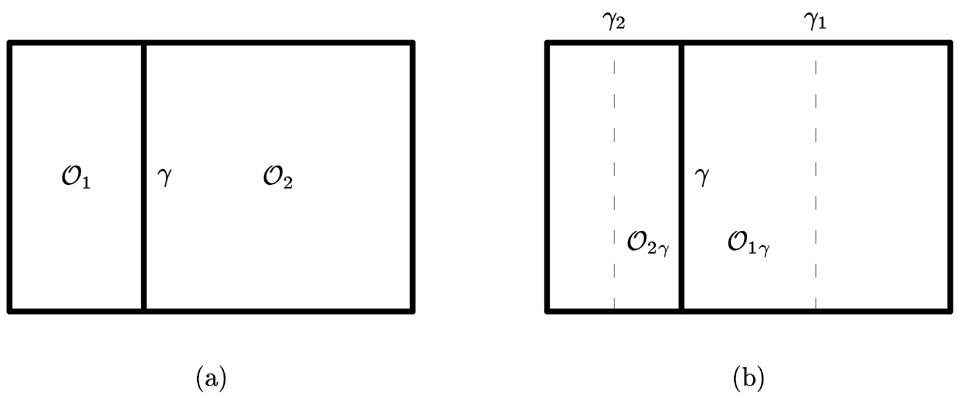
The construction of incomplete domain decomposition LU factorizations is presented for the case of two overlapping subdomains. The extension to several subdomains will be presented elsewhere due to space limitations.

Start by first subdividing Ω into two overlapping subdomains. Then the matrices constructed from the discretization of the restriction of the PDEs on these subdomains are used to construct a matrix G that has a dimension larger than that of A. The incomplete factorizations of A are obtained from the incomplete LU factorizations of G.

Decompose Ω into two non overlapping subdomains $\mathcal{O}_1$ and $\mathcal{O}_2$, and an internal boundary γ such that $\Omega = \mathcal{O}_1 \cup \gamma \cup \mathcal{O}_2$; see Fig. 1a. The internal boundary γ is extended to create subdomains Ω_1 and Ω_2 that overlap and cover Ω. Extend γ to the right and denote the new boundary by γ_1 and the region between γ and γ_1 by $\mathcal{O}_{1\gamma}$. Similarly, extend γ to the left and denote the new boundary by γ_2 and the region between γ and γ_2 is denoted by $\mathcal{O}_{2\gamma}$. The two overlapping regions are defined by $\Omega_1 = \mathcal{O}_1 \cup \mathcal{O}_{1\gamma} \cup \gamma_1$ and $\Omega_2 = \mathcal{O}_2 \cup \mathcal{O}_{2\gamma} \cup \gamma_2$, and the overlap between them is $\Omega_o = \gamma_1 \cup \mathcal{O}_{1\gamma} \cup \gamma \cup \mathcal{O}_{2\gamma} \cup \gamma_2$; see Fig. 1b. Also, let $\Omega_u = \Omega_2 \setminus \Omega_1$ and $\Omega_r = \Omega_2 \setminus \Omega_u$. Then Ω_1 and Ω_u are disjoint and cover Ω, and Ω_r and Ω_u are disjoint subdomains covering Ω_2. It can be seen that $\Omega_o = \Omega_r$.

Now let ω be the set of grid points introduced in Ω after discretizing Ω with mesh size h. Define by $\omega_1 = \omega \cap \Omega_1$ the set of grid points in Ω_1, $\omega_2 = \omega \cap \Omega_2$ the set of grid points in Ω_2, $\omega_u = \omega \cap \Omega_u$ the set of grid points in Ω_u, and $\omega_r = \omega \cap \Omega_r$ the set of grid points in Ω_r. Note that $\omega = \omega_1 \cup \omega_u$ since Ω_1 and Ω_u are disjoint subdomains covering Ω. Note also that $\omega_2 = \omega_u \cup \omega_r$ since Ω_u and Ω_r are disjoint subdomains covering Ω_2. Denote by n_1, n_2, n_u, and n_r the number of grid points in ω_1, ω_2, ω_u, and ω_r. The

Figure 1 (a) Nonoverlapping subdomains and (b) Overlapping subdomains



order of the matrix A of Equation (1), n, is equal to the number of grid points in ω, i.e. $n = n_1 + n_u$.

Let G_{11}, G_{22}, A_{22}^u, and A_{22}^r be the matrices arising from the discretization of the restriction of the PDEs on ω_1, ω_2, ω_u, and ω_r, respectively. The matrix G_{22} can be represented in 2×2 block form as

$$G_{22} = \begin{bmatrix} A_{22}^r & A_{22}^{ru} \\ A_{22}^{ur} & A_{22}^u \end{bmatrix} \begin{array}{c} \omega_r \\ \omega_u \end{array}$$

since ω_r and ω_u are disjoint subsets of ω_2 such that $\omega_2 = \omega_r \cup \omega_u$. Similarly, ω_1 and ω_u are disjoint subsets of ω such that $\omega = \omega_1 \cup \omega_u$, and hence, the matrix A can also be represented in 2×2 block form as

$$A = \begin{bmatrix} A_{11} & A_{12}^u \\ A_{21}^u & A_{22}^u \end{bmatrix} \begin{array}{c} \omega_1 \\ \omega_u \end{array} , \tag{2}$$

where

$$A_{11} = G_{11}, \quad A_{12}^u = \begin{bmatrix} 0 \\ A_{22}^{ru} \end{bmatrix} \begin{array}{c} \omega_1 \setminus \omega_r \\ \omega_r \end{array} , \quad \text{and} \quad A_{21}^u = \begin{array}{c} \begin{array}{cc} \omega_1 \setminus \omega_r & \omega_r \end{array} \\ \begin{bmatrix} 0 & A_{22}^{ur} \end{bmatrix} \end{array} .$$

Let I_k be the identity matrix of order k and $m = n_1 + n_2$. Consider P_1, P_2 and P, respectively $n_1 \times n_1$, $n_2 \times n_u$ and $m \times n$ matrices given by

$$P_1 = I_{n_1}, \quad P_2 = \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \begin{array}{c} \omega_r \\ \omega_u \end{array} , \quad \text{and} \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{array}{c} \omega_1 \\ \omega_2 \end{array} . \tag{3}$$

Let G_{12}, G_{21} and G be respectively $n_1 \times n_2$, $n_2 \times n_1$ and $m \times m$ matrices defined by

$$G_{12} = \begin{array}{c} \begin{array}{cc} \omega_r & \omega_u \end{array} \\ \begin{bmatrix} 0 & A_{12}^u \end{bmatrix} \end{array} , \quad G_{21} = \begin{bmatrix} 0 \\ A_{21}^u \end{bmatrix} \begin{array}{c} \omega_r \\ \omega_u \end{array} , \text{ and } G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{array}{c} \omega_1 \\ \omega_2 \end{array} . \tag{4}$$

Then, the identities hold

$$A_{12}^u = P_1^T G_{12} P_2, \quad A_{21}^u = P_2^T G_{21} P_1, \quad \text{and} \quad A_{22}^u = P_2^T G_{22} P_2.$$

Furthermore, it can readily be checked that the equality $A = P^T G P$ holds.

A is an M-matrix and so are G_{11} and G_{22} which are principal submatrices of A. It follows that the matrix

$$\widetilde{G}_{22} = \begin{bmatrix} A_{22}^r & 0 \\ A_{22}^{ur} & A_{22}^u \end{bmatrix}$$

obtained by setting some of the off-diagonal entries of G_{22} to zero is also an M-matrix. Therefore, there exist traditional splittings [BP94] of G_{11} and $\widetilde{G}_{22}$ such that

$$G_{11} = Q_1 - E_1 \quad \text{and} \quad \widetilde{G}_{22} = Q_2 - E_2,$$

where Q_1^{-1}, Q_2^{-1}, E_1 and E_2 are nonnegative matrices, i.e. the entries of Q_1^{-1}, Q_2^{-1}, E_1 and E_2 are all nonnegative. The matrices Q_1 and Q_2 are derived from the (block) ILU factorizations of G_{11} and $\widetilde{G}_{22}$ and have the form

$$Q_1 = (L_1 + B_1)B_1^{-1}(B_1 + U_1) \quad \text{and} \quad Q_2 = (L_2 + B_2)B_2^{-1}(B_2 + U_2),$$

where L_1 and L_2 are the strictly lower parts of G_{11} and $\widetilde{G}_{22}$; and U_1 and U_2 are the strictly upper parts of G_{11} and $\widetilde{G}_{22}$. The matrices B_1 and B_2 are M-matrices constructed during the factorization process.

Now let B, L, U, Q and $\widetilde{G}$ be matrices of order m defined by

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{matrix} \omega_1 \\ \omega_2 \end{matrix}, \quad L = \begin{bmatrix} L_1 & 0 \\ G_{21} & L_2 \end{bmatrix} \begin{matrix} \omega_1 \\ \omega_2 \end{matrix}, \quad \text{and} \quad U = \begin{bmatrix} U_1 & G_{12} \\ 0 & U_2 \end{bmatrix} \begin{matrix} \omega_1 \\ \omega_2 \end{matrix}$$

$$Q = (L + B)B^{-1}(B + U) \quad \text{and} \quad \widetilde{G} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & \widetilde{G}_{22} \end{bmatrix} \begin{matrix} \omega_1 \\ \omega_2 \end{matrix}. \tag{5}$$

The incomplete domain decomposition preconditioner of A is defined by

$$Q_{IDD} = (P^T Q^{-1} P)^{-1} \tag{6}$$

where Q and P are given in Equation (5) and (3), respectively. Note that the preconditioner has the feel of an LU factorization. Computing the action of Q_{IDD}^{-1} on a vector requires a forward elimination followed by a back substitution. From Equations (2), (3) and (4) it follows that the matrix PAP^T can be written as

$$\begin{aligned} PAP^T &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & \begin{bmatrix} 0 & 0 \\ 0 & A_{22}^u \end{bmatrix} \end{bmatrix} \\ &= G - G_{ur} - G_{ru}, \end{aligned} \tag{7}$$

where

$$G_{ur} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} A_{22}^r & 0 \\ A_{22}^{ur} & 0 \end{bmatrix} \end{bmatrix} \quad \text{and} \quad G_{ru} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & A_{22}^{ru} \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

Note that $G_{ur}P = 0$ and $\widetilde{G} = G - G_{ru}$.

3 Analysis

The matrix $\widetilde{G}$ was constructed, in the previous section, from principal submatrices of the matrix A which has been assumed to be a positive definite M-matrix. From these assumptions the following Lemma can be established [Kom96, DK97].

Lemma 1 *There exists a matrix E such that $\widetilde{G} = Q - E$ is a regular splitting, i.e. the entries of Q^{-1} and E are all nonnegative.*

The stability of the incomplete domain decomposition factorization is established in the following Theorem.

Theorem 1 *The preconditioned system $\mathcal{K} = Q_{IDD}^{-1} A$ is a principal submatrix of $Q^{-1}\widetilde{G}$ given by $\mathcal{K} = P^T Q^{-1} \widetilde{G} P$, and all of the eigenvalues of the preconditioned system $\mathcal{K}$ have positive real part.*

PROOF: First note that $P^T P = I_n$, $G_{ur}P = 0$ and $\widetilde{G} = G - G_{ru}$. Using these and Equations (6) and (7), it follows

$$\begin{aligned} \mathcal{K} &= Q_{IDD}^{-1} A = P^T Q^{-1} P A = P^T Q^{-1} P A P^T P \\ &= P^T Q^{-1} (P A P^T) P = P^T Q^{-1} (G - G_{ur} - G_{ru}) P = P^T Q^{-1} (G - G_{ru}) P \\ &= P^T Q^{-1} \widetilde{G} P. \end{aligned}$$

This establishes the first part of the Theorem.

Using Lemma 1 and the above result, $\mathcal{K}$ can be rewritten as

$$\begin{aligned} \mathcal{K} &= P^T Q^{-1} \widetilde{G} P = P^T Q^{-1} (Q - E) P = P^T (I_m - Q^{-1} E) P \\ &= I_n - P^T Q^{-1} E P. \end{aligned}$$

Since $\widetilde{G} = Q - E$ is a regular splitting, it follows that the spectral radius $\rho(Q^{-1}E)$ of $Q^{-1}E$ is less than unity ([BP94], page 181). Also, since $P^T Q^{-1} E P$ is a principal submatrix of the nonnegative matrix $Q^{-1}E$ it follows that $\rho(P^T Q^{-1} E P) \leq \rho(Q^{-1}E)$ ([BP94], page 28). Finally, the spectral radius $\rho(I_n - \mathcal{K})$ of $I_n - \mathcal{K}$ satisfies the inequality

$$\rho(I_n - \mathcal{K}) = \rho(P^T Q^{-1} E P) \leq \rho(Q^{-1} E) < 1,$$

which shows that all the eigenvalues of $\mathcal{K}$ have positive real parts. *QED*

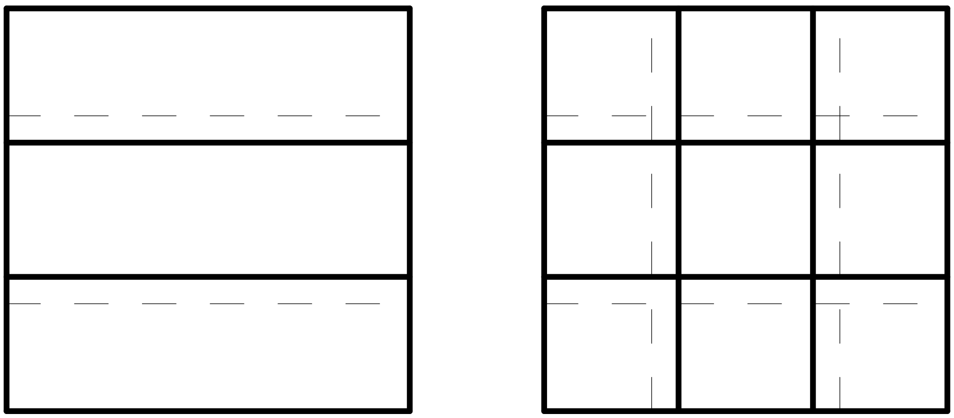
4 Numerical Experiments

The potential of the domain decomposition preconditioners is best illustrated by applying it to cases where the domain Ω has been decomposed into several subdomains. Both box and stripe decompositions of the computational domain Ω are considered; see Fig. 2.

The coefficient matrix of Equation (1) is obtained from the discretization of the PDEs on the unit square $\Omega = (0,1) \times (0,1)$. The following PDE is solved

$$-\Delta u + \gamma \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) + \beta u = f \quad \text{in} \quad \Omega$$

Figure 2 (a) Stripe decomposition (b) Box decomposition. Non overlapping and enlarged subdomains



with Dirichlet boundary conditions where $\gamma = 1000$ and $\beta = -100$. The function f is chosen such that the exact solution is $u = x(1-x)y(1-y)\exp(y)$.

The five-point finite difference scheme is used for the discretization of the PDE on a uniform grid. The first and second order derivatives are approximated using centered differences. Note that although this problem is highly non symmetric, its discretization matrix remains a positive definite M-matrix.

For $n = 32, 64, 128$, a uniform grid is introduced with spacing $h = 1/(n+1)$ in Ω. The matrix A arising from the discretization of the above PDE is a nonsingular M-matrix of order n^2 for each problem.

The linear system $Ax = b$ obtained from the discretization of the PDE is solved using preconditioned Bi-CGSTAB [VdV92] and GMRES(50) methods. The latter is the GMRES method [SSF95] that is restarted after every 50 iterations. The iterative solvers are considered to have converged when the initial residual is reduced by a factor of at least 10^{-6}, that is, the stopping criterion is $\| r_i \|_2 \leq 10^{-6} \| r_0 \|_2$, where $r_i = b - Ax_i$ is the i^{th} residual, x_i is the i^{th} approximation to the solution x, and $\| \bullet \|_2$ is the Euclidean norm. The initial guess is $x_0 = 0$ in all the test runs. The preconditioners used are the incomplete domain decomposition LU factorizations presented in this paper. To construct the preconditioner, compute the block ILU factorizations of the coefficient matrices derived from the discretization of the restriction of the PDE on the overlapping subdomains. The incomplete factorizations for these local matrices are their $INV(1)$ factorizations [CGM85, CM86, Meu89]. The ordering is the natural order. No effort is made to select a particular ordering for the grid points or for the subdomains.

The performance of the preconditioner Q_{IDD} is investigated. Throughout, the Bi-CGSTAB and GMRES(50) used in conjunction with a preconditioning matrix C will

Table 1 Number of iterations required for various grid sizes and overlaps

	Box Decompositions							Stripe Decompositions						
		$n = 32$		$n = 64$		$n = 128$			$n = 32$		$n = 64$		$n = 128$	
Ov	DM	Bi	GM	Bi	GM	Bi	GM	DM	Bi	GM	Bi	GM	Bi	GM
0h	1	4	5	6	8	11	14	1	4	5	6	8	11	14
2h	4	4	5	6	8	10	14	2	4	5	6	8	11	14
4h		4	5	6	8	10	14		4	5	6	8	11	14
6h		4	5	6	8	10	14		4	5	6	8	11	14
8h		4	5	6	8	10	14		4	5	6	8	11	14
2h	16	5	6	7	9	11	16	4	4	6	7	9	11	15
4h		5	6	7	9	11	16		4	6	7	9	11	15
6h		5	6	7	9	11	16		5	6	7	9	11	15
8h				7	9	11	16				7	9	11	15
2h	64	8	10	8	12	13	19	8	5	7	7	10	10	16
4h				8	12	13	19				7	10	11	16
6h				9	12	13	19				7	10	11	16
8h						13	19						11	16
2h	256			14	18	19	25	16			10	13	15	19
4h						19	24						16	19
6h						19	24						16	19

be denoted by Bi-CGSTAB$/C$ and the GMRES(50)$/C$, respectively.

A test is carried out for obtaining the solution of the above problem using the Bi-CGSTAB$/Q_{IDD}$ and GMRES(50)$/Q_{IDD}$ solvers. The numerical calculations were carried out in double precision on a Sun workstation. All calculations are serial. The numerical performance of the preconditioners is considered herein. Their parallel implementations which will be presented elsewhere.

Results

The test results are gathered in Table 1. The overlap between the subdomains is labeled `Ov` and is the same for any two subdomains that overlap. For instance, if `Ov` $= n_{ov}h$, where n_{ov} is a nonnegative integer, then the overlap between any two overlapping subdomains is $n_{ov}h$. In other words, the overlap between the grids corresponding to any two overlapping subdomains is n_{ov} grid lines. The number of subdomains is reported in the column labeled `DM`. The number of iterations taken by Bi-CGSTAB and GMRES(50) methods are reported in columns labeled `Bi` and `GM`, respectively.

In all the test runs, the case DM $= 1$ corresponds to using the $INV(1)$ factorization of A as preconditioner; i.e. Bi-CGSTAB$/INV(1)$ and GMRES(50)$/INV(1)$ methods are used.

In all the test runs, the number of iterations seems to be independent of the size of the overlap. On the other hand, Bi-CGSTAB$/Q_{IDD}$ and GMRES(50)$/Q_{IDD}$

require more iterations as the number of subdomains increases. Preconditioners based on box decompositions take more iterations than those derived from stripe decompositions. The coefficient matrices of the subdomains, however, are larger for stripe decompositions than for box decompositions. Therefore applying the preconditioners requires more computation on the subdomains in the stripe decompositions case than the box decompositions case.

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