THE DUALITY OF ZERO IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA [1]

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This article shows that the recognition of the dualities in equality (operator–equivalent) of the minus sign (unary – binary) and the zero (nullity – totality) during the transitional process from arithmetic to algebra by 12 – 13 year-old students constitutes a possible way to achieve the extension of the natural number domain to the integers.

LITERATURE AND THEORETICAL FRAMEWORK

Since the nineteen seventies, researches such as those carried out by Freudenthal (1973), Glaeser (1981), Bell (1982), Janvier (1985), Fischbein (1987), Resnick (1989), Vergnaud (1989) among others, have shown that students presented extreme difficulties related to the conceptualization and operating with negative numbers in the pre-algebraic and algebraic scope. The study of this topic, baseline for the mathematical education, continues to be in effect to date.

The results of the first steps of a research project in process that intends to go deeper in the problematic with negative numbers, through the study of zero, are reported in this article. Questions as the following lead this project in the transition of arithmetic to algebra:

a) How does zero contribute to the extension of the numerical domain of natural numbers to integers?
b) Do students consider zero a number?
c) Are they aware of the dual nature of zero?
d) Do they understand the addition, subtraction, multiplication and division by zero?
e) Does a historical-epistemological analysis of zero as a number would contribute to the understanding of the conflicts presented by students nowadays?
f) Which cognitive hanges are provoked in students by the teaching of integers through technological environments?

The theoretical foundation of the initial stage of the project is based on the ruling ideas of the following authors: Piaget (1933) expressed that when showing the child

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representations of the world, a "system of tendencies" are discovered in him, and of which the child had not have conscience, therefore, he had not been able to express them explicitly. In this same direction, Filloy (1999) explained that there are tendencies due to the cognitive structures of the subject that appear in every individual development state that give preference to several mechanisms to proceed, several ways to code and decode mathematical messages. These "cognitive tendencies" can be observed both in the classroom and during clinical interviews. In what is reported in this article, two cognitive tendencies identified by Filloy with numbers 2 and 8 were clearly brought up and which are described as follows:

Conferring intermediate senses. This tendency appeared in the solution of additions and subtractions of integers when assigning multiple senses to negative numbers that were corresponded to with the acceptance levels reported by Gallardo (2002). She showed that it is in the transitional process from arithmetic to algebra that the analysis of a student's construction of negative numbers becomes meaningful. During this stage students face equations and problems having negative numbers as coefficients, constants or solutions. She found that five levels of acceptance of negative numbers were abstracted from an empirical study with 35 pupils of 12–13 years old. These levels are the following: <u>Subtrahend</u> where the notion of number is subordinated to magnitude (in $a–b$, a is always greater than b where a and b are natural numbers); <u>Signed number</u> where a plus or a minus sign is associated with the quantity and no additional meaning of the term is necessary; <u>Relative number</u> where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry becomes evident in the continuous domain; <u>Isolated number</u> that of the result of an operation or as the solution to a problem or equation; <u>Formal negative number</u> a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers). This level is usually not reached by 12–13 years old students.

The presence of inhibitory mechanisms. The non-recognition of the subtraction of a greater number from a smaller number arose. This tendency of avoidance obstructed the concept of general number[2] in open sentences[3]. Likewise, the presence of negative solutions provoked the inhibition of syntactical rules which had already been mastered.

THE BEGINNING

The research began with a preliminary phase where The Algebraic Blocks Model [MB] was used, a teaching model for negative numbers, where zero expresses its dual feature, that is to say, *as a null element:* $a = a + 0 = a + 0 + 0 =\ldots$ and as *contained*

[2] General number is a symbol that represents an undetermined entity that can assume any value. Ursini & Trigueros, 2001.

[3] The open sentences belong to the type: $ax \pm bx \pm c =$, and the equations are in the form of: $ax \pm b = c$ and $ax \pm b = cx \pm d$, where a, b, c, d, are natural numbers.

by endless couples of opposites: a + (– a) = 0; where a is a natural number. The zero duality, *nullity – totality* contributed to the extension of the numerical domain in the tasks posed in this preliminary phase of the study.

The MB shows graphical representations with fixed arrangements of numbers, numerical operations and algebraic expressions. The black blocks represent positive terms, uncolored blocks represent negative terms, for example:

■ ■ ■ represents +3;	▯▯▯ ■ ■ represents (–3x) + (+2)

In this model, the count of positive numbers is extended to the negative numbers. Thus, the positive numbers lack their exceptional feature, that of being the only numbers with which you can count. There will be another type of numbers with which you can count with the peculiarity that when facing another positive with the same value, they void each other.

The adding action is represented by joining or linking the corresponding blocks, that is, colored and uncolored blocks. If in this link simultaneously appear blocks from both types, they are matched becoming void elements (main principle of the Model). As an example of the teaching carried out in the model, let us consider the addition 3x + (–2x) =.

> These numbers are represented in a diagrammatic way: ▮▮▮ + ▯▯
>
> They are added, that is, they are put together: ▮▮▮ / ▯▯ In this case, zeros are formed. The result is a rectangle with color that represents *x.:* ▮

The action of subtracting means to take out the elements that constitute the subtrahend from the minuend, marking the eliminated blocks with a cross. The alternate representation of the number: *a = a + 0= a + 0 + 0 =…* becomes relevant for the subtraction when the subtrahend is greater than the minuend. For example, 2 – 3 =, is chosen. The subtraction is represented as follows:

> ■ ■ – ■ ■ ■ It is observed that you can not take 3 from 2, then we continue to add one zero to number 2.
>
> The alternative representation of the minuend 2 + 0: ■ ■ ■ □ is obtained. Now, the subtraction can be made by marking with a cross the squares that have been taken out: ⊠ ⊠ ⊠ □ An uncolored square is obtained, which represents –1.

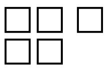
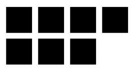
In this preliminary phase, this MB was used as a resource which exhibited the different cognitive tendencies of the students when they face new mathematical concepts and operations (Hernández, 2004). Questionnaires and individual clinical interviews were carried out to 16 students of 8th grade level who had already been taught with MB on the following topics:

1) To identify positive and negative numbers represented in MB.
2) To solve through arithmetical - algebraic language addition and subtraction operations represented in MB.
3) To solve additions and subtractions expressed in arithmetic – algebraic language.
4) To simplify open sentences.
5) To replace any numerical values in algebraic expressions.
6) To solve linear equations.

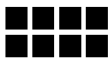
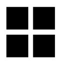
In this article, only the results on the performance of the best student Pamela (**P**), who was able to find a possible path towards the numerical extension during her performance in the interview, are reported. She had obtained the greater number of correct answers and been competent in the use of MB before her interview.

The most representative dialogues of the interview are shown as follows. What was expressed by **P** is written between quotation marks "… ". The interviewer (**E**) interpretations are written in brackets […]. The six topics listed above were approached.

In topic 1, **P** correctly represents algebraic expressions and numbers relating black color with positive numbers and no color with negative numbers. In the solution of operations represented in (Topic 2), the ambiguity between the negative number and the subtraction operation arose. Four cases are displayed as follows:

<u>1st Case</u>. Before addition:  add  , she writes: **–5 + 7=.**

Affirmation: *"It is as if I have seven minus five equals two"* [She reads the expression right to left and transforms the addition into one subtraction. Note that the equivalence: (–5) + (+7) = 7 – 5 is correct. However, the first member represents an addition of signed numbers and the second one a subtraction of natural numbers. In fact, **P** expresses the cognitive tendency that consists in giving the sense of subtrahend to the negative number -5].

<u>2nd Case</u>. Before subtraction:  –  . She writes: + 8 – + 4.

She observes the previous expression and affirms: *"It is the same as eight minus four, because this sign* [+8 – + 4] *is always minus, because minus by plus is minus"* [She considers the – + signs as a unique sign – , since she recurs to the rule of signs (–) (+) = – . This rule belonging to the multiplicative domain was learned without understanding it and she mechanically applies it in the additive mechanism. This fact leads her to consider a subtraction of natural numbers and not a subtraction extended to positive and negative numbers.)

3rd Case. Before representation: □□□□ □□□□ – ■■■■ ■■■ She writes:–8–(+7) = –1.

She observes the previous expression and says: *"It is as if I had minus 8 plus 7, equals minus 1"*. [**E** asks her: Why did not you make the subtraction?] She answered: *"I can not subtract because 7 is greater than negative 8. I made an addition"*. She writes: – 8 + 7 = –1. [**E** asks her again: Why is – 8 + 7 equals –1?] She responds: *"Because* –8 is equal to –1 *and –7, both forming the minus 8,* and she writes: – 8 + 7 = –1 –7 + 7 = –1. [In Fact, she learned through MB to disarrange numbers and to "make zeros". In this case, she transfers what is learned to the syntax and she forms zero: –7 + 7 = 0.]

The most outstanding issue on the 3rd case is that the written expression: –8 + 7; incorrectly considered by **P** equal to – 8 – (+7), has allowed us to discover that she accepts the addition of signed numbers in the case of: – *8* + 7 = –1. This acceptance was possible since she introduced the zero as the couple of opposites: – 7 + 7. The duality of zero has contributed to the extension of the addition beyond natural numbers. The above leads to the affirmation that she recognizes signed numbers, since she says: *"negative eight"*; relative numbers, since she writes: *"– 7 + 7"*, as well as the isolated number: "*–1"*, all of them intermediate senses of the negative and, consequently, manifestations of the cognitive tendency 2.

4th Case. Before representation: ■■■■ ■ ■■■■ ■ – □□ □□ She writes: 10– (– 4)=.

Explains: *"This sign* [10 – (– 4) =] *is from the operation and the other sign* [10 – (– 4) =] *belongs to the number"*. She adds: *"When there are two signs like these* [10 – (– 4)=], *it is as if we add"*. [It is observed that she distinguishes the sign of the number (unary) and the sing of the operation (binary), that is, she recognized signs but not the negative number – 4 per se. This is why, once she detaches from the model and is before the syntactical expression: 10 – (– 4) =, she applies the law of signs (–) (–) = +, disappearing the subtraction and, consequently, the possibility to extend the numerical domain of natural numbers to integers.]

Now we report the most relevant part of **P**'s performance when solving operations syntactically, that is, without the explicit presence of MB (topic 3).

Before the expression: (+ 8) – (+10) =. She writes: + 8 – 10 = 2. And expresses: *"No! It is minus two"* and corrects the result: +8 –10 = – 2. She explains: *"Because it is as if we had a subtraction, ten minus eight and the result is two, but since this number* [+8–10 = – 2] *is negative and greater than eight, the result is negative."* [Note that **P** considered a subtraction of natural numbers, orally expressed as: "*ten minus eight"* and, simultaneously, related the same minus sign to number ten in the written sentence: +8 – 10 = – 2.

In fact, she read the same expression twice, one from right to left for the subtraction *"ten minus eight"* and another from left to right where she recovers number 10 as a negative number, that is, as –10.

This *"incorrect duality of a unique minus sign"* can be interpreted as a progress, although incipient towards the extension of the numerical domain since she was able to subtract a greater number in absolute value from a minor number in absolute value through her "personal conception" of the minus sign duality.]

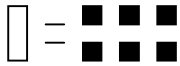
She interprets the unique minus sign [(+8) – (+10) = considering they have a dual nature, that is to say, she warns that the sign is linked to the number and also considers it as the sign of the operation and believes that this fact can occur simultaneously]. Although incorrect, this double meaning related to one sign allows her to subtract a greater number from a minor number, situation she did not accept in the 3rd case. Regarding the simplification of algebraic expressions (topic 4), **P** does not express any difficulties. In the replacement of numerical values in open sentences (topic 5), the inhibiting tendency regarding the subtraction operation appears once again, as it can be observed in the following dialogue corresponding to item x – 8 =.

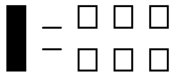
She writes: 3. And affirms: *"Here* [x – 8 =] *it would be three minus eight,…no!...it can not be done. I am going to choose another number since I can not subtract eight from three."* [Note that tendency 8 brought up in open sentences provokes an obstruction in the general number concept.]

It is important to point out that in the algebraic field, particularly before the expression x – 8 =, **P** presents an avoidance even with positive 3 since it is a number smaller than 8 that would allow the operation 3 – 8 = and she expresses: *"It can not be done."* During the replacement process, she does not extend the subtraction to the integers, despite she had done it in the numerical field in the case: 8 – 10 = –2 (topic 3). Therefore, when not accepting that *x* can take any values, **P** expresses that she does not recognize the variable as a general number in x – 8 = .

In the solution of equations (topic 6), **P** always uses the same procedure of adding or subtracting the additive inverse in both members of the equation. This has been the usual method she was taught to solve equations in MB and she transfers it to the algebraic language without difficulties. This fact shows that **P** recognizes the equality in its dual feature, that is, not only as an arithmetical equality that means carrying out one operation and obtaining a result, but also as an equivalent relationship between both members of the equality (Kieran, 1980). However, before the equations: 6 – x = 12, she is wrong as it can be verified as follows:

6 – x = 12 She adds number 6 to both members of the equation:

6 + 6 – x = 12 + 6. She affirms: "*I better solve with squares.*" [She uses MB and finally reaches the arrangement:  And explains: "*I am changing the color of the squares.*"

And shows the following arrangement:  **P** is quiet a few minutes… (question):*"Then x = –6?"* E: Why do you doubt?

She affirms: "*Because x is now positive, but six is still negative.*"

[**P** has difficulty to recognize a negative number as solution of the equation, when in some cases she had already recognized negative isolated numbers as results of operations (topic 2; 3rd case and topic 3), but not as a possible value for the unknown].

It is important to point out that the fact of considering the equality as one equivalence of expressions is what previously allowed **P** to solve equations which solutions were always positive. However, before an expression with a negative solution (x = – 6), P decodifies x as positive forgetting the equivalence of equality, fact recognized in the preceding tasks. The negative value unbalanced a knowledge that seemed to be consolidated in her.

FINAL DISCUSSION

From the analysis and interpretation of the interview, it can be concluded that during the transition of the arithmetic to algebra, the identification of the cognitive tendencies of **P** showed difficulties, but also possibilities of success leading to understanding the numerical extension. Therefore, if **P** only warns the subtrahend level and, besides, is not able to subtract a greater number from a minor number, or if she applies the laws of signs (–) (–) = +, (–) (+) = –, she will not be able to reach integers. But if **P** recognized the levels of the signed number, relative and isolated, as well as the extension of the subtraction operation (through the duality of the minus sign: unary – binary); besides, she can add signed numbers (through the duality of zero: nullity – totality) and knows a general method for the solution of equations (through the duality of equality: operator symbol – equivalence relation). **P** is before the possibility to accept other numbers different to the natural ones. We can affirm that these first results partially answer the three questions of the research: a), b) and c) posed in the introduction of this article, since zero contributed to the extension of the numerical domain through MB. **P** was able to carry out operations with zero, fact that proves that she considers it a number. Likewise, in her written productions she used, although without conscience of it, the three dualities above mentioned that she transferred from MB to the arithmetic – algebraic language. Now, we can neither affirm that these facts are completely stable in **P** nor that we know if it is about a transitory knowledge only linked to the teaching model she learned. She showed a good performance in "the elementary algebra of positive numbers". However, when negatives arose, persistent obstacles were present (inhibition of the subtraction operation, obstruction of the general number, non-acceptance of the negative solution) that prove the need of carrying out the theoretical analysis at a deeper level, as the one posed in the outgoing project mentioned at the beginning of this article.

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