STUDENTS' COLLOQUIAL AND MATHEMATICAL DISCOURSES ON INFINITY AND LIMIT

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The purpose of the study presented in this paper is to investigate how students deal with the concepts of infinity and limit. Based on the communicational approach to cognition, according to which mathematics is a kind of discourse, we try to identify the characteristics of students' discourse on the topic. Four American and four Korean students were interviewed in English on limits and infinity and their discourse is scrutinized with an eye to the common characteristics as well as culture, age, and education-related differences.

INTRODUCTION

Infinity is the conceptual basis for many mathematical topics such as the number line and infinite decimals. Since the 19th century, the concept of limit has been foundational to mathematical analysis. As is known to teachers and as confirmed by researchers, most students have considerable difficulty with both of these notions.

In this study, students' thinking about infinity and limit is investigated based on the *communicational approach to cognition*, according to which mathematics is a kind of discourse. There are several reasons why this kind of study may be important. First, there has been little research on the mathematical concepts of infinity and limit using discourse analysis as a methodology. Discourse analysis holds promise of answering some previously unanswered questions. Second, such investigations may lead to methods for helping students overcome their difficulties, something that will also have implications for teacher education and K-16 curriculum. Third, this approach may have implications for investigating advanced mathematical thinking in other areas. Finally, applying the communicational method to culturally different groups of students may shed light on the impact of culture on how students learn the most advanced of mathematical notions.

THEORETICAL BACKGROUND

Epistemology and History of the Notions of Infinity and Limit

The histories of the mathematical concepts of *infinity* and *limit* have been interwoven ever since their beginning. The story of infinity begins with the ancient Greeks. The Greek word '*apeiron*' meant *unbounded*, *indefinite*, or *undefined* (Boyer, 1949). For the Greeks, infinity did not exist in actuality, but rather as a potential construct. Although there was the notion of bounded processes, there was no concept of limit as a concrete bounding entity.

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In the Middle Ages, Christianity came to value infinity as a divine property. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find methods for calculating the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). Throughout the 18th century, calculus lacked firm conceptual foundations. At the end of the 18th century, mathematicians became acutely aware of inconsistencies with which the theory of infinitesimal magnitudes seemed to be fraught (Rotman, 1993).

Today's notion of limit emerged gradually in the 19th century as a result of attempts to remedy the uncertainties with which the mathematical analysis was ridden at that time. Cauchy and Weierstrass were pioneers of the movement toward a rigorous calculus (Moore, 1990). By mid-19th century the concept of limit became the basic concept of the calculus (Kleiner, 2001). At this time, limit turned into an arithmetical rather than geometrical concept, as it was before, in the context of infinitesimals. Infinity was now actual rather than potential. In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of the infinite set (Boyer, 1949).

In spite of the mutual interdependence of the concepts of limit and infinity, there has been little research to examine students' understandings and difficulties of both of them simultaneously.

Leaning the Mathematical Notions of Infinity and Limit

Various aspects of the learning about infinity and limit have been investigated over the last few decades.

Anchoring their research in the analysis *mathematical structure* of the notions, Cottrill et al. (1996) report that there are two reasons for student difficulties with limits. One reason is the need to mentally coordinate two processes ($x \to a,\ f(x) \to L$). The other is the need for a good understanding of quantification related to ε and δ. Borasi (1985) suggests several alternative rules about how to compare infinities based on students' intuitive notions (within this tradition, see also Cornu, 1992; Tall, 1992).

Other research focused on *misconceptions* and *cognitive obstacles* related to infinity and limit. Fischbein, Tirosh, & Hass (1979) and Tall (1992) emphasized the role of intuition. One source of difficulty is the belief that a part must be smaller than the whole. Other researchers (Cornu, 1992; Davis & Vinner, 1986) stress the influence of language. Students might have had many life experiences with boundaries, speed limits, minimum wages, etc. that involved the word "limit". These everyday linguistic uses interfere with students' mathematical understandings (Davis & Vinner, 1986). Przenioslo focuses on the key elements of students' *concept images* of the limits of functions. Still others have focused on *informal models* that act as cognitive obstacles (Fischbein, 2001; Williams; 2001). According to Williams, informal models based on the notion of actual infinity are a primary cognitive obstacle to students' learning.

Finally, some researchers address students' difficulties through the lens of the theory of actions, processes, objects, and schemas (APOS; see Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). Weller et al. speak about the cognitive mechanisms *interiorization*, *encapsulation*, and *thematization* that are used to build and connect actions, processes, objects, and schemas.

Conceptual Framework for This Research

The point of departure for the present study is the realization of the fact that when students come to the classroom to learn the notions of infinity and limit, they already have a certain amount of knowledge that comes from daily experience. The use of a given concept in everyday language can be crucial for students' future learning. For those who are supposed to teach the subject it is therefore important to find out how students use the notions of infinity and limit in colloquial discourse.

Most of the past research on learning limits and infinity was grounded in a neo-Piagetian, cognitivist framework which does not seem quite appropriate for this type of study as it underestimates not only the inherently social nature of student thinking, but also the role of discourse and communication in learning and other intellectual activities. Our project is guided by the conceptual framework within which school learning is equated to a change in ways of communicating. In particular, learning mathematics is seen as tantamount to becoming more skilful in the discourse regarded as mathematical. The word *discourse* signifies any type of communicative activities, whether with others or with oneself, whether verbal or not. Four distinctive features of mathematical discourses are often considered whenever discourses are being analysed, compared, and watched for changes over time: *words and their use*, *discursive routines, endorsed narratives*, and *mediators and their use* (Ben-Yehuda et al., 2004). In our ongoing study, particular attention has been paid to the participants' uses of the keywords *limit* and *infinity* in colloquial and mathematical discourses; to *discursive routines*, that is, repetitive patterns of both these discourses; and to *endorsed narratives* about limits and identity, that is, to propositions that the participants accepted as true.

DESIGN OF STUDY

Research Questions

Our interest in characterizing the mechanisms of students' thinking about infinity and limit led to the following research questions:

• What are the leading characteristics (in terms of word use, endorsed narratives, and routines) of students' colloquial and literate (school) discourse about infinity and limit?

• Do the students' colloquial and literate discourse on infinity and limit change with age and education?

• Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit? Can these differences be accounted for in terms of the differences in the colloquial uses of these words in English and in Korean?

Methodology

Each ethnically distinct group included one elementary student, one middle school student, one high school student, and one university undergraduate (to refer to groups' members, we use symbols such as A_5 for the American 5^{th} grader, K_{10} for the Korean 10^{th} grader, and A_U for the American undergraduate.) The four American students were English speakers from the United States while the four Korean students were non-native English speakers from South Korea whose first language is Korean. Since the interviews were conducted in English, the four Korean students who were selected had been living in the United States and attending US schools more than 3 years.

The interview questionnaire consisted of 29 questions, organized into eight categories. The first two categories aimed at scrutinizing students' *colloquial* discourses on infinity and limit, whereas the rest were targeted at investigating students' *mathematical* discourses on the topic. Examples of the interview questions are shown in Figure 1.

I. Create a sentence with the following word: (a) *Infinite*, **(b)** *Infinity*.

II. Say the same thing without using the underlined word.

(b) Eyeglasses are for people with **<u>limited</u>** eyesight.

III. Which is a greater amount and how do you know?

(d) A: odd numbers, B: Integers

IV. $\frac{1}{4} = 0.25$, $\frac{2}{8} = 0.25$, $\frac{3}{12} = 0.25$, …

How many such equalities can you write?

V. What do you think will happen later in this table? How do you know?

VI. (a) What is the limit of the following $\frac{1}{x}$ **when x goes to infinity?**

VII. Read aloud: $\lim_{x\to\infty} \frac{x^2+3x}{2x^2} = \frac{1}{2}$. **Explain what it says.**

VIII. (a) What is infinity?

Figure 1: Representative samples of questions from each category.

The interviews, which were conducted in English, lasted 30 to 40 minutes. The conversations were audio- and video-taped and then transcribed in their entirety.

Data were analyzed so as to identify and describe the three distinctive features of the respondents' discourses: word use, routines, and endorsed narratives. At the next stage, several comparisons were made: (a) for the two ethnic groups, we looked for similarities and differences between the group's colloquial and mathematical uses of the keywords *infinity* and *limit*; (b) we searched for differences in the mathematical discourses of different age groups within and across the ethnic groups (c) we compared the colloquial and mathematical uses of the words limit and infinity of the two ethnic groups. In this paper, we present the respondents' uses of the words *infinity* and *infinite*, and compare the results obtained in the two ethnic groups (comparison (c)).

SELECTED FINDINGS

The Use of the Words Infinity and Infinite

The Korean words *infinity* and *infinite* are mathematical and less often used in colloquial Korean than colloquial American. All students, with the exception of two undergraduates who took a calculus course, did not have the formal education about mathematical infinity.

In the first category, students were asked to create sentences with the words *infinite* and *infinity* (a separate sentence for each of these words). Their responses are summarized in Table. 1.

Students		I. Create a sentence with the following word: (a) Infinite		(b) Infinity	
American	A_5	[1]	They have an infinite amount of movies	[5]	The years go to infinity
	A_7	[2]	Outer-space is infinite and forever	[6]	Infinity is a concept…not a number
	A_{10}	[3]	There is an infinite amount of numbers	[7]	Infinity is the largest number
	A_U	[4]	There are infinite ways to spell certain things	[8]	I love you more than infinity
Korean	K_4	[9]	Numbers are infinite	[13]	One times infinity is infinity
	K_7	[10]	Numbers are infinite	[14]	A line goes to infinity
	K_{10}	[11]	We don't have infinite amount of natural resource in the planet	[15]	I don't think there is not just a thing in infinity.
	K_U	[12]	Some people like infinite space	[16]	We have to study infinity

Table 1: The summary of answer to the question that requires a sentence.

The first thing to note is that all the students, even those who are too young to have met the notion of infinity in the context of school mathematics, are capable of creating sentences with the words *infinite* and *infinity* – a fact that testifies to these words being a part of everybody's colloquial English discourse.

This said, there is a considerable difference between the American and Korean groups in the context in which the words are mentioned. In the American group, *infinite* is used in conjunction with *amount* in [1], [3], [11] and both words are applied mainly in the context of real-life phenomena involving large magnitudes: outer-space [2], ways to spell words, [4], number of years [5], love [8], etc. In the Korean group, the context of the sentences is predominantly abstract and mathematical (the sentences mention numbers [9], [10], lines [14], operations on infinity [13], and the infinity as an object of study [16]), whereas the relation to magnitudes and to large amounts is less pronounced.

There is also a delicate ontological difference between the groups in their application of the word *infinite* to numbers: While all three students who apply the word *infinite* to numbers (A_{10}, K_4 and K_7) seem to be saying the same thing – that numbers can be "infinite", only A_{10} makes it clear that he means the size *of the set* of all numbers. There is no reference to *the set* of all numbers in the utterances of the Korean students, and these utterances may be interpreted as saying that these are the numbers themselves (as opposed to the *set of numbers*, which is a second order construct) that are unlimited in their size.

The two differences noted above may be explained on the basis of the fact that the Korean mathematical words for *infinity* and *infinite* and *set*, with their origins in Chinese characters, do not appear in the Korean colloquial language and the Korean students do not associate them, or even their English counterparts, with anything in particular in the colloquial discourse. One can conjecture that for American students, the colloquial use of the English word *infinite* precedes the mathematical, whereas for the Korean students it may go the other way round.

The Use of the Word Infinity through Definitions

Students		VIII. (a) What is infinity?
American	A_5	Infinity will go on forever.
	A_7	Infinity is a concept that goes on forever
	A_{10}	Infinity keeps going on and increasing. Infinity has no limit.
	A_U	Infinity is never-ending…has no beginning and no end. There is not one thing that is infinite in the world. It's just a concept.
Korean	K_4	Infinity is like a number that never ends or something that never ends; infinity is not the number
	K_7	Infinity is like the furthest number keep going…like never-ending. Infinity is like it goes forever like there is no end.
	K_{10}	It's not a number…it's same like it's not limited…same that never end. The number system that never ends and keeps coming.
	K_U	It's *not a number* because it is a very large amount and cannot be explained to the number.

Table 2: The summary of the definitions about *infinity*.

In this part, the focus is on the *mathematical* discourse on infinity. The students were asked to define the notion and their responses are presented in Table 2.

The prevalent feature of the definitions given by the American students is that they take the object-like character of infinity for granted and characterize this object by saying what it is doing: "go on forever" (A_5, A_7), "keeps going and increasing" (A_{10}), "is never-ending" (A_U).

The Korean students begin with an attempt to specify the category to which infinity belongs, and they usually do it with the help of comparative or negative sentences, such as "It is like a number" (K_4, K_7) or "It's not a number" (K_{10}, K_U). Thus, a common property of all the answers in this group is that while stating some number-like properties of infinity, they also deny its being a number. Such explicit comparison to number (or to any other entity, for that matter) is absent from the American answers.

We have reasons, once again, to speculate that the Korean students' acquaintance with the English word *infinity*, unlike that of the American students, came primarily from the formal mathematical discourse. The more rigorous structure of their descriptions, which, unlike those of the American students, begin with the attempt to specify the general category and continue with the presentation of specific features, may be yet additional evidence that these students' were introduced to the discourse on infinity through mathematical, or as-if mathematical, definitions rather than through casual use.

CONCLUSION

Although our sample is too small to allow for generalizations, what we find in this study may serve as a basis for hypotheses to be tested in a future, more extensive project. On the grounds of our findings so far (and these were only sampled in the present report), we can conclude that colloquial discourse does seem to have an impact on mathematical discourse. This fact was evidenced by certain clear differences between the mathematical discourses on infinity of the American and Korean students, differences that we ascribed to the fact that only in English do the mathematical words *infinity* and *infinite* (as well as *set*) appear also in the colloquial discourse. With the colloquial discourse being the primary source of the American student's acquaintance with the notion, this discourse may have an impact not only on the students' later use of the mathematical keywords, but also on other aspects of their mathematical discourse, such as routines, use of mediators, and endorsed narratives. Our preliminary findings presented in this report justify additional attempts to test this conjecture.

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