GROWTH OF MATHEMATICAL UNDERSTANDING IN A BILINGUAL CONTEXT: ANALYSIS AND IMPLICATIONS

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The emergence of the Pirie-Kieren theory for the dynamical growth of mathematical understanding has inspired studies focusing on monolingual students. However, prior to my research, a study had yet to be undertaken that applied this theoretical model to a bilingual context. The purpose of this paper, therefore, is twofold: one, to illustrate an application of the Pirie-Kieren theory within a bilingual context, as a language for – and a way of observing, and accounting for – the growth of understanding; and, two, to examine within this bilingual context the subtle relationship between language switching and growth of mathematical understanding.[1] In order to pursue these purposes, the work of two Tongan bilingual students is analysed in order to apply the findings toward teaching and learning.

Since 1987, Pirie and Kieren have been more interested in investigating the "process of understanding" as an alternative way of looking at mathematical understanding, as "always under construction." (Kieren, Pirie & Reid, 1994, p. 49) The Pirie-Kieren theory was developed as a theory *for* the growth of mathematical understanding of a specific topic, by a specific person, over time. The analysis in this paper draws upon the Pirie-Kieren theory, a theory previously presented and discussed at a number of PME meetings (Kieren, Pirie & Reid, 1994; Martin & Pirie, 1998; Pirie, Martin & Kieren, 1996).[2] The Pirie-Kieren theory posits eight potential layers of understanding, attainable through either informal or formal actions, for a specific person, and for a specified topic. Beginning with the innermost layer, these layers are: *primitive knowing, image making, image having, property noticing, formalizing, observing, structuring,* and *inventising*. Each layer is embedded in all succeeding layers, and, with the exception of primitive knowing, each contains all previous layers. Hence, *growth of mathematical understanding* is observe as a continuous, back-and-forth movement through these layers of understanding, as the individual reflects on, and re-constructs, his or her current knowledge.

To illustrate the use of the Pirie-Kieren model within a bilingual context, a piece of video data is analysed from a case study with secondary school students in Tonga, located in the South Pacific. The Tongan bilingual context shares many features with other bilingual programs around the world, with the dominant English language being used to teach mathematics at the secondary level. Such dominant languages are considered to have "superior" mathematics vocabularies compared to indigenous

[1] The paper stems from my research that forms part of my PhD thesis on the aforementioned relationship, presently being submitted to the University of British Columbia under the supervision of Professor Susan E. B. Pirie.

[2] Features of the Pirie-Kieren theory were elaborated on in these meetings, and elsewhere (see Pirie & Kieren, 1994). Due to the scope of this paper, I will not elaborate on these features, but will explain each layer *through* the analysis.

languages. Yet, the Tongan students prefer to learn and talk about mathematics in their native language, especially in peer discussions. Largely because of the inadequacy of most indigenous languages in the language of mathematics, and the students' lack of proficiency in the language of instruction, bilingual students of the Tongan-type, as well as teachers, *switch* languages during mathematical discourse. This paper, which uses the Tongan bilingual context to study growth of mathematical understanding, offers an opportunity to appreciate, and encourages further study of, the unique characteristics of bilingual students in the South Pacific islands – a subject and a region traditionally marked by the absence of such useful educational research.

METHOD, SETTING AND TASK

The majority of this brief paper's analysis features a video case study of two Form 3 (13- to 14-year-old) Tongan bilingual students, Alani (A) and Maile (M), working together as a group (without the presence of any external observer). Video recording, widely used in case study research[3] (and most studies using the Pirie-Kieren model), was chosen as the most appropriate means of recording, collecting, and examining growth of mathematical understanding in a small-group setting. Accompanying the collected video data were students' work sheets, along with follow-up interviews about their recorded work. Using the Pirie-Kieren model, I will highlight a "mapping**",** a technique, for tracing the two students' growth of understanding of the chosen topic, "pattern".[4] At the same time, the analysis directs attention to the students' language use, particularly the role of their language switching. Language switching, also known as "code switching", is a unique feature of any bilingual situation, described by Baker (1993) as the way bilingual individuals alternate between two languages, in words, phrases, or sentences.

For the case study in question, Alani and Maile were given the first three diagrams of a continuing sequence in diagrammatical form (**Figure 1**).

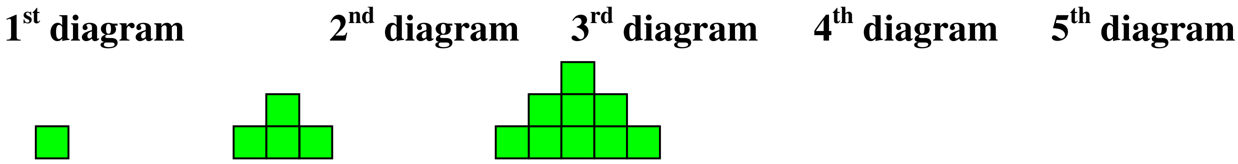


Figure 1: The task's pictorial sequence.

The task was designed for students to create, manipulate, test, and explore their ideas of or about patterns. To do that, the given pictorial sequence was accompanied by a set of questions to guide and validate their mathematical activity. The students were left to construct and draw the 4th and 5th diagrams (Questions 1 and 3), determine the difference between the 3rd and 4th diagrams (Question 2), and then later asked to discuss and predict totals for the 6th, 7th, 17th, and 60th diagrams in the given pictorial

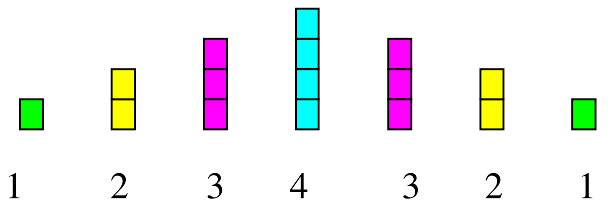
[3] Bottorff (1994) and Pirie (1996) discuss, in-depth, the nature and use of video recordings in qualitative research.
[4] *Tracing* involves examining students' constructed images. An "image" can be any physical or mental representation students may have about the topic, and that is, by its nature, specific, mathematically limiting, and context dependent.

sequence (Questions 5, 6, 9, and 10.) In particular, the students were asked to identify the patterns in the number of square blocks they added each time (Question 7) and the total number of square blocks used in each diagram (Question 8).

DATA ANALYSIS: MAPPING

When Alani is given the task, he reads the first question aloud, then immediately switches languages to work in Tongan [**LS1**] as he points out the total of the first two diagrams, "So it's one there [1st diagram] --- four there [2nd diagram] --- Hold on --- one there; one, two, three, four --- add three to that [1st diagram]?"[5] Alani's actions indicate evidence of **image making** [**G1**], as he first engages directly, through counting, in an activity associated with constructing an image of what he sees, then he reviews his work in an attempt to make sense of it.

In his part, Maile responds to Alani in Tongan: "Do it quickly --- you just draw it [4th diagram] --- you don't have to waste time." Then Maile sketches the 4th diagram and explains in Tongan his pictorial image of the pattern as a set of ascending and descending vertical columns of square blocks. (**Figure 2a**) Starting with four square blocks in the middle, Maile explains, "Just do it like this: four, three, two, one [middle column to left] --- three, two, one [right columns] --- finish!" In response to Maile's constructed image [**G2**], Alani explains in Tongan his constructed image for the pattern along the base layers [**G3**]. He says, "Five there [base of 3rd diag.] --- so it's five there and seven at the bottom there [base of 4th diag.]. Right? Seven there and then five there, and three there. Right? Yes!" (**Figure 2b**) Both students, therefore, move out in their growth of understanding to work at the **image having** layer and are also able to articulate the features of the sequence in Tongan.



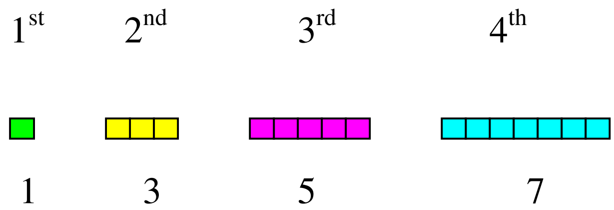


Figure 2a: *Maile's image (columns).* **Figure 2b**: *Alani's image (base layers).*

Next, Alani uses his constructed image to draw the 4th diagram by "stacking" the horizontal layers together, arranged appropriately from the bottom to top in ascending order, meanwhile speaking only in Tongan. [**G4**] (See **Figure 3**)

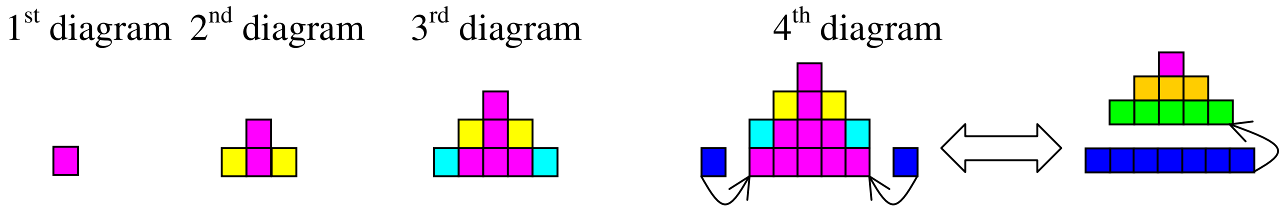


Figure 3: Alani's images for the pattern along the base layers, and at the corners.

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[5] The **underlined** words are the English translation of the bilingual students' Tongan discourse. The codes "**LS**" and "**G**" refer to evidences of language switching (or no language switching) and growth of understanding respectively.

Then Alani moves to answering Question 2, and after reading it, Alani **extracts** the key words, “extra square” [**LS2**] by saying “Extra square --- so it’s seven there [base of 4th diagram]; and then nine the base there [5th diagram] --- and then eleven the base there [6th diagram]. Alani reflects and articulates on his constructed pictorial image for the pattern along the base layers from the 4th to the 6th diagram (**Figure 2b**), as evidence of his continued working at the **image having** layer [**G5**].

During his work at the image having layer, Alani consciously sees the relationship between the layers, and concludes that the pattern in his pictorial images – the *common difference* – along the base layers *and* between horizontal layers is “Add two extra squares” (see **Figure 3**) [**LS3**]. Alani has moved out to **property noticing** [**G6**] by constructing a context-specific property, based on his knowledge in manipulating and combining aspects of his constructed pictorial images. In this instance, Alani associates the key phrase, “extra square”, from the question with the common difference (of two square blocks) between any two consecutive layers. This extracted phrase appears, therefore, to *dictate* the way Alani approaches his constructed images, and at the same time *directs* him toward outer-layer thinking sophistication.

However, Alani continues describing his constructed images in Tongan, then he sketches the base layer of the 3rd diagram and asks Maile, “What is it called that one at the bottom there?” Maile responds, “Base!” [**LS4**]. Still, Alani is not satisfied as he continues to use equivalent English words such as “row” and “step” as metaphors for the extra square blocks being added on both ends of the base layers. However, Maile intervenes and expresses his intention of “thinking” about the task [**Extract 1**[6]**]:**

1	M:	Hold on while I think. You move from that one there while I think myself.
2		You do number three while I think it [*Question 2*] myself.
3	A:	No! Look here: add two to the last block.
4		Which is that [*base of 2nd*]; add two to the last step.
5		--- get then that five there. [*Base of 3rd*]
6		Add two to the last step [*base* of 3rd] --- and get then a seven [*base of 4th*]
7		--- what’s an explanation for it? Two extra square blocks ---
8	M:	Oh --- I already know it --- add the prime number.
9		--- Look here: it’s three; it’s five; it’s seven; it’s nine ---

Alani continues to work at the **property noticing** layer [**G7**] but still struggles to find the “right” English label for his noticed property, although he seems comfortable expressing it in Tongan [3-7]. He associates the *Tonganized*[7] equivalent words “sitepu” (step) and “poloka” (block) with the phrase “poloka fakamuimuitaha” (last block) [3] and “sitepu faka’osi” (last step) [4]. These “verbal associations” have no effect on his growth of understanding. [**LS5**] But Maile’s statement, “Hold on while I think ---” [1], shows *stepping back* and *looking* for any connection within or among his constructed image(s). This stepping-back process allows Maile to notice a

[6] Because of the length of this paper, selected video transcripts represent segments of the entire video recording, but the comments and discussions draw on considerable additional data.
[7] Tonganization is a form of “conventionalisation” in which English words have been borrowed and used in Tongan.

numerical pattern along the base layers when he declares, "I already know it --- add the prime numbers" [8]. This evidence of moving out to **property noticing** [**G8**] is accompanied by a **shift** in language to using the non-equivalent English word, "prime" [**LS6**]. In this situation, Maile's growth of understanding *brings about* the act of language switching using the term, "prime numbers". However, Maile's mathematical meaning for this label becomes apparent, in relation to Alani's constructed image (**Figure 2b**), when Maile explains, "Look here: it's three, it's five, it's seven, it's nine" [**LS7**]. This incident is discussed further in the conclusion.

Meanwhile, Alani answers Question 3 by finding the total number of square blocks in the first four diagrams. He shifts to working only in Tongan as he reflects on his previous constructions. Maile continues to "think" silently while Alani finishes drawing the 4^{th} diagram. Alani then moves back out to discussing, in Tongan, the total for the 5^{th} diagram with Maile (**Question 4**), and together the two students come up with a total of "Ua-nima --- Twenty five square blocks". [**LS8**] As he attempts to draw the 5^{th} diagram, Alani finds himself unable to come up with the diagram. This prompts him to *fold back* from property noticing [**G9**], where he was currently working, to image having [**G10**], and to work with his constructed images for the pattern. Pirie and Kieren (1994) call this action "folding back", because Alani is faced with a challenge that is not immediately solvable, prompting him to return to an inner mode of understanding in order to re-construct and extend his current *inadequate mathematical* understanding. In this situation, language switching is not involved, as he speaks only Tongan. Alani goes back to his image along the base layers (**Figure 2b**), and uses it to build the 5^{th} diagram as a stack of horizontal layers.

While Alani draws the 5^{th} diagram, Maile uses "trial-and-error" method to manipulate random numbers arithmetically in order to match the diagrams' totals, again through thinking aloud in Tongan. Then Maile suddenly notices another property [**Extract 2**]:

M: I already know it. Look here; here it is:
[*Points to the relation between diagrams' ordered number and totals*]
One by one is one; two by two is four; three by three is nine; four by four
--- is sixteen; five by five, twenty-five…seven by seven is forty-nine.
How about that? Square number. How about that? Got it!
I just thought of it and get it. Five to that --- [*draws the layers of 5^{th} diag.*]
M: So it's seventeen by seventeen --- yeah!
A: One, two, three, four, five, six, seven; and then next is nine.

In this instance, Maile finds a relationship between the diagrams' numbers and their corresponding totals [10-13], again illustrating working with his constructed numerical images at the **property noticing** layer [**G11**]. He identifies this noticed property of the numerical totals as "square numbers" [14] [**LS9**], then quickly applies it to the 17^{th} diagram [16]. He later verifies the total for the 6^{th} diagram (Question 5), as evidence of a constructed, organized scheme for the pattern. The two students continue using Maile's newly found arithmetical rule to determine the total for the 7^{th} diagram (Question 6). In explaining their answer, Maile simply says, "No need to draw it!" [**LS10**]. Such a declaration suggests Maile's readiness to move outward in his growth of understanding, to **formalizing**. [**G12**]

The two students then move to answering Question 7 about the pattern in the extra number of square blocks. Alani, after reading the question, **shifts** languages again to discussing his generalization of the "extras" (additional square blocks) in Tongan. [**G13**] He explains and *reformulates* his formalized understanding of the pattern as, "Add two to the last step and total them." [**LS11**] But his peer, Maile, reflects on his own prior construction by saying: "You just add the prime number to the last row". Furthermore, Maile associates the mathematical label, "prime numbers", with the set {1, 3, 5, …}, which stands for the number of square blocks along the "last row or the base". [**LS12**] Both students are observed to have moved out to work at the formalizing layer. [**G14**] Maile then goes on to read Questions 9 and 10 in predicting the totals for the 17th and 60th diagrams [**Extract 3**]:

M: Square the seventeen. What number? Seventeen by seventeen.
A: Seven by seven, forty-nine; One by seven --- eleven (*add 7 and carried 4*)
One, seven, zero --- Nine, eighty-two --- Two eighty-nine.
Do that --- seventeen times seventeen --- just the same ---
M: Square --- (*Writes as* "289. Explanation is, $17^2 = 289$")
A: Square --- square the seventeenth diagram.
M: Ten --- "Can you group predict the 60th diagram?" Sixty by sixty.
A: Sixty times sixty --- six by zero is zero --- zero ["3600, 60^2 or square 60"]
six by six is thirty-six --- Three thousands, six hundreds!

In this extract, Alani and Maile are observed to move out to work at the **formalizing** layer [**G14**], through their quick application of the generalized arithmetical rule for calculating the totals for the 17th and 60th diagrams. Each time, the students **shift** back-and-forth between reading the question in English, to calculating the totals in Tongan – the language they feel most comfortable in expressing themselves [**LS13**].

DISCUSSION AND IMPLICATIONS

Figure 4 shows the mapping of Alani and Maile's growth of understanding of "pattern". (Alani's mapping in **bold** line, and Maile's in *thin* line.) Like all Pirie-Kieren mappings[8], each "point" marks a significant incident in the students' growth of understanding. This example, therefore, highlights the power of the Pirie-Kieren theory as an observational tool, not just in monolingual cases, but in bilingual situations as well. As a result, four specific relationships between language switching and growth of mathematical understanding emerged: one, language switching can still occur without growth of mathematical understanding (see discussion on **Extract 1, LS5**); two, growth of mathematical understanding can take place in the absence of language switching (see examples in **G2**, **G3, G11-G12**, and folding back in **G9-**

[8] The tabular format of this mapping reveals very little of the complex nature of growth of understanding as viewed through the Pirie-Kieren theory. But it helps to see clearly the back-and-forth movements discussed in this example. [**Keys:** primitive knowing (**PK**), image making (**IM**), image having (**IH**), property noticing (**PN**) and formalizing (**F**)]

G10); three, language switching can *enable* growth of mathematical understanding (example **LS3/G6**); and four, growth of mathematical understanding can *bring about* language switching (see examples in **LS6/G8** and **LS9/G11**).

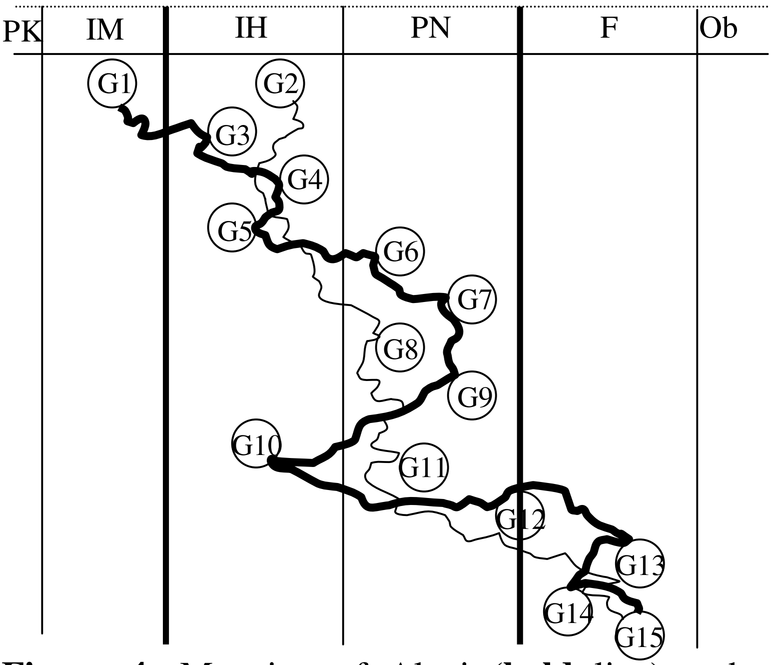

Figure 4: Mapping of Alani (**bold** line) and Maile's growth of understanding of "pattern".

In the first case, the ability of a bilingual learner to express his or her mathematical understanding through language switching comes from his or her underlying *bilingual* language capacity, which, in Pirie-Kieren terms, is a part of the learner's primitive knowing – the prior knowledge he or she brings *into* the task.

But in the second case, the example of Alani and Maile shows two students progressing in their understanding of the topic "pattern" without the necessity or effect of language switching. This finding challenges the assumption that bilingual students will find mathematics harder if the language of instruction is in their second language, and that such students are therefore naturally disadvantaged in mathematics in comparison to monolingual students.

More importantly, Alani and Maile are able to work mainly with mathematical ideas and images to further their growing understanding of "pattern", in spite of their weak English skills. Based on these two students' example, mathematical understanding does rest with the ideas and images, and not with the words (Lakoff & Núñez, 2000).

The case of Alani and Maile also shows they continued to progress in their growth of understanding without needing to use English, instead resorting to their native language while *working* and *thinking* mathematically. Such an observation offers a profound implication for teaching in bilingual situations, because it refutes two naïve assertions: first, that bilingual students' first language is irrelevant to their understanding of mathematics, and second, that indigenous language learning is actually detrimental to a student's education.

Finally, Maile's incorrect *borrowing*[9] of the phrase, "prime numbers", did not deter him from his understanding of the mathematics, or his growing understanding of the topic. This example explains why Tongan-type bilingual students do not necessarily

[9] I described "borrowing" as a form of language switching, involving mixing of *non-equivalent* English words.

have to be good at their second language, in order to be better in mathematics. Teachers, however, often assume students literally misunderstand whenever teachers "hear" wrong mathematical labels, and consequently "believe" they have encountered a lack of mathematical understanding. Thus, teachers ought to pay attention to distinguishing wrong mathematical labels from a lack of mathematical understanding, particularly in bilingual situations where an act of language switching can be easily overheard, identified, and likely *evoke* various instantaneous images.

In conclusion, there is a dilemma within the bilingual mathematics classrooms between "teaching language" versus "teaching mathematics" (Adler, 1998). If, in a bilingual situation, mathematics is the goal of teaching, then mathematical ideas, concepts, and images, have to be the focus of teaching; teachers must also be aware of their appropriate use of the mathematical language, terminology, and convention.

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