DEVELOPING STUDENTS' UNDERSTANDING OF THE CONCEPT OF FRACTIONS AS NUMBERS

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Research has shown that many students have not fully developed an understanding that fractions are numbers. The purpose of this study was to investigate the effects on the understanding of fractions as an extension to the number system of a teaching programme focusing on mixed numbers. Significant differences were found in favour of the programme with greater emphasis on mixed numbers. The study suggests that a programme involving multiple representations for mixed numbers may help students realise that fractions are numbers.

INTRODUCTION

While students may have some facility with fractions, many of them appear not to have fully developed an understanding that fractions are numbers (e.g., Kerslake, 1986, Domoney, 2002 and Hannula, 2003). Kerslake (1986) emphasises the need for students to understand fractions at least as an extension of the number system. Her report presents some of the difficulties 12 to 14 year old students have in connection with fractions. The suggestion is made that many of those difficulties occur because students see fractions as only parts of a shape or quantity and not as numbers. The part-whole model was the only interpretation familiar to all students who took part in her study. Kerslake thinks that the problem starts in primary school when fractions are first introduced merely as parts of geometric pictures. She argues that school practice does not give enough hints to students that fractions are numbers. The work with graphs, algebraic equations and number patterns usually involves only integers.

Research has also shown that students have difficulties in identifying the unit in part-whole diagrams showing more than one unit (e.g., Dickson et al., 1984). When a fraction greater than one is represented in a diagram like the one in Figure 1, many students respond 7/10 rather than 7/5. Similar problems arise when separate part-whole diagrams are used to illustrate addition of two proper fractions (Figure 2) or when the total is greater than one unit (Figure 3).

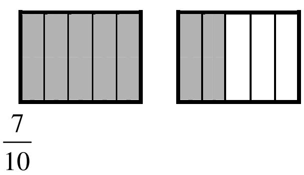

Figure 1

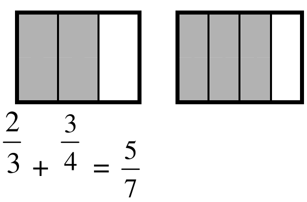

Figure 2

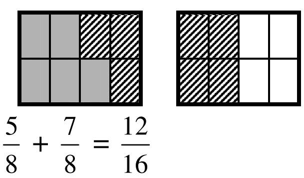

Figure 3

In the CSMS investigations, Hart (1981) noticed that diagrams sometimes helped in the solution of problems with fractions, or were used to check whether the answer

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found was feasible. However, the process of interpreting a part-whole diagram often involved: (i) counting the number of pieces which were shaded, (ii) counting the total number of pieces, and (iii) then writing one whole number on top of the other. In the interviews, just after the students answered the fraction shaded in a part whole diagram for 3/5, they were asked to give the fraction not shaded. Hart reports that few subtracted the fraction shaded from one (1 − 3/5) they often used again the counting process just mentioned. It may be here conjectured that those students gave the correct fractions without realising the connection between the fraction 5/5 and the whole number 1. In fact, this counting process of naming a fraction does not require the application of any concept of fractions as parts of a whole. The fraction is interpreted as a pair of whole numbers. Research has also shown that students have difficulties in identifying a proper fraction in a number line showing two units instead of one unit of length (e.g., Kerslake, 1986 and Hannula, 2003). A common misconception is to place the fraction 1/n at (1/n)th of the distance from 0 to 2. So the identification of the unit in number lines seems to be as problematic to some students as in part-whole diagrams.

Although part-whole diagrams are thought to be misleading and a possible inhibitor of the development of other interpretations for fractions (e.g., Kerslake, 1986), Pirie and Kieren (1994) present how 10 year old Katia achieved "a new understanding" (p. 174) of addition of unrelated fractions (halves and thirds) by drawing part-whole diagrams (pizzas) for the fractions and later dividing both into sixths. There is also some agreement that fractions should be introduced as parts of a whole (e.g., English and Halford, 1995). Probably because it is the first aspect of fractions met in a child's life. So more research needs to be done about how a move from the part-whole aspect to the aspect of fractions as numbers could be achieved (Liebeck, 1985 and Kerslake, 1986). This move was the focus of the present research.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

English and Halford (1995) have developed a psychological theory of mathematics education which combines psychological principles with theories of curriculum development. They discuss the importance of representations and analogical reasoning in helping students construct their mathematical knowledge from prior knowledge. Yet the choice of representation and the actions to be performed upon it can have important consequences for mathematical learning. Some representations can even obscure or distort the concepts they are supposed to help students learn. Certain representations like fictitious stories such as "mating occurs only between fractions, so mixed numbers - 1¾ - become improper fractions - 7/4 ..." may help students remember procedures but do nothing to develop conceptual understanding (Chapin, 1998, p. 611). Some important pedagogical and physical criteria for selecting representations are suggested in the literature (e.g., Skemp, 1986 and English and Halford, 1995), but only the pedagogical versatility criterion will be discussed in this paper.

Skemp (1986) advises teachers to choose versatile representations which can be used to construct long-term schemas. Such schemas are applicable to a great number of mathematical concepts and so make the assimilation of later concepts easier than a short-term schema which will soon require reconstruction. English and Halford (1995) call the criterion of selecting versatile representations “the principle of scope”. They consider the part-whole model to be a representation with scope as it can illustrate many fraction concepts and operations. The idea is to use the same type of representation to communicate several concepts and operations which are related among themselves. It is not just a matter of economy, but of allowing more relationships to become exposed.

Bell et al. (1985) think that some misconceptions may result from new concepts not being strongly connected with the student’s previous concepts. On the other hand, some other misconceptions may result from “the absence of some actually essential detail of the knowledge-scheme which has been overlooked in the design of the teaching material” (p. 2). Therefore, certain misconceptions may also be related to instructional constraints which may result in students’ construction of a schema in a more limited way. Naming improper fractions (Figure 1) or adding the numerators and denominators in addition of fractions (Figures 2 and 3) may be the result of a more limited schema for fractions. The student may see fractions merely as a pair of two whole numbers, one written on top of the other. In order to develop a conceptual knowledge of rational numbers, students should be able to both differentiate and integrate whole numbers and fractions. Yet versatility of a representational model does not imply uni-embodiment. It seems important to use several models for each concept, but two or more related concepts, whenever possible, should be represented together so that their relationship becomes clear. An example which concerns the present study involves using multiple representations to work simultaneously with whole numbers and fractions in order to highlight the relationships between those two sets of numbers.

METHODOLOGY

The purpose of the study was to investigate the effects on the understanding of fractions as an extension to the number system of a teaching sequence for fractions which places emphasis on fractions of the type n/n ($n \neq 0$) and on mixed numbers since from the beginning of the instruction. The study was also concerned with ways of helping students to move from the part-whole aspect to the aspect of fractions as numbers. Each of two teaching sequences was administered to a group of around 60 students of 11 years of age drawn from six schools in England (Amato, 1989). Group X used multiple representations (contexts, concrete materials, pictures and diagrams, spoken languages and written symbols) to represent proper fractions and mixed numbers from the beginning of instruction. Group Y used multiple representations to represent only proper fractions at the beginning of instruction. However, at the end of instruction part-whole diagrams for mixed numbers were also presented.

Some cheap concrete materials which are used to teach place value with whole numbers, like coloured plastic straws, can easily be extended to fractions and decimals through cuts of the unit. For example, the number 135¾ can be represented with straws as in Figure 4. Hundreds, tens and units can be represented together with fractions of those units in both enactive and iconic ways. This type of representation may help students to visualise fractions and decimals as an extension to the right side on a place value system and so as an extension to the number system. The terminology employed in some textbooks does not seem to help students to associate fractions with an extension to the number system. When learning about whole numbers, they read words like units, tens, hundreds, etc. However, when learning about fractions, the word "unit" is substituted by the word "whole". So not many attempts are made to associate fractions with the previously learned numbers by an appropriate use of language.

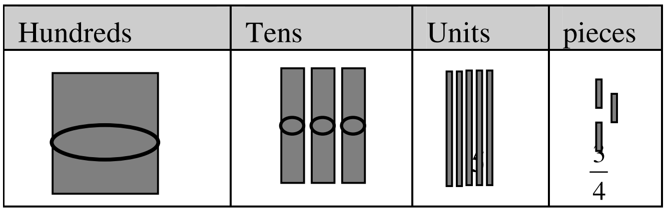


Figure 4

Part-whole diagrams can also be helpful in the development of the concept of fractions as numbers if used in a way that highlights the unit and the connections between fractions and whole numbers. Soon after working with concrete materials and part-whole diagrams for fractions less than one unit (e.g., 1/4, 2/4 and 3/4, Figure 5), diagrams for fractions equal to one unit (e.g., 4/4, Figure 6) and mixed numbers (e.g., 2 units and 3/4, Figure 7) are presented.

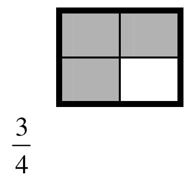


Figure 5

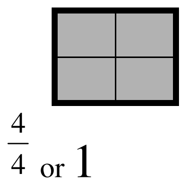


Figure 6

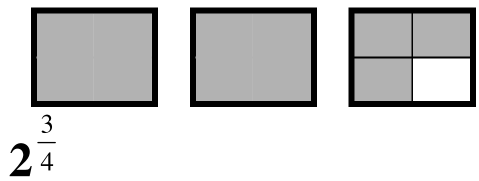
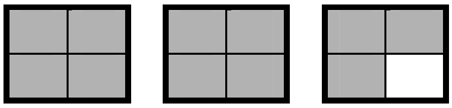


Figure 7

The presence of whole unsliced units in those diagrams may help some students realise that the proper fractions in the mixed number notation are numbers smaller than one. Often mixed numbers are introduced much later in the book or in one of the following books and together with improper fractions. The equivalence between the two notations is usually presented with the help of diagrams where all the "wholes" are cut into equal pieces (Figure 8). This kind of representation does not seem to emphasise the two units as much as when they are not cut (Figure 7).



$\frac{11}{4}$ or $2\frac{3}{4}$

Figure 8

In this study, not only fractions were added in similar manner to that of whole numbers but also the "carrying" process was extended to fractions in a way that reinforces the relation between fractions of the type n/n and the whole number 1. So the study was concerned with ways of helping students to move from the part-whole aspect to the aspect of fractions as numbers and it sought to answer the question: "Does the use of mixed numbers from the beginning of instruction concerning fractions help the development of the concept of fractions as numbers?"

The main activities included in both teaching sequences were:

(1) representing numbers with straws and recording in figures the number being represented with pictures of straws;

(2) counting forward and backwards with fractions: (a) shading diagrams to represent numbers, (b) recording in figures the number being represented with diagrams, and (c) following in figures only a counting number pattern;

(3) using part-whole diagrams for recording the number being represented by the shaded part and the missing number (unshaded part);

(4) using part-whole diagrams to represent three-dimensional divisible units and to help solving sharing problems with whole numbers for both dividend and divisor;

(5) adding fractions: (a) adding fractions in a similar algorithm to the one used for whole numbers (vertical position), and (b) recording resulting fractions of the type n/n as the whole number "1";

(6) multiplying a whole number by a fraction: (a) using part-whole diagrams for changing multiplication into repeated addition and to help combining fractions that together would be equivalent to one unit or a whole, (b) changing multiplication into repeated addition only in figures, and (c) using multiplication tables in a way similar to that which is used when a whole number is being multiplied by another whole number (the sequence of products would form a number pattern); and

(7) Working with number lines associated with the idea of measuring.

The teaching sequences were evaluated by a pre-test, an immediate post-test and a five weeks delayed post-test. The questions on the tests involved the use of fractions in number contexts similar to those in which whole numbers are often used. The questions were extracted from the tests in the projects "Concepts in Secondary Mathematics and Science" (Hart, 1981) and "Strategies and Errors in Secondary Mathematics" (Kerslake, 1986). Covariance analyses were performed on the scores of each post-test, and in both cases, the scores on the pre-test were used as covariate.

SOME RESULTS

The types of activities, the fractions and the quantity of items involving fractions on the worksheets were the same for both groups X (mixed numbers) and Y (no mixed numbers). However, group X spent more time on the worksheets (about 4½ hours) than group Y (about 4 hours). This was expected as group X had at the beginning of instruction three extra worksheets revising place value with whole numbers. Also when group X worked with mixed numbers at the beginning of instruction, they not only had to count pieces and write fractions but also to count units and write whole numbers. A sample of 148 students took the pre-test and started the instructional sequences. Eight of them did not manage to finish 10% of the sequence. On the days when the immediate and delayed post-tests were administered, totals of nine and eleven students were absent respectively. Therefore, the experimental sample was composed of 120 students who had done the three tests and finished 90% of the teaching sequence.

Analysis of covariance with one regression line was used to investigate the effects of using mixed numbers from the beginning of instruction on the acquisition of the concept of fractions as numbers and to allow for initial differences between the experimental groups on the pre-test score. First, it was used to test the operational hypotheses and employed the score on the immediate post-test as the dependent variable. In a second instance, covariance analysis was used for both re-testing the hypotheses and investigating the achievement over time of the two groups. In the latter case the delayed post-test was taken as the dependent variable. The main variable which were thought to relate to the dependent variable in both instances were the pre-test score. The operational hypothesis was tested with differences at the .05 level considered significant.

The majority of students did not perform well on the pre-test. More than 90% of the experimental sample scored less than half of the maximum possible score in the pre-test. It could be noticed that some students had little knowledge about fractions, especially their notation. They could easily talk about halves and quarters but questions like “How do I write one quarter in figures?” were asked in the pre-test and in the initial worksheets. The effect of “Mixed Numbers from the Beginning of Instruction” was significant in both the immediate post-test (scores without covariate adjustment: $F = 13.56$ and $p = .000$ and scores adjusted for pre-test scores: $F = 10.73$ and $p = .001$) and in the delayed post-test (scores without covariate adjustment: $F = 15.01$ and $p = .000$, and scores adjusted for pre-test scores: $F = 12.88$ and $p = .000$).

Student teachers’ understanding of the concept of fractions as numbers has also been found to be limited (Domoney, 2002). More recently, I have been using the idea of focusing on fractions of the type n/n and mixed numbers since the beginning of instruction with student teachers (Amato, 2004a). The idea has proved to be effective in helping them overcome their difficulties in relearning rational numbers conceptually within the short time available in pre-service teacher education (80 hours). I have greatly reduced the number of activities for place value and operations with whole numbers alone. However, through activities involving multiple and versatile representations for concepts and operations with mixed numbers and decimals (e.g., 35¾+26¼ or 24.75-12.53), student teachers have been provided with many opportunities to: (a) revise whole numbers as the representations for mixed numbers and decimals include a whole number part, and (b) make important relationships between rational numbers interpretations and between operations with whole numbers and operations with fractions and decimals. I am also using a similar program to help Brazilian 10 year olds construct rational numbers concepts and the connections among whole numbers, fractions, decimals and percentages.

CONCLUSIONS

Significant differences were found in favour of those students who used mixed numbers from the beginning of instruction. Students' understanding of fractions as an extension to the number system appear to benefit from the use of multiple representations for fractions equal to one unit (n/n) and mixed numbers. It was not difficult to teach the mixed number notation at the beginning of instruction soon after the students had learned the notation for proper fractions. It was interesting to note a student using his fingers to find the solution to "2½ + 2½". He represented 2½ by showing 2 whole fingers and ½ of the third finger. He then covered the other half with his second hand and hide the fourth and fifth fingers behind the palm of his hand and said "2½". After that he showed the 2½ fingers which were hidden and said "plus 2½ makes 5". The process of adding whole numbers with fingers was extended naturally to the addition of mixed numbers with halves. In order to understand fractions as an extension to the number system, students need a variety of experiences with fractions equal to one unit and mixed numbers as well as with numbers between zero and one unit.

Kerlake's suggestion (Kerslake, 1986) that the geometric part-whole interpretation of fractions inhibits the understanding of fractions as numbers and other interpretations of fractions appears to be justified. Part-whole diagrams may be interpreted as a particular way of representing two whole numbers and not as a representation of a single number. The relationship between one whole shape and the whole number 1 may not be recognised by some students. On the other hand, the type of part-whole diagrams used to represent mixed numbers in the activities performed by the students who participated in the present study were seen as beneficial to the understanding of fractions as an extension to the number system. The presence of whole unsliced units in those diagrams may have helped students realise that the proper fractions in the mixed number notation were numbers smaller than 1.

Mixed numbers are often used in everyday life: traffic signs (e.g., 3¼ miles), recipes (e.g., 1½ pints of milk) and ages (e.g., 9½ years). Using decimals in such instances would be more complicated language. To Liebeck (1985) the concept of mixed numbers arises naturally from measuring objects (e.g., 1 metre and 2 tenths of a metre). She thinks that recording a length between 1m and 2m as 1½ m is a strong "hint" that there are numbers between two consecutive whole numbers. Approaches such as "1 + ¼ can be written as 1¼ " and "3½ = 3 + ½ = 6/2 + 1/2 = 7/2" (p. 33) are too formal for the introduction of mixed numbers and improper fractions respectively. Hannula (2003) found that mixed numbers were much easier to locate on a number line than proper fractions. Yet little emphasis appears to be given to mixed numbers. Many textbooks introduce fractions first with pictures of real objects where pieces are missing and then with geometric part-whole diagrams, but normally only fractions "less than one whole" (proper fractions) are presented. Few textbooks work extensively with fractions "equal to one unit" (n/n with $n \neq 0$) and mixed numbers. These fractions may provide the initial link between fractions and whole numbers.

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