ANALYZING STUDENT MODELING CYCLES IN THE CONTEXT OF A 'REAL WORLD' PROBLEM

Roberta Y. Schorr — Rutgers University-Newark

Miriam Amit — Ben Gurion University

Many students do not apply their real world intuitions and sense-making abilities when solving mathematics problems in school. In an effort to better understand how to help students draw upon these valued resources, we investigate the manner in which the solution to a particular problem activity is repeatedly re-interpreted by a student. This is done within the context of a models and modeling framework in which we discuss the modeling cycles and associated models that were used. We suggest that the nature of the problem activity combined with the time and support needed to cycle through multiple models contributed to this student's ability to move beyond his initial, simplistic solution, toward a more complex solution, one that ultimately fit well within his own "real world" intuitions and experiences.

INTRODUCTION

The fundamental premise of this research is that students should be able to learn mathematics with understanding. A central component of understanding involves reflecting upon one's own solution processes, and then refining and revising the solution as appropriate in order to produce solutions that make sense. (Hiebert et.al., 1997). Unfortunately, this act of understanding does not occur as often as we would hope. Indeed, the mathematics education research literature is replete with instances in which young students provide solutions to mathematical problems that make little or no real world sense (Carpenter, Lindquist, Matthews, & Silver, 1983; Greer, 1997; Yoshida, Verschaffel, & De Corte, 1997; Vinner, 2000). The same is true for college level students. Verschaffel, De Corte, & Borghart, (1997) report that college students "revealed a strong tendency …to exclude real-world knowledge from their own spontaneous solutions of school word problems" (p. 339). Inoue, (2002) also found that college students responded to mathematical problems with unrealistic answers, even when specifically asked to use their real world sense making skills.

Wyndhamn and Saljo (1997) speculate that one reason for the lack of sense making is that students often interpret word problems by "follow[ing] rules and use[ing] symbols without reflecting on, or analyzing, what these rules and symbols imply in the specific context in which they are used." (p. 362). In this research, we suggest that an important component of helping students to make meaningful sense of the mathematics they encounter involves building a learning environment in which meaning is highly valued, and where students are consistently encouraged to reflect on their own problem solving processes, to test their ideas in the context of the problem, and then to refine and revise their solutions accordingly. We contend that

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 137-144. Melbourne: PME.

this must happen over the course of many cycles (modeling cycles, as will be described later in the paper). In such an environment, simplistic or nonsensical responses can become increasingly refined thereby resulting in mathematically sensible solutions. In this paper, we document one such instance along with the corresponding stages of revision.

FRAMEWORK

A models and modeling perspective was used to guide all levels of this research. We refer to the development of mathematical ideas in terms of "models" and "modeling cycles" (c.f. Lesh & Doerr, 2000; Schorr & Koellner-Clark, 2003). Briefly stated, a model can be considered to be a system for describing, explaining, constructing or manipulating a complex series of experiences. An individual can interpret a situation by mapping it into his or her own descriptive or explanatory system for making sense of the situation. Once the situation has been mapped into the internal model, transformations within the model can occur, which in turn can produce predictions, descriptions, or explanations for use in the problem situation (Schorr & Koellner-Clark, 2003). Models tend to develop in stages where early models are often fuzzy or distorted versions of later, more advanced models. We contend that in many cases, students never cycle through multiple models, and their first or second cut solution reflects that. It is our hypothesis that when given the opportunity to cycle through multiple models in a supportive learning environment, students can develop mathematically more sophisticated and thoughtful solutions (Schorr & Lesh, 2003).

A learning environment consists of at least two critical and interrelated components. The first relates to the classroom atmosphere, and the second relates to the nature and type of problem solving experiences that the students encounter. We contend that classrooms that encourage students to talk about their ideas, reflect on the reasonableness of their solutions (orally and in writing), listen to the solutions of others, discuss different representations of the same problem and the relationship among representations, and share, defend and justify their solutions—orally and in written form, are more likely to result in sense making. In such classrooms, ideas are embraced, reflective activity is expected, and personal experience is valued.

Since a main purpose of this study was to investigate students' modeling cycles, it was not only important to encourage an atmosphere and learning environment in which sense making was valued, but also to find a problem activity that had the potential to elicit a thoughtful, sensible solution. The activity that was chosen was designed to encourage problem solvers to produce products that were not simply answers to specific questions; but in addition entailed constructions, descriptions, and explanations, that revealed many aspects of the thought process that goes into the final solution (Amit, Kelly & Lesh, 1994). Solutions to activities like the one that was chosen often involve sequences of modeling cycles in which the "given" information is systematically re-interpreted in a variety of ways (Lesh & Doerr, 2000; English, 1997).

In the sections that follow, we will provide evidence of the models and modeling cycles that occurred. For each cycle, we will offer our interpretation of the meaning of the particular model, the influence of real world sense making, and the implications of the changes in the final solution.

METHODS AND PROCEDURES

The context for this research was a course that was designed by the author (who was the classroom teacher) in order to help poorly performing students to succeed in college level courses. This particular class consisted of eight students, all of whom were recent graduates of local urban high schools. The students met with the teacher and a teaching assistant twice weekly for approximately one hour per session for a total of 14 weeks.

On this particular occasion, which occurred midway through the term, all students were asked to solve the "Radio Problem" (see below). They were given the option of working alone or with a partner. They were all asked to keep a written journal in which they included reflections on their work, and what, if anything, they might change when they resumed their work. They were also told that there was more than one solution path that could be taken to solve the problem. All students worked on the activity for a total of three hours spread over as many sessions. When the students completed their solutions, they were asked to formally present their work to the class. Selected students were interviewed after their presentations about their solutions and strategies. All sessions, presentations, and interviews were videotaped. The teacher also kept careful field notes. Data include all of the written work, videotapes and field notes.

The Radio Problem Activity: The activity that follows is adapted from a problem developed by the Educational Testing Service as part of the PACKETS® program[1]. The problem was designed to relate to similar experiences that the students might have had when purchasing portable radios with headsets. Note that the final solution is not simply a specific solution that relates to the unique set of data, but rather one that can be generalized to other radios with different attributes.

> The editors of Consumer Reports want to make a new consumer guide for products that are important to teenagers. The first items that they want to rate are portable radio-cassette players with headsets. They need your help to develop a rating system…The editors want a rating system that readers can use to rate any model (even if it is not listed on the attached list), and compare the models to determine which are the "best buys". The editors have also gathered the attached information for some models. They plan to use these as examples to show readers how to use the rating system. To help the editors, please: I) Develop a rating system for these players. Be sure that the system can be used

[1] The problem was taken from the PACKETS® program for Middle School Mathematics which was developed by the Educational Testing Service.

to identify overall "best buys" which take into account the factors that the survey indicates are important. Also, readers should be able to use the rating system with ANY other players, including those not listed in the guide, so include any tables or charts that are part of your system. II) Write clear step-by-step instructions that make it easy for readers to use your rating system. III) Write a letter to the editors explaining why you decided on your rating system and describe its advantages and disadvantages.

Included was a comprehensive data table which listed information for each of 11 brands of radios: The chart below represents only two of the brands, (due to space limitations).

Brand	Price (dollars)	Dimensions (inches)	Weight (ounces)	Tape Sound Quality	Radio Sound Quality	Battery Life (hours)	Number of AA batteries	Comments (on a separate list)
Aiwa	$49	51/4x 31/2 x 1 3/8	good	good	good	12	2	A,B,C,1
Sony	$69	5 3/8 x 4x 1 3/4	14	fair	poor	10 1/2	4	B,F,G,H,J, 1,4,7,9,10

RESULTS

This paper focuses on one particular student, James, whose work is chosen to be representative of the class. James began by constructing a model that represented his solution to the task. The model served as a means by which he could consider the feasibility and utility of his solution as a rating system. As James solved the problem, he often experienced a conflict between his own personal experience of listening to radios and the choices he had made as a result of applying his model (as noted in his reflections and comments). This pattern of considering the solution and assessing its utility in a real world context occurred several times until James reached what he considered to be a useful and generalizable solution. James noted that when he first began, he "…felt this project couldn't be done." but then had what he termed a "breakthrough". He said "Once I got that, it made me want to progress." (noted in his written reflection). Below, we briefly describe what he did, how he reflected on his work (taken from his written reflections, oral comments, and final presentation).

First Model: Rating List. James counted the number of advantages and disadvantages (as provided in the last column of the data table) for each radio. He added the number of advantages (each advantage was assigned the value +1) to the disadvantages (each disadvantage was assigned the value –1) thereby getting a positive number when the number of advantages exceeded the disadvantages and a negative number when the disadvantages exceeded the advantages. He then paired each radio with its corresponding outcome and listed them in an ascending and sequential order. In this "rating list" values ranged from +2 to –6. As part of his written reflection he noted that, "*My first attempt was rushed, kind of a scapegoat,* *and mainly left too many questions open…"* He then discussed the usability of the solution, e.g., the rating list, and as he did, he expressed his dissatisfaction with it. He noted that it would not be "accommodating to the teenage crowd" since it was not " what teenagers are looking for" (per his written reflection on his initial solution). James acknowledged that he had not really found a solution that addressed the needs of the teenaged crowd--the target population for the problem, and those most likely to purchase the radios. More specifically he stated*"…you must take into consideration what teen-agers are looking for including: good sound quality, low price, low cost of running and lightweight."* From this point on, he consistently referred to "the teenage crowd" as being the important factor as he reflected upon the changes that needed to be made.

Second Model: checklist. James built a new physical representation (model) using many of the information provided in the data table. He selected categories such as a radio's weight, price, and battery life to be included in the model, and ignored such categories as size of radio and sound quality. He then sorted the data by magnitude in ascending order, e.g. price was arranged from cheapest to most expensive, also taking into account the frequency of each value's occurrence. James determined what he considered to be a "good" range of data per category as well as a "not good" range, (a price that ranged from $39 to $69 was identified by James as a "good"). A numerical value was allocated to each data range: "good" data received a higher number of points (3) and all the rest received 0 points. James then attempted to test his model by rating a subset of the radios according to the above criteria. This resulted in a "best buy" list where the radio that had the highest rating was deemed as the "best" radio.

The transition from the first to the second "model" was rather dramatic. Instead of continuing to use only one dimension, i.e. the advantages and disadvantages of the radios (as he had previously done), he adopted a multi-dimensional approach in which he selected information from the data (table), intentionally ignored some of the other information (such as brand name), and then defined ranges of "good" with associated numerical values. One piece of information that he chose to ignore, namely, the brand name, proved to be very important as it allowed him to consider the rating of radios in a more generalizeable way (a key aspect of his next model).

The supportive learning environment provided him with an opportunity to take the time to consider "what is important for teenagers" and reflect on his work. Building upon his reflections, he proceeded to select relevant categories and eliminate non-relevant categories. His decision was based on personal beliefs and preferences. For example, he noted that he did not pay attention to the category of "size of radio" because he thought that size did not play an important role in teenagers' purchasing decisions. He also felt that for most people, a less expensive price is a good price. However, a really cheap price may be indicative of poor quality. He stated "if it is too cheap it 's probably not good". Following this logic, he decided exclude the cheaper prices (such as $24, $33 and $35) in the range of "good".

As with his first model, the end result of this new model involved verification and utilization of using a subset of the radios. Although the new model was more comprehensive than the first, James was still dissatisfied. He concluded by saying: *"it does not capture everything, it does not balance*". This statement marks the transition point into the next model.

Third Model: "prototype chart" – refined and expanded checklist. James "fine tuned" and revised the boundaries of the data ranges within the existing category boundaries. He added an intermediate range of "medium" and assigned numbers: 3 point for good, 2 for medium and all the rest 0 points. He continued using a method in which different categories were "weighted" differently. For example, the rating corresponding to the price or weight of the radio was done on a scale of 0 to 3, whereas the rating corresponding to the "life of a battery" category was based on a 0 to 2 scale. These were justified and explained by James as he noted *"price is important and battery life is important, but from a teenagers perspective, price is more important."* Next he expanded the scope of the categories to be included in the decision making process by adding categories with qualitative data. A sound quality category was added to his checklist along with a rating of 2 for good sound quality, 1 for fair sound quality and 0 for poor sound quality were assigned.

In the new scaling system, it appears as if James' allocation of weights to the different categories represented his own way of conceptualizing what is important in the purchasing of a radio. For example, price is more important then battery life; therefore a "good" price contributes 3 points to the rating of the radio while the longest battery life contributes only 2 points. As a final step, James applied the model to the rating of the radios. He was pleased with the results and commented that *"it balances pretty well"*. At this time, James realized that the task called for rating instructions that could be applicable to *any* radio and not limited to the 11 described in the task. James felt that his current model did not fully comply with that (see the statement of the activity). This realization marked the transition point that led James to the fourth and final model.

Fourth Model: General Chart and Operating Manual. James expanded the data boundaries to account for *any* radio's price and weight. He did so by adding the phrases *less* or *over* at the end of the "good" range. In addition, James created a table that included all of the categories, all data values, both quantitative and qualitative, and a list of advantages and disadvantages. He also added a "key" so that the user could easily discern how to use the point value. For 3 points he used brackets, for two points he used a curled line, and so on. He wrote guidelines to account for special case scenarios. For instance, if the overall rating of the product is –6, James decided that the radio should be penalized with the loss of another 2 points. This was done because a rating that was that that low meant that the radio was of poor quality, and should have –2 points added to the overall rating. James also attached a rather detailed manual so that a novice rater could easily use his guide. As part of his finished product he wrote

> You simply place your name of brand walkman into the column (price), scroll right to the next column identifying where the characteristic of your walkman falls under, recognize the point value, and scroll down placing the point value in the void. Continue this process. After all point values for each category are in the voids, add them all together to get your total worth. In regards to tape and radio sound quality, point values are as so: 'Good' is worth 2 points, 'Fair' is worth 1 point, and poor has no worth. All of these components in sync will result in total worth or a "best buy'. Special considerations are present on the chart but hold no real dilemmas.

In addition, James recommended that one should rate the sound quality and the overall tape quality by playing actual music on the radio. After the rating process was complete, James further recommended that all points should be summed up for the different radio brands, and based upon this summation, one could choose the radio of his/her liking. In the end, James checked the final model by rating each of the radios on the original list, and creating a new list. Both the model and the new list were to his satisfaction and he even expanded the targeted audience, claiming that this new "best buy" list could be useful to those interested in purchasing a new radio as well as those who sell radios because the list illustrates and summarizes each product's performance. James summed this up by saying, "With this rating system, the consumer's task will be virtually effortless and seem more inviting, leaving the buyer with no other option but to take advantage of it."

CONCLUSIONS

James went through several cycles in order to solve the problem, cycles that reflected a progression from simplistic to more complex and generalizeable. We suggest that the first or second solutions that James produced are more typical of the solutions that one would expect in many classrooms, solutions that do not fully build upon students' personal, sense making capabilities. It was only through repeated reflection and revision (in which James experienced a conflict between his own personal experience and his mathematical solution), that James was prompted to revise, test, and refine his work. This type of reflection and revision was consistently encouraged within the classroom environment, in conjunction with the use of carefully chosen problem activities. This particular task was specifically designed to capitalize on students' personal experiences with purchasing radios, thereby providing a context in which sense making could be applied. Further, the problem called for a solution that was more than a specific solution for a unique set of data (involving a concrete and local situation), but rather one that could be generalized to include many different types of situations, and whose processes could relate to a whole class of structurally similar problems involving quantifying qualitative data; working with extreme and diversified situations, some of which are directly related (for example, the longer the battery life, the higher the rating), while others within the same problem are not (for example, the higher the price, the lower the rating); the invention and application of weighted scales; etc. We believe that this type of mathematical activity is critical if students are to experience the types of "breakthroughs" that James described.

References

Amit, M, Kelly, E., and Lesh, R., (1994). Characteristics of effective model eliciting problems," In Ponte, J.P., and Matos, J.F., (eds.) *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Portugal, Vol. 3, pp. 160-167.

Carpenter, T.P., Lindquist, M.M., Matthews, W., and Silver, E.A. (1983). Results of the third NAEP mathematics assessment: secondary school. *Mathematics Teacher,* 76, p. 652-659.

English, L. (1997). Children's reasoning processes in classifying and solving computational word problems. In L. English (Ed.) *Mathematical reasoning*. Lawrence Erlbaum, Hillsdale, NJ. P191-220.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction.* 7(4) p. 293-307.

Hiebert, J., Carpenter, TP., Fennema, E., Fuson., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: teaching and learning mathematics with understanding.* Heinemann, Portsmith, NH.

Inoue, N. (2002). The role of personal interpretation in mathematical problem solving. *Unpublished doctoral dissertation.* Teachers College, Columbia University.

Lesh, R., and Doerr, H. (2000). Symbolizing, communicating, and mathematizing: key components of models and modeling. In P. Cobb, E. Yackel, and K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms.* Lawrence Erlbaum Publishers p. 361-384.

Schorr, R.Y. & Lesh, R. (2003) A models and modeling perspective on classroom-based Teacher Development. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.

Schorr, R.Y. & Koellner-Clark, K. (2003). Using a modeling approach to consider the ways in which teachers consider new ways to teach mathematics. *Mathematical Thinking and Learning*

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Verschaffel, L., De Corte, E. and Borghart, I. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modeling of school word problems. *Learning and Instruction.* 7(4) p. 339-360.

Vinner, S. (2000). Mathematics Education-procedures, rituals and man's search for meaning. Paper presented at the Ninth International Congress of Mathematics Education. Tokyo Japan.

Yoshida, H., Verschaffel, L., and De Corte, E. (1997). Realistic considerations in solving problematical word problems: do Japanese and Belgian children have the same difficulties? *Learning and Instruction.* 7(4) p. 329-338