MODELING STUDENTS' LEARNING ON MATHEMATICAL PROOF AND REFUTATION

Fou-Lai Lin

Department of Mathematics

National Taiwan Normal University

Based on a national survey and some further studies of mathematical proof and refutation of 7th through 9th graders, this paper will show evidence of the existence of continuity between refuting as a learning strategy and the production of conjectures, and between a more effective teaching strategy and the traditional teaching strategy. A detailed analysis of students' refutation schemes will be presented, and a model of their refuting process will be described based on both their refutation schemes and an expert's thinking process on refutation.

INTRODUCTION

Connecting Teaching with Students' Cognition

Research on students' mathematics cognition usually aims to investigate students' thinking and the strategies used, and further to show what guides students' thinking and why the strategies are used. Information about students' cognition can then naturally be applied to redesigning teaching strategies for enhancing students' learning in mathematics classrooms. Both the students' mathematics cognition and the related teaching modules associated with empirical evidence on its effectiveness are meaningful resources for teachers to learn teaching. Indeed, results of research on students' mathematics cognition proved to be key resources for redesigning teaching modules and reforming curriculum to ensure effective learning (Hart, 1980, 1984; Lin, 1991, 2000; Harel, 2002; Boero et al., 1998, 2002; Duval, 2002).

This paper focuses on investigating teaching and learning strategies to connect students' mathematics cognition for enhancing learning on mathematical proof and refutation. We will analyze cognition on proof and refutation in a specific group of students (about one third of their age population). And, for easy implementation in school practices, we chose the coloring strategy for learning proving, and the refuting strategy for learning conjecturing; both strategies are economic and innovative with new thinking. The evidence of using refuting as a learning strategy to generate innovative conjectures shall be presented.

A Research Program on Argumentation and Mathematics Proof

An ongoing two-staged research program on the development of proof and proving is the main reference in this paper. The first stage (2000~03) studied junior high students' understanding of proof and proving. The second stage (2003~07) is studying teaching and learning of mathematics proof. Three phases were carried out during the first stage: instrument development, pilot study, and national survey. Six

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 3-18. Melbourne: PME.

booklets comprising of algebra and geometry questions for 7th, 8th, and 9th graders were developed for the national sampling survey, and the survey involved 1181 seventh, 1105 eighth, and 1059 ninth graders respectively from 61, 60 and 61 classes in 18 sample schools. Most of the items developed in the English study (Healy & Hoyles, 1998) were adopted and modified based on Taiwan students' responses in the pre-pilot study during the first phase of the first stage. In addition, some new tasks were evolved from our interviews, which enabled the features of students' pre-formal reasoning to come through in both the instrument and coding system.

The second stage, teaching and learning mathematics proof, is comprised of an integrated project and four subprojects focusing on algebra (Lin, et al., 2004), geometry (Cheng & Lin, 2005), reading comprehension of geometry proof (Yang & Lin, 2005), and teaching and learning the validity of conditional statements (Yu Wu et al., 2004). The studies are strongly influenced by the work of many current researchers, such as the classification of student proof scheme (Harel & Sowder, 1998) and its application on teacher education (Harel, 2002), the cognitive analysis of argumentation and mathematical proof (Duval, 1998, 1999, 2002), the framework of proof and proving (Healy & Hoyles, 1998), the complexity of students understanding proving (Balacheff, 1987), the function and value of proof (Hanna, 1996, de Villiers, 1991, Hanna & Jahnke, 1993), and the theoretical validation approach of the Italian school (Garuit, Boero & Lemut, 1998).

ONE MORE STEP TOWARD AN ACCEPTABLE PROOF

The Incomplete Proof Group

When the national survey was administered in December 2002, the 9th graders had just learned formal proof in geometry for three months, while the 7th and 8th graders had not yet learned it. Based on the detailed coding schemes, students' performances on geometry proving were regrouped into four types: acceptable, incomplete, improper and intuitive proof. Students missing one step in their deductive reasoning is a typical incomplete proof. Students reasoning non-deductively or based on incorrect properties or with correct properties that do not satisfy with the given premises are codes of the improper proof. Students reasoning based on visual judgment or authority are typical codes of the intuitive proof.

The terminology "acceptable proof" derived from a statement by Clark and Invanik (1997): "Writing, for both students and researchers, is not just about communicating mathematical subject matter. It is also about communicating with individual readers, including powerful gatekeepers such as examiners, reviewers and editors." We took into account teachers' views for assessing whether a proof was acceptable or not.

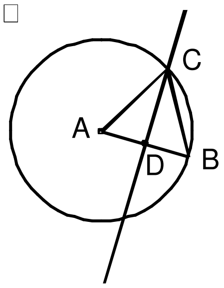
Students in the incomplete proof category were able to recognize some crucial elements for their reasoning (Kuchemann & Hoyles, 2002). They were able to distinguish premises from conclusions in the task setting. Particularly, on the two-step proof items, they were even mindful to check conditions of the theorems applied, i.e., micro reasoning (Duval, 1999.) They were also able to organize statements according to the status, premise, conclusion and theorem into a deductive step. Duval (2002) named such competency as the first level in geometrical proof. The second level is the organization of deductive steps into a proof. From the first step conclusion to the target conclusion, valid deductive reasoning generally moves forward through either successive substitution of intermediary conclusion or coordination of some conclusions. Duval (2002) pointed out that students might have "gaps in the progress of reasoning which makes the attempt of proving failed." This arises either from misunderstanding of the second level organization or from the context of the problem. We shall carefully examine Duval's statement above for the group of students who performed incomplete proofs in the two-step proof tasks.

The data from our national survey showed that one quarter of 9th graders could construct acceptable proofs in a two-step unfamiliar item; approximately one third was able to perform incomplete proofs; and one third did not have any responses at all.

It is obvious that educators would like to focus on this one third of 9th graders who were able to perform incomplete proofs, and to develop a learning strategy for them to fill the gap, i.e., develop one more step toward an acceptable proof. An effective learning strategy should promise that nearly a half of 9th graders will be able to construct a two-step unfamiliar geometry proof.

Incapability of Students with Incomplete Proof Performance

The two-step unfamiliar question used in the survey is as follows.



A is the center of a circle and AB is a radius. C is a point on the circle where the perpendicular bisector of AB crosses the circle. Please prove that triangle ABC is always equilateral.

Two types of incomplete proofs were observed. One type was missing the ending process. Students showed that AC=BC and AC=AB, but did not conclude that the three sides were equal. From a deductive point of view, they were ritually incomplete with the ending process, i.e., if a=b and b=c then a=b=c. Do these students who performed two valid deductive steps still have difficulty in the ending process, a classical syllogism? Or might these students simply be thinking that the two conclusions were too obvious for implying the target conclusion? Should one write this obvious step down? Would this be just an issue in the conventions of mathematical writing? Studies of students' understanding of proof by contradiction (Lin et al., 2002) and mathematical induction (Yu Wu, 2000) showed that senior high students who concluded their proofs without the ending process using either method, very often developed a ritual view about the methods. And the principle of the methods was not understood (Lin et al., 2002). If a teacher considers the two valid deductive steps as an acceptable proof, would the teacher create learning difficulties on mathematical proof for some students? A general question can be asked: How many students who can perform every valid deductive step necessary for a proof task also have difficulty organizing the deductive steps into a proof? Interview data showed that there were students behaving as such.

The other type of incomplete proof was missing one step, either AB=AC or AC=BC. The information "AC is a radius" was implicitly situated within the given premise. This information was invisible for students who did not conclude AB=AC. The property of the perpendicular bisector of a segment seemed unclear for students who did not draw the conclusion AC=BC. Some students of this type might not be aware of the need to derive the equality of all three sides for an isosceles triangle. Thus, the group of students with incomplete proof performance might not be able to:

(1) organize the deductive steps into a proof, or

(2) visualize some implicit information in the given premise, or

(3) recognize a needed mathematics property, or

(4) be aware of all necessary statements/deductive steps.

These four cognitive gaps are due not only to:

(1) misunderstanding of the organization of deductive steps into a proof,

(2) the content of a problem, but also

(3) the context knowledge, and

(4) the epistemic value, i.e., the degree of trust of an individual in a statement, from likely or visually obvious, to a statement becomes necessary (Duval, 2002).

For teaching experiments, one needs to rethink a learning strategy to ensure that students can cross these cognitive gaps.

A Learning Strategy for Promoting One More Deductive Step

Using X as learning strategy for students within their mathematics proof activities is an active research issue. Fifteen paper presentations that dealt with this issue in PME 22~28 are reviewed. The different Xs used in those papers include: arranging the context of proof situations (Garuti et al., PME26) and encouraging interactive discursion to create students' cognitive confliction (Boufi (PME26), Krummheuer (PME24), Douek et al. (PME24), Sackur et al. (PME24), Antonini (PME28)), learning within an ICT environment for conjecturing (Miyazaki (PME24), Gardiner (PME22), Hoyles et al. (PME23), Sanchez (PME27), Hadas (PME22)), emphasizing teachers' questioning as scaffolding (Blanton et al. (PME27), Douek et al. (PME27)), and using metaphors (travel) for setting target goals (Sekiguchi (PME24)). Note that the notation (PME24) indicates the paper appeared in the Proceedings of PME24. We exercised a "thought experiment"(Gravemeijer, 2002) with each of those strategies in addition to typical geometry teaching strategies used in Taiwan secondary mathematics classroom, to match the characterization of the incomplete proof group and enhance them to move one more deductive step. Finally, we chose two strategies that are commonly observed in typical Taiwanese 9th grade geometry classrooms and tested them for helping students achieve one more deductive step. The reading and coloring strategy means that students are asked to read the question, label the mathematical terms, and draw or construct this information on the given figure by color pens. The analytic questioning strategy means that students are asked to reply on what the question asked you to prove, and what conditions in the premise can be useful.

Several phases were conducted in our teaching and learning study:

- *Phase (1):* A three-item diagnostic assessment paper was developed for identifying sample subjects of the focus group. All three items share a common feature with implicitly necessary information.
- *Phase (2):* An instructional interview was conducted on 9 samples individually to examine the effectiveness of implementing both learning strategies simultaneously.
- *Phases (3) and (4):* A small group teaching experiment was carried out to study the effectiveness of only implementing one of the two learning strategies.
- *Phase (5):* A set of learning tasks on geometry proving was developed.

Based on the data resulting from phase (3), we will analyze the function of coloring the mathematical terms in proving. Turning implicit information into explicit information is definitely one function of the strategy. What else happened so that the subjects were able to complete an acceptable proof? It is noteworthy to interpret this with the data collected in the phase (3).

The three items, including the two-step unfamiliar item (G2) used in the national survey, were used in both phases (1) and (2). Nine samples were identified and interviewed. Their performances before the instructional interviews (Pre-I) and after intervening with the reading and coloring strategy (R-C) and analytic questioning strategy (A-C), respectively, during the interviews are presented in Table 1.

The notation (31) denotes the sample who performed an incomplete proof without the ending process due to omission (sample 02) or students' epistemic value that the ending process is unnecessary (sample 05, 06, 09). The notation 31* indicates that sample 01 would not agree with the syllogistic rule "if a=b and b=c then a=b=c" during the interviews, but agreed that "a=b and b=c" are the conditions for an equilateral triangle with sides a, b and c. The behavior of sample 01 on the syllogistic rule reveals one kind of reason for missing the ending process.

Sample	Performance	G1	G2	G3	G1	G2	G3	Sample
01	Pre-I	0	32	0	4	32	32	06

	R-C	4	32	4		32	32	
	A-Q		31*			(31)	(31)	
02	Pre-I	0	(31)	0	0	32	0	07
	R-C	4	4	32	4	32	32	
	A-Q			4		4	4	
03	Pre-I	0	0	32	0	32	4	08
	R-C	4	32	4	4	32		
	A-Q		4		4			
04	Pre-I	32	21	32	0	(31)	32	09
	R-C			4			4	
	A-Q							
05	Pre-I	4	(31)	32				
	R-C			(31)				
	A-Q							

Note: Definition of codes: 4 denotes an acceptable proof; 31 denotes incomplete, missing the ending process; 32 denotes incomplete, missing one deductive step; 21 denotes improper, using an incorrect property; 0 denotes no response.

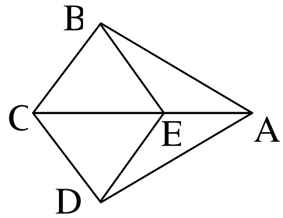
Table 1: Students' performance with/without the learning strategies R-C and A-Q

Table 1 shows that among the 24 (27-3) positions of students' performances which need to move towards an acceptable proof, 15 positions were successfully moved before or after the intervening of only the reading and coloring strategy. Since this coloring strategy is procedural in nature, the cognitive demand on learners for using this strategy is much lighter than using the analytic questioning strategy, which demands quite heavy analytical thinking. So, it is worthy to further explore the extent to which the reading and coloring strategy can enhance students' proving performance. Which kind of proof content will be effective by using this strategy? And a further interpretation of the effectiveness also seems interesting. This is the phase (3) study.

Effects of the Coloring Strategy

During the phase (3) study, four two-step unfamiliar new items were developed for 8 new participants. Before intervening with the reading and coloring strategy, out of 32 (8×4) performances, 10 were acceptable proofs and 22 were unacceptable, i.e., incomplete, or improper or had no response. Each participant had at least two unacceptable performances. One week later, 8 participants worked on the same items after intervening with the reading and coloring strategy. As a result, 16 out of the 22 unacceptable proofs had progressed to acceptable proofs. However, 4 out of 10 acceptable proofs became unacceptable, in which 3 out of 4 negative effects were coded from the same item 3.

Item 3.

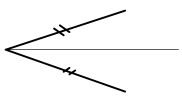


Points A, E, C are collinear,

and △ABC is congruent to △ADC.

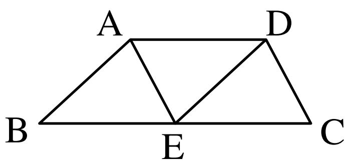
Show that: BE=DE

Two students misinterpreted the equality signs labelled on ∠ABC and ∠ADC as ∠CBE =∠CDE. The other student associated the sign around point C, with the angle bisector theorem and applied it improperly. Indeed, colored signs labelling on sub-figures which cross each other would generate a disturbance that affects visualizers' interpretation on the explicit information transmitted from the sub-figures.



Among the non-effected performance, all six were collected from item 2.

Item 2.



Points B, E, C are collinear,

and △ABE is congruent to △DEC.

Show that: AD//BC

When the equality signs were colored on the six elements, sides and angles of each triangle, the colored signs produced superfluous relations among the elements. Whenever a relation matching his/her target goal was observed by a student, it became active and operational. Students then applied it without justifying deductively. This seemed to be the pattern among those non-effected unsuccessful performances. Analyzing the negative effects and non-effects of the coloring label strategy, a criterion could be used by teachers to restrict the tasks on using the strategy. If a disturbance or superfluous relation from the coloring strategy were intentionally generated onto an item, it may backfire and result in negative effects or non-effects; in this case, the strategy may not be suitable for this item.

Transmission of the Subfigure with Relation to the Theorem Image

In spite of the negative and non-effects of the coloring strategy, we are interested in how the effectiveness (16/22) of the reading and coloring strategy takes place. From neuro-psychological perspectives, "Learning occurs… when transmitter release rate increases make signal transmission from one neuron to the next easier. Hence learning is, in effect, an increase in the number of 'operative' connections among neurons" (Lawson, 2003).

Learning was indeed achieved by those subjects who applied the coloring strategy and were able to perform an acceptable proof. How were the operative connections increased among the statements according to specific status and the use of theorems? The necessary theorems existed previously in the subjects' mental structure, but were inoperative before they applied the coloring strategy. The result of the coloring process revealed subfigures with notable relations that may also correspond to the theorem. If this happens, then learners have increased the relation between the subfigure and the needed theorem. To make it clear, we shall use the term theorem image, similar to the term concept image (Tall & Vinner, 1981), to describe the total cognitive structure that is associated with the theorem, which includes all the mental pictures and associated examples, relations, process and applications. A theorem image is built up over years of learning experiences. It is personal and constantly changing as the individual meets new stimuli. Different stimuli can activate different parts of the theorem image. The stimulus resulting from coloring of mathematical terms in the premise is functioning to lead the transmitter of the revealed subfigure with relation to the corresponding part of his/her theorem image. This leads the effect of the organization of one deductive step.

MAKING DECISIONS ON FALSE CONJECTURES

Some items in each of the six booklets were connected to how students reason to make their decisions on a given false conjecture. Students were asked to make a decision among two (three) choices – agree, disagree, or uncertain (algebraic item) – and then give explanations on their choices. A unity of coding schemes was evolved for both geometry and algebra surveys. The coding schemes were used to analyze the students' performances. Based on this coding scheme, a model of refuting will be discussed. Firstly, for researchers to make sense of the thinking process in mathematical refutation, an expert was interviewed.

Mr. Counter-Example's Thinking Process on Refutation

A mathematician, nicknamed Mr. Counter-Example by his peers during his graduate studies, was interviewed to reveal the thinking process of an expert on refutation.

> "Suppose an unfamiliar mathematics proposition is proposed by myself or peers. Reading it and without having much sense with the proposition, the doubtfulness of its truth usually does not arise in my mind. To make sense of the proposition, very often I'll substitute some individual examples. Then, I will find more and more examples to satisfy the premise. Naturally those examples will be classified according to certain mathematical property. As long as the property is grasped, all kinds of examples will be considered. Finally, a specific kind of example will be identified to counter the conclusion if the proposition is false."

According to Mr. Counter-Example's description, his refuting process covers five sequential processes:

1. Entry
2. Testing some individual examples point-wisely for sense making
3. Testing with different kinds of examples
4. Organizing all kinds of examples
5. Identifying one (kind of) counterexample when realizing a falsehood

This expert's thinking process on refutation can be inferred to analyze students' reasons on refuting.

On Geometrical False Conjectures

Two conjectures in geometry were adopted from the English study (Healy & Hoyles, 1998):

> "Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the center of circle?" (7G1, Geometry)
>
> "Whatever quadrilateral I draw, at least one of diagonals will cut the area of the quadrilateral in half?" (8G1, Geometry)

Three false conjectures were evolved from the interviews carried out during the pilot study phase of the first stage. The following one was included in geometry booklets for both 7th and 8th graders who were the subjects concerned in this section.

> "A quadrilateral, in which one pair of opposite angles are right angles, is a rectangle." (7&8 G5, Geometry)

This coding scheme was evolved according to the performances of the national representative sample and the expert's thinking process on refutation, and is more detailed than the schemes developed in the English study (Hoyles & Kuchemann, 2002), which only focused on high-attainers (top 20~25% of the student population).

On geometrical false conjectures, students either confirmed or refuted it. Comparing the frequency on G5 of 7th and 8th graders' performances, there is no evidence of progress with correct decisions over the year (37% for 8th graders, even more than 26% for 7th graders). Based on the words provided by students who ticked disagree, we classified them into three subcategories: rhetorical argument, correcting the given information, and generating counterexamples. Duval (1999, 2002) classified the relationship between a given statement A and another statement B into two types – the derivation relationship and the justification relationship. For each type, there are two kinds of reasoning that are practiced or required in mathematics teaching and learning. Semantic inference and mathematical proof support the derivation relationship; heuristic argument and rhetorical argument support the justification relationship. In our code scheme, codes c2, c3, c4, g1, g2 are the so-called heuristic arguments that take into account the constraints of the situation in the task. Generally, an argument is considered to be anything that is advanced or used to justify or refute a proposition. This can be the statement of a fact, the result of an experiment, or even simply an example, a definition, the recall of a rule, a mutually held belief or else the presentation of a contradiction (Duval, 1999). Reasons relative to the person spoken to or beliefs of the interlocutor are the rhetorical arguments. Therefore, code d4 is a rhetorical argument, and d3 is a heuristic argument.

False Conjectures if P then Q				
Code	Frequency (%)			
	7G1	7G5	8G1	8G5
Confirmation	44	26	31	37
d_0 – Misunderstanding the given information	1	2	2	1
d_1 – Much ado about nothing	23	5	12	9
d_2 – Confirm Q with incorrect reason	9	3	8	6
d_3 – Giving P' s.t. P'→Q	3	12	3	17
d_4 – Authority	0.1		0.2	1.1
Refutation	52	67	68	59
Rhetorical argument	8	8	17	11
Correcting the given information	15	51	12	33
c_0 – Criticizing the given information	9	13	3	5
c_1 – Non-example	3	3	5	9
c_2 – Providing alternative Q		32		16
c_3 – Characterizing Q s.t. P' $\rightarrow Q$	2	3	3	2
c_4 – Empirical decision	0.3		0.5	0.1
Generating (a) counterexample(s)	24	4	34	11
g_0 – Do not believe it is always true	3	1	5	3
g_1 – Giving the possibility of a counterexample	5	0.6	13	4
g_2 – Giving the way of generating a counterexample	4	0.3	4	1
g_3 – Explicit, clear counterexample	12	2	10	3
g_4 – Counterexample with mathematical proof		0.1	0.9	0.1

Note: Non-responses are not included

Table 2: 7th and 8th graders Code Frequencies on items G1 and G5 (N7=1146, N8=1050)

Our coding scheme with code frequencies cover three out of four kinds of reasoning practiced by our 7th and 8th graders on refuting false conjectures: rhetorical argument, heuristic argument and mathematical proof (clear counterexample counts). The relatively high frequency of code c2 in 7G5 was contributed by students who reasoned that under the assumption, a quadrilateral can be either a square or a rectangle. This reason reflects the prevalence of students who misunderstand the inclusion relationship between squares and rectangles. Putting the number of students with codes c2, c3, c4, g1 and g2 together, and computing its frequency, we found that 11% and 36% of 7th graders and 20% and 24% of 8th graders were able to make a heuristic arguments for refuting G1 and G5, respectively.

On Algebraic False Conjectures

Three false conjectures in algebra survey for 7th and 8th graders were chosen for discussion.

A3 "If the sum of two whole numbers is even, their product is odd?" (Both 7th and 8th graders, adopted from Küchemann & Holyes, 2002.)

A6b "The sum of a multiple of 3 and a multiple of 6 must be a multiple of 6?" (8th graders)

The data (3,6,6) in A6b was replaced by (3,6,9) in A6c for 8th graders, and respectively by (2,4,4) and (2,4,6) in A6b and A6c for 7th graders. Students' works on algebraic false conjectures were analyzed with this code scheme: "g3: explicit, clear counter example, can be distinguished into three subcodes," "g31: counterexample without reason," "g32: both supporting and counterexamples," and "g33: counterexample with analytic reasons," which often is a rule for generating a specific counterexample. Referring to the expert's thinking process on refutation, both processes (2) and (5) will be coded by g31. Thus, without words, code g31 could result from primitive or advanced thinking.

Instead of presenting the national survey data, we'll present a brief description of the students' words to model their refutation schemes on algebra. On confirmation: (1) "I believe that only true statements will be presented in my learning" (code d_1); (2) "I consider it correct, because its familiar format is akin to statements in textbooks" (code d_4); (3) "I had supporting examples, e.g., 3+6=9 and 3×2+6×2=18, they are multiples of 9" (A6c) (code d_3). On uncertain responses: (1) "I am not certain because the multiple is not given," students interpreted the term multiple in "a multiple of 3" as specific numbers, a misconception (code r_1); (2) "I had both supporting and counterexamples," in ordinary language, this statement is uncertain (code g_{32}). On refutation performances: (1) "The statement is so elegant, I must have learned it before. But, I did not. So it can't be always correct" (code g_0); (2) Simply adding a negation without reasons (code r_1). Beyond the above beliefs and rhetorical arguments, the students' refutation schemes are coded by g1, g2, g31, g32, g33 and g4. Their thinking process then is similar to certain points in the expert's thinking process.

Refuting Generates Conjectures

When students gave their explanations for refuting, many gave heuristic arguments and explicit counterexamples with reasons, and we observed that some of these students had even produced relations, known properties evidences, general rules, etc. Buying the notion of "Cognitive Unity of Theorems" from the Italian school (Garuti et al., 1998; Boero, 2002), instead of the concerns of the possible continuity between some aspects of the conjecturing process and some aspects of the proving process, we would like to investigate the possible production of conjectures derived from the aspect of the refuting process.

The activity of refuting in mathematics is considered an economic way of helping students to develop competency in critical thinking. Competency of critical analyses has been recognized as a deficit in Taiwan education and is now emphasized in the school curriculum (Ministry of Education, 2003). Two refuting-conjecture tasks in algebra and geometry respectively were developed for the investigation. Each task is comprised of several items. The first item is making decisions on relatively easy false conjectures that aim to motivate students to be aware that the task is on refuting. The second item is given some false conjecture used in the national survey for refuting. The third and fourth items ask students to produce one conjecture and more conjectures, based on their refuting processes.

All nine 7^{th} graders who participated in the investigation with the algebra task produced meaningful conjectures. Three of them even produced a general rule for a whole number m that is divisible by the linear combination of whole numbers $ax+by$.

Seventy-five 9^{th} graders from two classes were asked to participate in the geometry task investigation. The four false conjectures used in the tasks were 7G1 (denotes item G1 in the 7^{th} grade survey), 8G1, 8G5, 9G6, respectively. According to the code of frequencies of refutation schemes, 76%, 73%, 53%, and 60% of their performances were in the category "generating counterexamples" with respect to those false conjectures 7G1, 8G1, 8G5, and 9G6 respectively. The conjectures produced by this group are presented in Table 3.

%	7G1	8G1	8G5	9G6
Thm.	33	20	52	7
New statement	17	8	7	1
Innovation	5	33	8	56
Total	55	61	67	64

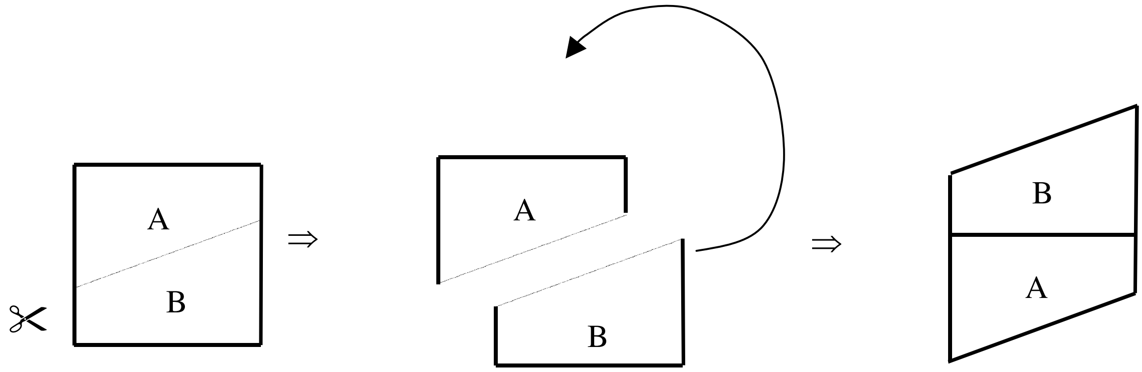
Note: Thm. denotes the conjecture is a theorem. New Statement denotes the conjecture is a new writing of learned properties. Innovation denotes the conjecture is an innovative one.

Table 3: Frequency (%) of different type of conjectures. N=75, 9^{th} graders

Table 3 shows that the success rate for producing correct conjectures on these four tasks was approximately 60% or more. Different frequencies of each type of conjectures imply that 8G1 and 9G6 are excellent for creating brand new conjectures by 9^{th} graders. The item 9G6 is quoted here.

9G6.

A square is cut along the dotted line, then inverted. Is the resulting figure a rhombus?



The conjectures produced by students were further distinguished into "correlating" or "not correlating" to their explanations for refuting.

The relatively high percentages in Table 4 show the continuity of the refuting process and conjecturing process. This claims that refuting is an effective learning strategy for generating conjectures. To create innovative conjectures, the content in the given false conjecture needs to be well-designed, and 9G6 is a good example.

	7G1	8G1	8G5	9G6
T1	40	57	38	69

Table 4: The percentages of conjectures that correlate to refuting

Boero (2002) reported that the Italian school has identified four kinds of inferences, intervening in conjecturing processes: (1) inference based on induction, (2) inference based on abduction, (3) inference based on a temporal section of an exploration process, and (4) inference based on a temporal expansion of regularity. Reading students' productions in the refuting-conjecture tasks, we observed that false conjectures in numbers 7A3 and 8A6 can enhance the generation of conjectures that are inferences based on induction, abduction (e.g., a narrative) and even deduction (e.g., 3h+6k=3(h+2k)); the task with figure dissection 9G6 can generate conjectures that are inferences based on a temporal section of an exploration process (the dissection), and tasks with 7G1 and 8G1 are relatively effective on generating conjectures that are based on the expansion of regularity (such as new statements of some properties). The following excerpt is from 9G6.

> If a line cuts a rectangle along the pair of longer sides into two parts so that the cross segment is equal to the longer side, then the two parts can be inverted to form a rhombus.

This conjecture is produced in association with sequential operations on a rectangle.

CONCLUSION

Based on our study, there is evidence showing the existence of continuity in different aspects of mathematics education. In the mathematics learning aspect, a rather high percentage of students were able to produce correct conjectures when working on refuting-conjecture tasks; this shows the existence of continuity between the refuting process and the production of truth statements. For some students, this continuity can even extend to their proving process. Indeed, some students have already provided counterexamples with analytic or mathematical proofs to refute false conjectures. In the mathematics teaching aspect, the effectiveness of the reading and coloring strategy on geometrical two-step proving shows that teachers can keep their traditional teaching approach, in which they can encourage students to label meaningful information within the given premise and conclusion and then seek linkages between the premise and the conclusion. Without disturbing their approach but suggesting students to use color pens for labelling, teachers can enhance students' proving competencies. This demonstrates continuity between a more effective teaching strategy and the traditional teaching strategy. In the aspect of research in mathematics education, there is continuity between the investigating processes by educators in mathematics education research and by mathematicians in mathematics proving. The six phases of mathematicians in proving identified by Boero (1999) is indeed shared by mathematics educators in their studies, such as the study presented in this paper. Formulating on-going investigating issues is always considered to be connected with reflections on previous phases.

Carrying out more testing on the effectiveness of the refuting-conjecture tasks will create an equilibrated set of conjecturing tasks suitable for activating different types of inferences.

Several phases of research in mathematics education presented in this paper are rather traditional, such as (1) Identifying 1/5~1/3 of students in their age population, whose mathematics understanding are more likely to be enhanced. (2) Characterizing those students' competencies. (3) Carrying out an experimental study with a redesigned learning strategy that connects to the characteristics of their cognition.

This approach can frame local (geological and societal) education issues in the wider context of collaborative international studies, for the purpose of improving mutual education. The experience seems to be a very healthy and effective approach.

Acknowledgement

This paper is supported by the National Science Council of Taiwan (# NSC 93-2521-S-003-009). Any opinions expressed here are those of author and do not necessarily reflect the views of the sponsoring institution. The author would like to thank the research team members for their extensive contributions to the study: C.Y. Chen, Y.H. Cheng, I.E. Chen, J.Y. Wu, and K.L. Yang, C.L. Lin, H.C. Hao, and C.H. Chang.

References

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics* 18(2), 147-176.

Boero, P. (2002). The Approach to Conjecturing and Proving: Cultural and Educational Choices. *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand,* pp. 248-254.

Boero, P. (1999). Argumentation and mathematical proof: a complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on Teaching and Learning in Mathematics Proof.* (990708)

Clark, R., & Ivanik, R. (1997). *The politics of writing.* London: Routledge.

Cheng, Y. H. & Lin, F. L. (2005) *One more step toward acceptable proof in geometry.* Symposium Proposal EARLI 2005, Cyprus.

Duval, R., (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.) *Perspectives on the Teaching of Geometry for the 21st Century* (pp. 37-52). Dordrecht/Boston: Kluwer Academic Publishers.

Duval, R. (1999). Questioning argumentation. *International Newsletter on Teaching and Learning in Mathematics Proof.* (991112)

Duval, R. (2002). Proof understanding in MATHEMATICS: What ways for STUDENTS? *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand,* pp. 61-77.

Garuti, R., Boero, B., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. *Proceedings of PME22,* Vol.2, 99345-352.

Gravemeijer, K. (2002). Developmental research, a course in elementary data analysis as an example, *Common Sense in Mathematics Education*, pp. 43-68.

Hanna, G., & Janke, N. (1993). Proof and application *Educational Studies in Mathematics* 24(4), 421-438.

Hanna, G., & Janke, N. (1996). Proof and proving. In: Bishop A. *et al.* (Eds.) *International handbook of mathematics education* (pp. 877-908). Dordrecht: Kluwer Academic Publishers.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In: Schonfeld, A., Kaput, J., & E. Dubinsky E. (Eds.) *Research in collegiate mathematics education III.* (Issues in Mathematics Education, Volume 7, pp. 234-282). American Mathematical Society.

Harel, G. (2002). DNR-Based Instruction: Its Application in Developing Mathematics Teachers' Knowledge Base, Particularly their Proof Schemes. *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand*, pp. 78-90.

Healey, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students in the UK.* Research Report Mathematical Sciences, Institute of Education, University of London.

Hoyles, C., & Küchemann, D. (2002). Students' explanations in geometry: insights from a large-scale longitudinal survey. *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand,* pp. 9-22.

Lawson, A. (2003). *The Neurological Basis of Learning, Development and Discovery*. Kluwer: Academic Publishers.

Lin, F. L. (1991). Characteristic of 'Adders' in Proportional Reasoning. *Proceedings of the National Science Council Part D: Mathematics, Science, and Technology Education,* Vol.1 (1), pp.1-13.

Lin, F. L. (2004). Features and Relationship of Reasoning, Proving and Understanding Proof in Number Patterns. *International Journal of Science and Mathematics Education, 2*(2), 227-256.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to limit and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Yang, K. L. (2005). Conceptualizing reading comprehension of geometry proof. *Proceedings of 2005 International Conference on Reading and Writing in Science and Mathematics.* p. 61

Yu, Jya-Yi Wu (2001). *Reasoning Difficulty Related to Validity of Argument and Truth of Assertion in Mathematics.* Unpublished Doctoral Dissertation, University of Pittsburgh.

Yu, Jya-Yi Wu, Chin E.-T., & Lin C.-J. (2004). Taiwanese Junior High School Students' Understanding about the Validity of Conditional Statement. *International Journal of Science and Mathematics Education, 2*(2), 257-282.