RF02: GESTURE AND THE CONSTRUCTION OF MATHEMATICAL MEANING

Ferdinando Arzarello*, Laurie Edwards **

Dipartimento di Matematica, Università di Torino, Italy (*)

Saint Mary's College of California, USA (**)

The role of gestures in mathematical thinking and learning is examined from the perspectives of cognitive science, psychology, semiotics and linguistics. Data from situations involving both children and adults, addressing mathematical topics including graphing, geometry, and fractions, are presented in the context of new theoretical frameworks and proposals for the analysis of gesture, language, signs and artefacts.

INTRODUCTION

Recent research in mathematics education has highlighted the significance of the body and, specifically, perceptuo-motor activities in the process of mathematics teaching and learning (Lakoff & Núñez, 2000; Nemirovsky *et al*., 1998). The analysis of the role of the body in cognition takes place within a wide multi-disciplinary effort, involving neuroscience, cognitive science, experimental psychology, linguistics, semiotics and philosophy. These disciplines offer complementary tools and constructs to those who wish to investigate the complex interactions among language, gesture, bodily action, signs and symbols in the learning and teaching of mathematics. The goal of the Research Forum is to examine the role that gesture plays in the construction of mathematical meanings. More specifically, we are concerned with the following questions:

- How can we describe the phenomenology of gestures in mathematics learning (e.g.: What kind of gestures are there? Is the classification created by McNeill (1992) adequate for mathematical gestures?)
- How does gesture function in the processes of learning mathematical concepts?
- Can gesture provide evidence about how mathematical ideas are conceptualized?
- Are gestures context-dependent? In particular, how do they change when students interact with artifacts?
- Which theoretical frameworks are suitable for analysing gestures in mathematics learning taking into account work on gesture carried out within disciplines outside of mathematics education?
- What consequences of the research on gesture can be drawn for mathematics students, teachers, and prospective teachers?

The analysis of gesture, both within and outside of mathematics education, takes place within the broader framework of recent work in embodied cognition and

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 123-154. Melbourne: PME.

cognitive linguistics. As applied by Lakoff and Núñez (2000), this framework holds that human bodily experience, as well as unconscious mechanisms like conceptual metaphors and blends, are essential in the genesis of mathematical thought. In this view, mathematics is a specific powerful and stable product of human imagination, with its origins in human bodily experience. As noted by Seitz (2000, emphasis in the original), "In effect it appears that we *think* kinesically too [….] and has been postulated [….] that the body is central to mathematical understanding (Lakoff & Nunez, 1997), that speech and gesture form parallel computational system (Mc Neill, 1985, 1989, 1992)." In a similar vein, R. Nemirovsky (2003) has emphasized the role of perceptuo-motor action in the processes of knowing:

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the subject itself". [As a consequence,] "the understanding of a mathematical concepts rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context. (p. 108)

Furthermore, attention is now being paid to the ways in which multivariate registers are involved in how mathematical knowing is built up in the classroom. This point is illustrated by Roth (2001) as follows:

> Humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment. (p. 9)

This attention to the body does not negate the fact that mathematics and other forms of human knowledge are "inseparable from symbolic tools" and that it is "impossible to put cognition apart from social, cultural, and historical factors": in fact cognition becomes a "culturally shaped phenomenon" (Sfard & McClain, 2002, p. 156).

The embodied approach to mathematical knowing, the multivariate registers according to which it is built up, and the intertwining of symbolic tools and cognition within a cultural perspective are the basis of our frame for analysing gestures, signs and artefacts. The existing research on those specific components finds a natural integration in such a frame.

GESTURES VIEWED WITHIN PSYCHOLOGY

Within a psychological perspective, we begin with the seminal work of McNeill (1992), who stated that, "gestures, together with language, help constitute thought" (p. 245). McNeill (1992) classified gestures in different categories: *deictic* gestures (pointing to existing or virtual objects); *metaphoric* gestures (the content represents an abstract idea without physical form); *iconic* gestures (bearing a relation of resemblance to the semantic content of speech); *beat* gestures (simple repeated gestures used for emphasis). Since his study, much research has analysed how gesture and language work together and influence each other. Alibali, Kita and Young (2000) develop McNeill's view that gesture plays a role in cognition, not just in communication, in the Information Packaging Hypothesis (IPH):

> Gesture is involved in the conceptual planning of the messages, helps speakers to "package" spatial information into verbalisable units, by exploring alternative ways of encoding and organising spatial and perceptual information…gesture plays a role in speech production because it plays a role in the process of conceptualisation (p. 594-5)

According to the IPH, the production of representational gestures helps speakers organise spatio-motoric information into packages suitable for speaking. Spatio-motoric thinking (constitutive of representational gestures) provides an alternative informational organisation that is not readily accessible to analytic thinking (constitutive of speaking organisation). Analytic thinking is normally employed when people have to organise information for speech production, since, as McNeill points out, speech is linear and segmented (composed of smaller units). On the other hand, spatio-motoric thinking is instantaneous, global and synthetic (not analyzable into smaller meaningful units). This kind of thinking, and the gestures that arise from it, is normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery.

Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought. (Goldin-Meadow, 2003, p. 178)

Another important aspect of the analysis of gesture concerns the relationship between the content of the speech and the gesture. We can speak of a *gesture-speech match* (M) if the entire information expressed in gesture is also conveyed by speech. If not, that is, if different information is conveyed in speech and gesture, we have a *gesture-speech mismatch* (Goldin-Meadow, 2003). This information is not necessarily conflicting but possibly complementary, and may signal a readiness to learn or reach a new stage of development (Alibali, Kita & Young, 2000; Goldin-Meadow, 2003). According to Goldin-Meadow, mismatch is "associated with a propensity to learn" (p. 49), "appears to be a stepping-stone on the way toward mastery of a task" (p. 51); and may place "two different strategies [for solving a problem] side by side within a single utterance" highlighting "the fact that different approaches to the problem are possible" (p. 126). In general gesture-speech mismatch reflects "the simultaneous activation of two ideas" (p. 176).

GESTURES VIEWED WITHIN SEMIOTICS

The fact that gestures are signs was pointed out many years ago by Vygotsky, who wrote:

> A gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture. (Vygotsky, 1997, p. 133)

Semiotics is a useful tool to analyse gestures, provided that a wider frame, which takes into account their cultural and embodied aspects as well, is considered. An analysis of this kind has been carefully developed by Radford, who introduces the notion of *semiotic means of objectification* (Radford, 2003a):

> The point is that processes of knowledge production are embedded in systems of activity that include other physical and sensual means of objectification than writing (like tools and speech) and that give a corporeal and tangible form to knowledge as well....These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (p. 41)

Gestures can be important components of semiotic means of objectifications, whether used when communicating directly with others, or to highlight aspects of artefacts and symbolic representations of mathematical concepts.

Psychologists now distinguish between *linguistic* and *extralinguistic* modes of expression, describing the former as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs (Bara & Tirassa, 1999). When students are learning the signs of mathematics, they often use both their linguistic and extralinguistic competence to understand them; e.g. they use gestures and other signs as semiotic means of objectification. Of course, in all these means of objectification both modalities (linguistic and extralinguistic) are present, with different strengths and in different ways depending on the dynamics of the situation.

SUMMARY OF THE RESEARCH FORUM

The papers of the Research Forum address the main questions and themes summarized above. F. Arzarello *et al.* present an example involving geometric visualization to illustrate a new theoretical framework for analysing gesture and speech in mathematics learning environments. M.G. Bartolini Bussi analyses the genetic links between artefacts and gestures in pupils (9 years old) who use real and virtual artefacts. L. Edwards utilizes data from adult students discussing fractions to argue that the original narrative-based classification of gestures should be adjusted and modified for analysing gestures in mathematical discourse. R. Nemirovsky and F. Ferrara approach gestures from the point of view of perceptuo-motor thinking, showing the connections between parallel strands of bodily activities, in a microanalysis of gestures and eye motions during a graphing activity. L. Radford explains the role of semiotics in analysing gestures as means of semiotic objectification, illustrating his framework with data from modeling activities.

SHAPING A MULTI-DIMENSIONAL ANALYSIS OF SIGNS

F. Arzarello*, F. Ferrara*, Ornella Robutti*, D. Paola**, C. Sabena* ^
Dipartimento di Matematica, Università di Torino, Italy (*)
Liceo Issel, Finale Ligure, Italy (**)
Université Laurentienne, Sudbury, Ontario (^)

INTRODUCTION AND BACKGROUND

Recently the analysis of gestures and their role in the construction of meanings has become relevant not only in psychology, but also in mathematics education. Gestures are considered in relation with speech, and with the whole environment where mathematical meanings grow: context, artefacts, social interaction, discussion, etc. Mathematics, as an abstract matter, has to come to terms with our need for seeing, touching, and manipulating. It requires perceivable signs and so the environment is crucial in the teaching-learning process.

In this paper, we elaborate on two different ways to look at the cognitive processes of students when they communicate and reason during a mathematical activity. We propose a theoretical frame shaped by the encounter of certain perspectives, developed in the disciplines of mathematics education, psychology, neuroscience, and semiotics. In particular, the theoretical notions we use here are the following: from psychology, the *Information Packaging Hypothesis* (Alibali, Kita & Young, 2000); from semiotics, the idea of *semiotic means of objectification* (Radford, 2003a) and that concerning the different functions of signs, i.e. *iconic, indexical* and *symbolic* (Peirce, 1955; Radford, 2003a), and from psycho-linguistics, the distinction between *linguistic* and *extra-linguistic* modes of expression (Bara & Tirassa, 1999). Let us sketch them here for our purpose; a more detailed account is given in the introduction of the present research forum.

In psychological research, the Information Packaging Hypothesis (IPH) describes the way that gesture may be involved in the conceptual planning of the messages, by considering alternative "packagings" of spatial and visual information, so that this information can be verbalized in speech (Alibali, Kita and Young, 2000). Within the similar perspective that gestures play an active role not only in speaking, but also in thinking, *gesture-speech matches* and *mismatches* are defined (Goldin-Meadow, 2003). A match occurs when all the information conveyed by a gesture is also expressed in the uttered speech; a mismatch happens in all the other cases. Mismatches are the most interesting since they indicate a readiness for learning, conceptual change or incipient mastery of a task. But gestures are also significant from the side of semiotics if seen as *signs*. Vygotsky (1997) already pointed out that "a gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture" (p. 133). Nevertheless, semiotics is useful to analyse gestures only if does not forget their cultural and embodied aspects. Such a direction has been followed in mathematics education by Radford (2003a) with the introduction of the so-called semiotic means of objectification. These semiotic means are constituted by different types of signs, e.g. gestures, words, drawings, and so on. They have been introduced to give an account of the way students come to generalise numeric-geometric patterns in algebra. Different kinds of generalisation have been detected. Among them is the so-called *contextual generalisation*, which still refers heavily to the subject's actions in time and space, within a precise context, even if he/she is using signs that could have a generalising meaning. In contextual generalisation, signs have a two-fold semiotic nature: they are becoming symbols but are still indexes. These terms come from Pierce (1955) and Radford (2003a). An *index* gives an indication or a hint of the object: e.g. an image of the Golden Gate, which makes you think of the city of San Francisco. A *symbol* is a sign that contains a rule in an abstract way: e.g. an algebraic formula. As relevant in communication (in thinking as well) gestures can be considered with respect to linguistic and extra-linguistic modes of expression. The former is characterised as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs: "linguistic communication is the communicative use of a symbol system. Language is compositional, that is, it is made up of constituents rather than parts... Extra-linguistic communication is the communicative use of an open set of symbols. That is, it is not compositional: it is made up of parts, not of constituents. This brings to crucial differences from language..." (Bara & Tirassa, 1999; p. 5). In communicative acts the two modes co-exist. Students who learn the signs of mathematics, often rely on both their linguistic and extra-linguistic competences to understand them: for example, they use gestures and words as semiotic means of objectification. Typically, gestures are extra-linguistic modes of communication, whereas speech is on the linguistic side.

A NEW FRAMEWORK: THE PARALLEL AND SERIAL ANALYSIS

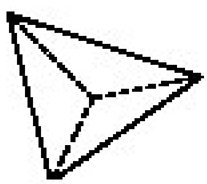


Figure 1

We show a brief example from the activity of some 8th grade students involved in approaching a geometrical problem. They have been asked to describe the geometric solid formed when two square pyramids are placed side by side (with one pair of base sides touching). The solution, which must be visualized by the students, is a tetrahedron seen from an unusual point of view.

Consider the following utterances by Gustavo, and one of his concomitant gestures:

Gustavo: Yeah, it is a solid, made of two triangles placed with the bases below, which are those starting in this way and going up, and two triangles with the bases above that are those going in this way [see Fig. 2].

We can analyse data like these in a double way, using what we call *parallel* and *serial* analysis. Both analyses take into consideration the dynamics of what we think of as the major components of processes of objectification: not only speech and gestures (respectively ***s*** and ***g*** in Fig. 3), but also written words and mathematical signs (respectively, ***w*** and ***x*** in Fig. 3). The latter, even if not directly part of the communication acts, are a product of them, and often arise from gestures and words used by the involved subjects (Gallese, 2003; Sfard & McClain, 2002).

Figure 2

The components of objectification processes may develop according to two types of dynamics. We call the first dynamics *Parallel Process of Objectification (**PPO**)*; it results when (some of) the different components are seen as a group of processes synchronically developing (e.g. when one talks and gestures simultaneously). They can match or mismatch with each other in the way they are encoding information.

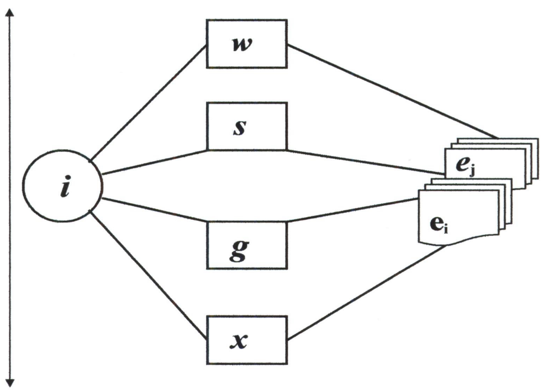


Figure 3: The *PPO*

In such a case, we are interested in a parallel analysis of the components (see the vertical arrow in Fig. 3), which focuses on the mutual relationships among them, where all components refer to the same source ***i*** and possibly to different encoding e_i's. The main elements of a parallel process of objectification are: (i) the idea of semiotic means of objectification; (ii) the Information Packaging Hypothesis; (iii) Match and Mismatch (Goldin-Meadow, 2003).

We call *Serial Process of Objectification (**SPO**)* a second type of dynamics, which results when two different components are spread over time and happen in different moments, as steps of a unique process. An example is given by a sign produced as a frozen gesture (Vygotsky, 1997), or by a gesture embodying some features of a previous sign. In this case, we are interested in a serial analysis (see the horizontal arrow in Fig. 4) focusing on the subsequent transitions from different sources ***i*** to different encoding e_i's.

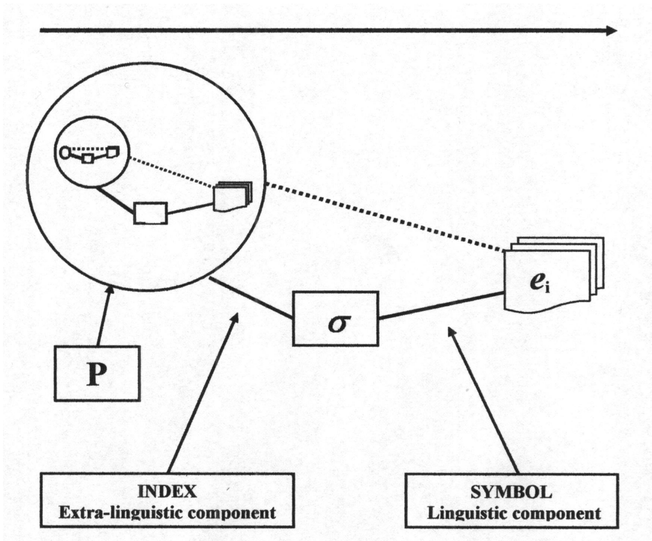


Figure 4: The *SPO*

The Serial Process of Objectification is shown in Fig. 4. Its main elements are again: (i) the semiotic means of objectification; and (ii) the Information Packaging Hypothesis. But there are also two other elements: (iv) the indexical-symbolic functions of signs; and (v) the linguistic and extra-linguistic modes of communicative acts. A serial process of objectification happens when one (or more) serial (or parallel) process(es) **P**, represented in the circle of Fig. 4, is (are) the grounding for the genesis of a new sign (indicated by $\boldsymbol{\sigma}$).

For technical reasons, just one component appears in the circle, but there could be more. The sign $\boldsymbol{\sigma}$ is the pivot of the process; it can be any kind of sign: a drawing, a word, a gesture, a mathematical sign, etc. It is generated by the previous process(es) P and produces an encoding of **P**. The relationships between $\boldsymbol{\sigma}$ and **P** are mainly extra-linguistic, whereas the relationships between $\boldsymbol{\sigma}$ and $\boldsymbol{e_i}$ ***are*** mostly linguistic. In other terms, the sign $\boldsymbol{\sigma}$ has an indexical function with respect to P, but it has also a fresh symbolic function with respect to the encoding $\boldsymbol{e_i}$. Thus, the ***SPO*** could be the basis for a new serial process, and so on, in an ongoing series of nested generalisations. Examples of ***SPOs*** are given by the learning of speech in kids or by that of reading written texts in young pupils. Mathematical examples are the processes undertaken by students who are learning Algebra or some other chunks of mathematical ideographic language, from Arithmetic to Calculus.

Generally both types of dynamics, ***PPO*** and ***SPO***, can support the genesis of signs. As a consequence, each process of objectification may be analysed from both points of view, that is, as a parallel process and as a serial process. We call *parallel* and *serial* the two resulting types of *analysis*. Let us go back to the initial example that we can now interpret through the two analytical lenses. The parallel analysis points out the conflict between the two pieces of Gustavo's theoretical knowledge concerning the 2D and 3D figures. The serial analysis shows that Gustavo's gestures are mediating the transition from the 2D features of the triangles to the 3D ones of the solid. After this episode, the experiment goes on and culminates with the acknowledgement by students of the tetrahedron as a "triangular pyramid". Parallel and serial analysis allow us to focus properly on the dynamics of what is happening. As such they are useful tools of investigation. In fact, parallel analysis reveals itself as a tool suitable for identifying conflicts, even before they appear to block or slow students' activities. On the other hand, the serial analysis represents a tool suitable for focusing on the dynamics through which the subjects try to overcome obstacles met in their activities.

Acknowledgments: Research program supported by MIUR and by the Università di Torino and the Università di Modena e Reggio Emilia (COFIN03 n.2003011072).

WORKING WITH ARTEFACTS: THE POTENTIAL OF GESTURES AS GENERALIZATION DEVICES

Maria G. Bartolini Bussi & Michela Maschietto[1]

Dipartimento di Matematica (Modena)

Università degli studi di Modena e Reggio Emilia

INTRODUCTION

We shall summarize some findings of two studies (Bartolini et al., 1999; Bartolini et al. in press) concerning primary school. In the former we have studied the genesis of a germ theory of the functioning of gears. In the latter we have studied the construction of the meaning of painting as the intersection between the picture plane and the visual pyramid. The studies have been carried out in a Vygotskian framework that has been gradually enriched with contributions of other authors. As a result, classroom activity has been designed and orchestrated by the teacher in order to foster the parallel development of different semiotic means (language, gestures, drawing), which form a dynamic system (Stetsenko, 1995, p. 150).

In both studies, concrete artefacts came into play. Wartofsky's distinction between primary, secondary and tertiary artefacts proved to be useful (1979). *Primary artefacts* are "those directly used" and *secondary artefacts* are "those used in the preservation and transmission of the acquired skills or modes of action". Technical tools correspond to primary artefact whereas psychological tools are the individual counterparts of secondary artefacts. *Tertiary artefacts* are objects described by rules and conventions and not strictly connected to practice (e. g. mathematical theories, within which the models constructed as secondary artefacts are organised).

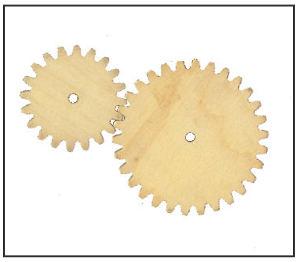
WHEN THE ARTEFACT IS A GEAR.

The role of gestures when concrete tools are into play is obviously very large. Wartofsky himself emphasizes mimicry, among the different representations used to preserve and transmit the modes of action. Gestures are essential to use the artefact, as 'a machine is a device that incorporates not only a tool but also one or more gestures' (Leroi-Gourham, 1943). We found that, from 2nd grade on, when the teacher

[1] Abridged version of a study (in preparation) carried out together with Maria Alessandra Mariotti, and Franca Ferri, within the National project Problems about the teaching and learning of mathematics: meanings models, theories (PRIN_COFIN 03 2003011072).

designs suitable activities aiming at constructing a germ theory of the functioning of gears and supports pupils' work, there is a parallel and intertwined development of three different semiotic means: gesture – drawing - speech (in oral and written forms): the development is towards the appropriation of the meaning of motion direction, represented by a sign ('arrow') with an appropriate syntax, that also allows students to solve difficult problems concerning trains of any number of gears.

Our findings are summarized in Table 1, adapted from (Bartolini et al., 1999, p. 79) which relates the findings of that study to issues discussed in this forum.



The *primary artefacts* are given, in this case, by tools with gears and toothed wheels inside. In the figure, a pair of toothed wheels is represented (courtesy of R. Nemirovsky, TERC). To start the gear a gesture is needed: it creates an action scheme that 'enables students to tackle virtually any particular case successfully' (*factual generalization,* Radford, 2003a, p. 47).

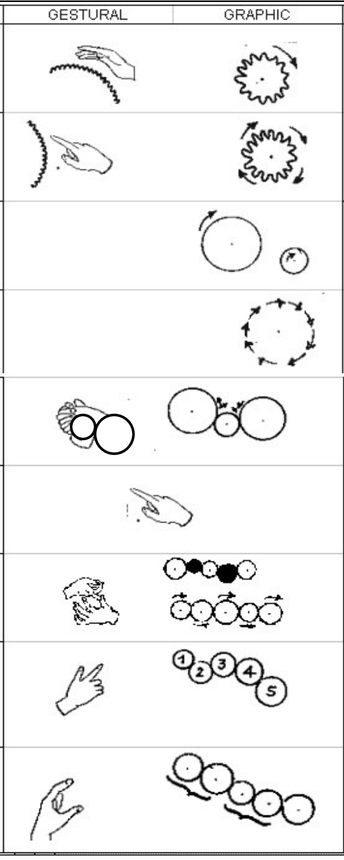


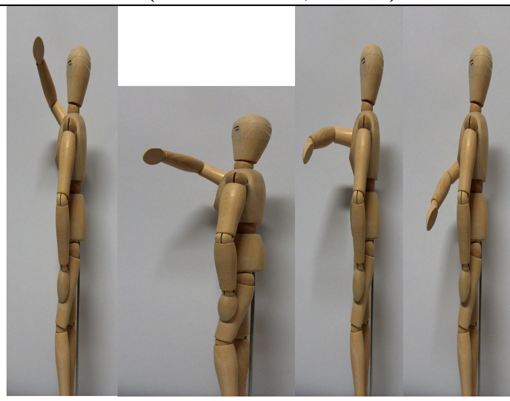
Table 1. *From gesturing to signs (Bartolini et al. 1999, p. 79)*				*Wartofsky*	*Edwards / McNeill*	*Radford*
n	GESTURAL	GRAPHIC	VERBAL			
1			push this wheel this way this wheel goes this way	PRIMARY Gesture on a primary artefact to turn the wheel as a whole or pushing a point.	Iconic physical	*Factual generalization*
2			this tooth goes this way		Iconic physical	*Factual generalization*
3			clockwise anticlockwise	Construction / appropriation of secondary artefacts	No gesture	*Contextual generalization*
4			left - right up - down		No gesture	*Contextual generalization*
5			wheels turn in pairs	Gesture to represent a primary artefact (secondary)	Iconic physical	*Contextual generalization*
5'			this wheel pushes that wheel	Construction / appropriation of secondary artefacts	Iconic physical	*Contextual generalization*
6			white-black A-B thumb-index this - the other way	*Towards tertiary artefacts* Gesture to represent a mathematical model	metaphorical	*Contextual generalization*
7			one-two-three-four-...		metaphorical	*Symbolic generalization*
8			wheels are paired		metaphorical	*Contextual generalization*

When young pupils (e.g., 2nd grade ones) are asked to represent this experience by drawing, they spontaneously introduce the sign 'arrow' (a semiotic mean of objectification) that seems to objectify on paper the gesture of the hand. Later the sense of the sign changes together with the parallel evolution of drawing and speech. In Table 1 we have related our findings with those of other authors.

When the artefact is a sentence evoking a concrete artefact

In a 4th grade classroom (Bartolini et al. in press), a complex activity about perspective drawing has been started. The first step has been the exploration and the interpretation of an artefact (Dürer's glass) built in wood, metal and Plexiglas, where one observes through the eyehole the perspective drawing of the skeleton of a cube put behind the glass. Some months later, at the beginning of the 5th grade, when the concrete artefact is no longer in the classroom, a very short sentence from L. B. Alberti (De Pictura, 1540) is given to interpret in classroom discussion: "*Thus painting will be nothing more than intersection of the visual pyramid*". Gestures are very important in the interpretation: gestures mime planes and lines and constitute a fundamental support to imagine a pyramid.

<table>
<tr><td colspan="2">Table 2
"Thus painting will be nothing more than intersection of the visual pyramid"
L.B.Alberti (De Pictura, 1540).</td></tr>
<tr><td>▸ You have to imagine it. I understood this, if you saw it near the object you obtain a large image; if you saw it near the eye you get a smaller image. [With gestures, many children saw the visual pyramid].

▸ If you go down straight, because with our hands we form a kind of plane parallel to the one of the objects [With his hands he traces two parallel planes in space]. In this way you certainly obtain a figure which is exactly the same as the base of the pyramid, but smaller.
1a</td><td>
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[…] A visual pyramid is a kind of pyramid 'made by you', that is the pyramid helps you to see what you see in different ways, in fact, as I have drawn, it makes you see the sun in several ways. I have drawn that drawing, because it clarifies how a visual pyramid is and also how it must be shaped. I have enjoyed making the sun, bigger and bigger, because it makes one understand much. Anna's eye is open and the other is closed, it is not visible but if you notice there is her arm pointing close to the other side of her face to close the other eye.</td><td>2a
</td></tr>
</table>

The pupils do not seem troubled by this imaginary context, as the following exchange shows:

Luca: How can you possibly saw the visual pyramid, which is a solid that does not exist?

Alessandro B.: Exactly how you imagine it. If you see it because you imagine it, you can saw it as well. You have to work with the mind.

Three months after this discussion, the pupils are asked to comment individually, in writing (using also drawing if they wish), about the same sentence by Alberti (Maschietto & Bartolini, submitted). In Table 2 some exemplary protocols from the above activities are presented: 1a. The transcript (with comments) of an oral exchange between two pupils in classroom discussion; 1b. The simulation of gesture by means of a dummy; 2a. A drawing produced to explain Alberti's sentence; 2b. An excerpt of the written text, added as a commentary of the sentence and of the drawing.

The right way to produce the gesture ('straight down' i.e. vertically) is verbally explained immediately by the second speaker. This way of cutting an 'imaginary' pyramid in the air becomes a shared action scheme in the classroom, repeatedly used by the pupils and by teacher as well. The gesture works in any position (*contextual generalization*, Radford 2003a). Three months later most pupils prove to have internalized the meaning of the visual pyramid and produce meaningful drawings. In the one reported here there is another instance of *contextual generalization*, which concerns the possibility of tilting any 'imaginary' picture plane in non-vertical position. We know from the history of perspective that this was not a trivial problem.

DISCUSSION

Wartofsky's elaboration of artefacts refers to 'external' objects. He discusses the secondary artefacts as follows:

> Such representations […] are not 'in the mind', as mental entities. They are the products of direct outward action, the transformations of natural materials, or the disposition or arrangement of bodily actions […].

In the classroom pupils construct/appropriate these cultural products by means of social activity carried out together with their peers under the teacher's guidance. We have shown in two cases concerning spatial experience with concrete artefacts how internalization of social activity, is realised by semiotic means of objectification (Radford, 2003a) that are used in parallel and intertwined with each other.

THE ROLE OF GESTURES IN MATHEMATICAL DISCOURSE: REMEMBERING AND PROBLEM SOLVING

Laurie D. Edwards

Saint Mary's College of California

The purpose of this analysis is to examine the role of gestures within the context of a particular setting involving mathematical discourse, specifically, an interview where students were asked to describe how they learned certain mathematical concepts and to explain how they solved problems involving fractions. The overall goal of the study was to examine both the form and function of gestures within a context of mathematical communication and problem solving, and to begin to develop an analytic framework appropriate to understanding gesturing within the domain of mathematics.

Previous research has examined the role of gesture in a number of different mathematical contexts, including learning to count (Alibali & diRusso, 1999; Graham, 1999), classroom communication (Goldin-Meadow, Kim & Singer, 1999), ratio and proportion (Abrahamson, 2003), motion and graphing (Nemirovsky, Tierney & Wright, 1998; Radford, Demers, Guzmán. & Cerulli, 2003, Robutti & Arzarello, 2003), and collaborative problem solving (Reynolds & Reeve, 2002; also see Roth, 2001, for a review of research on gesture in mathematics and science). Gesture is defined as "movements of the arms and hands ... closely synchronized with the flow of speech" (McNeill, 1992, p. 11). In contrast with speech, which is linear, segmented and composed of smaller units, gesture is global and synthetic; it can express meanings as a whole and one gesture can convey a complex of meanings (McNeill, 1992). Gesture can be seen as an important bridge between imagery and speech, and may be seen as a nexus bringing together action, imagery, memory, speech and mathematical problem solving. The investigation of gesture in mathematics takes place within a theoretical context that sees cognition as an embodied phenomenon, and that examines how both evolutionary constraints and individual bodily experience provide a foundation for the distinctive ways that humans think, act, and speak about mathematics (Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999).

The data for the study comprise a set of gestures displayed by twelve adult female students while talking about their memories of learning fractions, and during and after solving problems involving fractions. The participants were prospective elementary school teachers, and the interviews were carried out in pairs. A corpus of more than 80 gestures was collected. The majority of the gestures were displayed in response to questions asking the students to recall how they first learned about fractions.

Figure 1: "I think we did, like, just a stick or a rod…"

These gestures generally fell into four categories, representing an extension of McNeill's original typography of gestures into iconics and metaphorics:

(1) Iconic gestures referring to physical manipulatives or actions (e.g., "a stick or rod" or "cutting a pie")

(2) Iconic gestures referring to inscribed representations of physical manipulatives (e.g., "a pie chart")

(3) Iconic gestures referring to specific written algorithms (Figure 1b)

(4) Metaphoric gestures (referring to an abstract idea or action, e.g. Figure 2)

In Figure 1a, the student describes a manipulative (possibly fraction bars), and goes on to talk about "dividing it again and again," moving her right hand in a chopping gesture toward the right to indicate the iteration of this division. This chopping motion can also be categorized as an iconic gesture referring to a physical action.

Figure 1b shows an example of a student displaying an "iconic-symbolic" gesture: gestures that refer not to a concrete object but to a remembered written inscription for an algorithm or mathematical symbol; that is, an "algorithm in the air" (Edwards, 2003). The importance of written algorithms for mathematics, and for students memories of learning mathematics, would seem to require this expansion of the typology of gestures that McNeill originally developed to analyze narrative discourse.

Figure 2 shows a part of a gesture made by a student responding to a question about how she would introduce fractions to children. The gesture began with the two hands close together, with whole hands slightly curled and facing each other, and ended with the hands opening out and moving to the right. These somewhat vague metaphorical gestures about generic mathematical operations contrast sharply with the very precise iconic-symbolic gestures used when describing specific arithmetic algorithms with fractions.

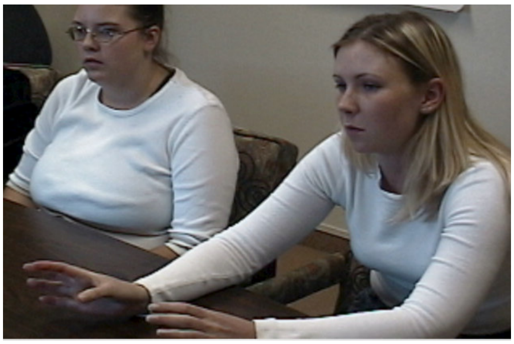

Figure 2: "Like the different formulas"

In addition to gestures displayed in response to the interviewer's questions, one student displayed a complex sequence of gestures associated with a description of how she solved a problem involving comparing two fractions. She and her partner had worked out which was larger, 3/4 or 4/5, and the student was explaining her solution after the fact. The student's spoken words are below (underlining indicating words synchronized with a gesture):

S2: Well, I mean it's like I'm thinking if I had a pie and I had 5 people versus 4 people then,[R: Ah.] you know, we're each kinda getting less of a piece [R: Ah.] because there's a fifth piece we have to like, put out to the other four people.

The four gestures corresponding to the underlined words or phrases consisted of (1) pointing with right index finger to right temple ("thinking"); (2) moving the first two fingers of the right hand from right to left at chest height ("less"); (3) a diagonal chopping motion with the whole right hand at face height ("fifth piece"); which continues into a (4) circular movement of the whole hand in front of and parallel to the face and chest ("put out to the other four"). This use of gesture did not seem to be a static illustration of remembered objects or inscriptions, as some of the other gestures were. Instead, the sequence of gestures was fully synchronized with the description of the problem solution, and may have played a facilitating role in solving the problem. The first gesture would be described as an emblem (a conventionalized gesture for "thinking" by pointing to the temple), but the other four gestures highlighted important aspects of the solution: the relative size of the fractions; i.e., the denominators ("getting less of a piece"), the number of pieces, i.e., the numerators ("a fifth piece") and a sharing operation ("put out to the other four people").

The current study elicited a wide variety of gestures, primarily associated with students' memories of learning fractions, but also occasionally in connection with current problem solving and reasoning. In either context, the gestures were not simple illustrations, but reflected important aspects of the materials and representations present while the students were learning. These findings are similar to those in a study of bodily motion and graphing, in which the authors stated, "The way students describe functions shows deep traces of their actions and interactions with instruments and representations. Such traces are not complementary to the concept but are an essential component of its meaning" (Robutti & Arzarello, 2003, p. 113).

The analysis of the gesturing in mathematical contexts has provoked a re-examination of the categories developed by McNeill for describing gestures elicited in association with narrative descriptions. The initial analysis of the fraction data stimulated a division of McNeill's category of iconic gestures into two sub-categories: iconic-physical and iconic-symbolic. However, the nature of mathematics as a discipline may require an even more refined categorization of gestures. This is because while in everyday life, concrete objects do not "refer" to anything beyond themselves, in mathematics teaching, many concrete objects have been designed to "represent" more abstract mathematical objects. So when a student gestures in a circle when talking about fractions, she may be referring simply to the plastic fraction pieces she remembers from elementary school, or she may be thinking about those pieces in regards to a particular fraction or operation. Furthermore, outside of mathematics, written symbols are not usually manipulated as if they were objects. Thus, descriptions and analyses of gesture in mathematics should take into account these features of mathematical practice and discourse. Furthermore, the analysis of gesture may help to illuminate the relationships and developmental path among physical actions, speech, internalized imagery, written symbols, and mathematical abstractions.

CONNECTING TALK, GESTURE, AND EYE MOTION FOR THE MICROANALYSIS OF MATHEMATICS LEARNING

Francesca Ferrara*, Ricardo Nemirovsky**
Dipartimento di Matematica, Università di Torino, Italy (*)
TERC, Cambridge, MA (**)

INTRODUCTION AND BACKGROUND

In the last years deep changes have characterised the study of thinking and learning based on ongoing research in neuroscience, psychology, and cognitive science. These changes were supported by the availability of new technologies, which allow for a fine-grained recording of human activity. Different areas of cognition (such as language, vision, motor control, reasoning), which in the past were considered largely autonomous, have started to be studied as integrated and working in unison. This trend entails that research can get a wider and more detailed viewpoint to analyse thinking and learning processes. Examples come from the psychological research on gestures since the '80 (see Kita, 2003) and from vision science (e.g., Tanenhaus *et al.*, 1995). These emerging studies are generating new insights on the nature of thinking in educational research and the study of mathematics learning. For instance, Nemirovsky (2003) argues that "thinking is not a process that takes place "behind" or "underneath" bodily activity, but it is the bodily activities themselves". Within this viewpoint, even "the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending on the circumstances" (ibid.). The integrated study of bodily activity calls for a type of analysis, which is sometimes called "microgenetic analysis"; that is, a detailed examination of the genesis of ideas and approaches by a subject over short periods of time (minutes or seconds), while they are occurring. Microanalytic studies can document variability and actual processes of local change. Furthermore, the advent of digital video and other tools (portable eye trackers are an example), made microanalysis practical and more widespread.

EYE MOTION AND PERCEPTION

Perception and motor control (main constitutive aspects of thinking) are inextricably related in eye motion. Contrary to common belief, the eyes do not take whole snapshots of the surroundings onto our brains. Studies in eye motion provide evidence for Gibson's (2002/1972) thesis that visual perception is not an all-at-once photographic process of image-taking from the retina to the brain but a "process of exploration in time" (p. 84). Since "perception is *not* supposed to occur in the brain but to arise in the retino-neuro-muscular system as an activity of the whole system" (ibid.; p. 79), eye motion is crucial for such a process. Our study focuses on a type of eye motion, the saccadic one, consisting of rapid transitions ("saccades") between "fixations". A fixation is a point in the field of view around which the eyes stay on a relatively long period of time, commonly in the range of tenths of a second. The exploration in time results in some repeated cycles or trajectories formed by the successive fixations, the so-called scanpaths (Norton & Stark, 1971). The scanpaths clearly depend by the circumstances, are idiosyncratic to the individual seeing, and reflect the questions one has in mind. As a consequence, our eyes are constantly and actively traversing the surroundings. They do not record the environment, but they interrogate it, as Yarbus (1967) pointed out in the case of subjects looking at paintings. Other researchers have studied eye motion in context as a means to analyse the strategies different subjects activate when involved in a mathematical activity Some studies (Epelboim and Suppes, 2001) show that eye motion is central not only to seeing what is out there, as it were, but also for imagining things that are not present in the field of view. Therefore given that imagination and visualisation are essential for mathematical understanding, eye motion can be an important tool to reveal thoughts in catching a solution or grasping a meaning.

We will examine the coordination of talk, gesture, and eye motion, moment-by-moment, for a subject interviewed on graphs of motion. In our example, graphs describe a motion story read and interpreted by the subject, who wears an eye tracker recording his eye motions while a second camera films his gestures.

AN EXPLORATIVE EXAMPLE

The example briefly considered is based on an exploratory interview we conducted with a graduate student wearing a state-of-the-art portable eye tracker. The battery-operated eye tracker was carried within a small backpack connected to a head-mounted pair of miniature cameras (for the image of the scene, and for eye motion on the scene: see Fig. 1*a*, where at any time the cross represents the fixation). An external camera recorded gestures and hand motion (Fig. 1*b*). The interview included a "Motion Story" telling the imaginary motion of a person:

Figure 1

> I was quietly walking to the bus stop. I looked back and saw that the bus was fast approaching the stop. Then I ran toward the bus stop. However, the bus went by me and did not stop. I slowed down and kept walking toward the bus stop to wait for the next one. But, I forgot to put a letter in the mailbox, which is placed just a few metres behind where I was. So, I walked quickly toward the mailbox and I posted my letter. As soon as I realized that the next bus was coming, I ran back and I waited for it at the bus stop.

The interviewee (L) was asked to draw on a whiteboard a graph of position vs. time relative to the story and then the corresponding velocity vs. time and acceleration vs. time graphs. The ensuing conversation was about the characteristics of these graphs, maxima and minima, etc. Our analysis strives to trace the process of graph construction over time. For reasons of space we can just sketch the dynamics. At first, L is looking in the story for information to use for drawing the position vs. time graph. His eyes go back and forth from the right side (see Fig. 2*b*) where he has to draw, to the story placed on the left side (Fig. 2*a*). Fixations are located in the written text on places useful to gather important information to be translated in pivotal points of the graph. After L determines the points in time, he draws straight lines connecting them.

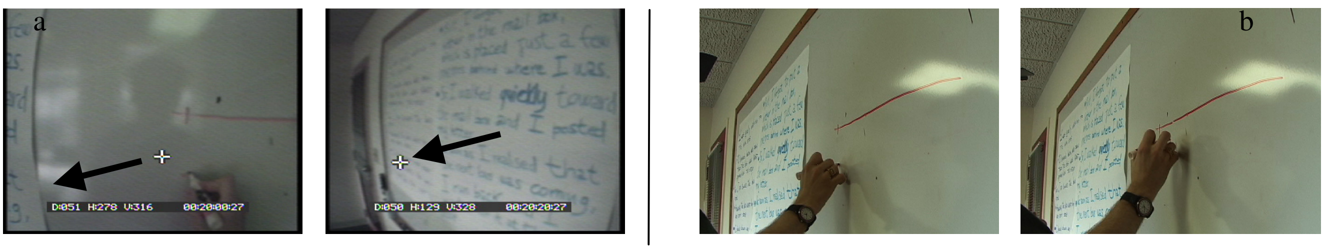

Figure 2

For example at time 3.55.09, L focuses in the story (Fig. 3*a*) on the speed feature (fixation on "quickly") of the piece of the graph he is starting to trace (Fig. 3*b*).

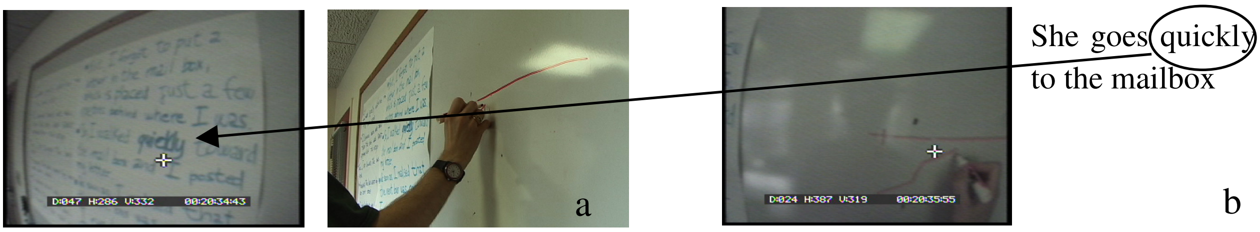

Figure 3

The resulting graph is shown in Fig. 4, where the position of the bus stop is set by L as the zero for the distance axis. Then a second phase started, in which the drawing is checked in relation to the story. The hand is kept still on a graphical element as to not lose the reference in the drawing, while the fixation goes to the text at the corresponding moment (Fig. 5). Then the eye comes back on the graph to traverse, together with the hand (Fig. 6), the motion started at that moment; moreover, L joined this description with his utterance ("She [the character of the story] ran back").

Figure 4

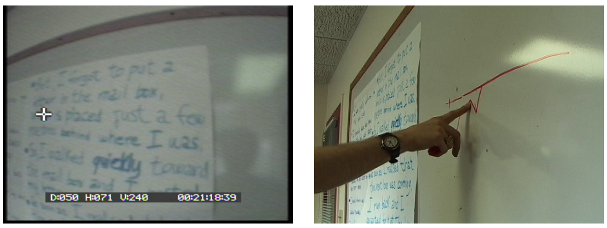

Figure 5

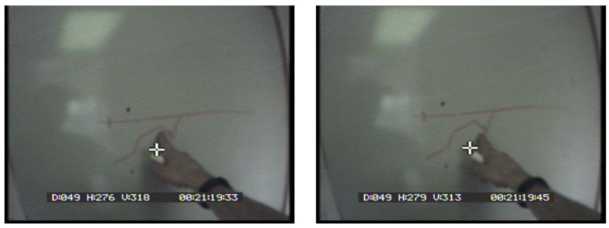

Figure 6: She ran back

In an ensuing phase L gathers from the distance vs. time graph information needed to draw the velocity vs. time graph. L's eyes and hands moved to relate the two graphs, their relations, and the physical quantities related to motion (a sequence of fixations and gestures is shown in Fig. 7).

Figure 7: L draws the velocity vs. time graph

Then a question by the interviewer (F in the following) marks the beginning of a reflection on the shape of the two graphs:

F: So, you suppose that in these three time intervals [hand pointing to the three pieces at the same height on the velocity vs. time graph] she has the same velocity?

To answer L goes back to the story. Then his eyes go from velocity vs. time to the distance vs. time to check the relations between the graphs and the motion described in the motion story; checking leads L to erase and redraw part of the graph (Fig. 8).

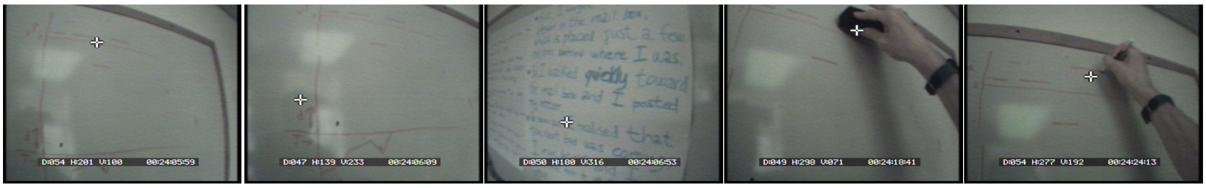

Figure 8: checking relations between graphs and motion

The dialogue between L and F developed further as L justified his changes or choices for the drawing, in trying to assess whether the pieces of distance vs. time indicated by F have the same slope:

L: I mean, I guess, I gave the word quickly the same magnitude basically as the running, so…

F: So, that's the reason because on this graph this part and this part have the same slope [hand pointing to the two pieces on the whiteboard]

L: Yeah.

F: That's the reason. What about these two parts? [hand pointing to the other pieces of the graph with same slope]

L: Those are the same, I think, because… although I guess maybe I'm not so good in drawing. I guess this one [L is pointing to the first segment] could be a little faster than this one… 'cause it says quietly walking [L is pointing to the second segment]… quietly walking versus walking

There seems to be three major functions of L's fixations: *locating*, e.g. when L needs essential information in the story, or when he has to choose where to draw a critical point; *checking*, e.g. when he goes back and forth from one source of information (say, the story) to another (say, the graph) to make sure they cohere; *directing*, e.g. when the eye helps the hand to get the (approximately) correct height of the critical points for the velocity vs. time graph (later for the acceleration vs. time graph). Furthermore, although each completed graph is in some sense a static object, L's eye motion shows that at any given time he is focusing on a very particular aspect, either coordinating with elements of the written story or of another graph. Each visual focusing appears to always have a question motivating it (e.g. should it be steeper? longer? Are these two the same speed?). Each graphical segment has to comply with numerous demands (consistency with the time interval, steeper than another one, etc.) and often his drawing of a segment complies with one or some of them but not with all of them. L goes through an iterative process of repair and re-drawing. As he draws and redraws he also becomes increasingly familiar with the motion story, needing less direct consulting of the text. Examining every single fixation as an effort to address a certain question is significant to a microanalysis of the situation. The sense of the whole for a graph (or a narrative) emerges gradually out of repeated focusing on particular events and shapes. In this sense, knowing how to graph a distance vs. time graph, or deriving velocity and acceleration from it, entails an intuitive sense of what to look at and how to look at it over time, in order to address ongoing questions.

WHY DO GESTURES MATTER? GESTURES AS SEMIOTIC MEANS OF OBJECTIFICATION

Luis Radford
Université Laurentienne, Canada

One of the most intriguing aspects of gestures is that in such varied contexts as face-to-face communication, talking over the phone, and even thinking alone, we all make gestures but we still do not know why. Explanatory models have been proposed by neuro-psychology, information process theories, etc. Our problem here is narrower. We are interested in understanding the role of gestures in the mathematics classroom. However, before going further, we should ask: why do gestures matter? Contemporary forms of knowledge representation are challenging the cognitive primacy with which the written tradition has been endowed since the emergence of printing in the 15th century. The audio and kinesthetic dimensions of oral communication of the pre-print era –dimensions that were replaced by the visual and linear order of the written text– are nowadays viewed with a revived and rejuvenated cognitive interest. Current studies on gestures and perceptual-motor activity belong to this stream.

Now, the way in which each one of us, as mathematics educators, may understand the role of gestures is naturally linked to the theoretical framework underpinning our research. From the semiotic-cultural approach that I have been advocating (Radford, 1998, 2003b), gestures are part of those means that allow the students to objectify knowledge -that is, to become aware of conceptual aspects that, because of their own generality, cannot be fully indicated in the realm of the concrete. In a previous article I have called those means *semiotic means of objectification* (Radford, 2003a). In addition to gestures, they include signs, graphs, formulas, tables, drawings, words, calculators, rules, and so on.

Our answer to the question: “Why do gestures matter?” can then be formulated as follows. Gestures matter because, in learning settings, they fulfill an important function: they are important elements in the students’ processes of knowledge objectification. Gestures help the students to make their intentions apparent, to notice abstract mathematical relationships and to become aware of conceptual aspects of mathematical objects.

However, considered in isolation, gestures have -generally speaking- a limited objectifying scope. We have tried again and again the following experiment: we have turned off the volume of many of the hundreds of hours of our video-taped lessons and, even though we *see* the students making gestures and carrying out actions, our understanding of the interaction is very limited. The same can be said of other semiotic systems. Thus, we have also turned off the image and, even though we *hear* the discussion, our understanding of the interaction is again very restricted. We have also stopped both the sound and the image and limited ourselves to *reading* what the students *wrote*, and the result has been as poor as in the previous cases. The reason behind the poor understanding of the students' interaction that results from isolating one or more semiotic systems present in learning is that knowledge objectification is a multi-semiotic mediated activity. It unfolds in a dialectical interplay of diverse semiotic systems. Each semiotic system has a range of possibilities and limitations to express meaning. The conceptuality of mathematical objects cannot be reduced to one of them, not at least in the course of learning, for mathematical meaning is forged out of the interplay of various semiotic systems.

SEMIOTIC NODES

The theoretical construct of *semiotic node* (Radford *et al*. 2003) is an attempt to theorize the interplay of semiotic systems in knowledge objectification. A *semiotic node* is a piece of the students' semiotic activity where action and diverse signs (e.g. gesture, word, formula) work together to achieve knowledge objectification. Since knowledge objectification is a process of becoming aware of certain conceptual states of affairs, semiotic nodes are associated with the progressive course of becoming conscious of something. They are associated with layers of objectification.

Let us illustrate these ideas through a story-problem given to a Grade 10 class. In the story-problem two children, Mireille and Nicolas, walk in opposite directions, as shown in Figure 1. The students were asked to sketch a graph of the relationship between the elapsed time and the remaining distance between the children.

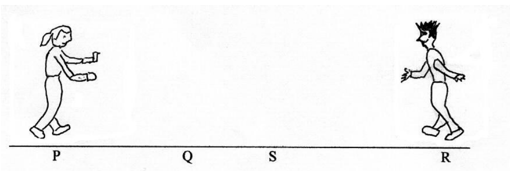

Figure 1. Mireille walks from P to Q. Nicolas walks from R to S

Supported by the students' previous experience, one of the Grade 10 students, Claudine, proposed a compelling -although incorrect- argument: the graph, she suggested, is something like an "S". Ron did not agree, but could not counter Claudine's argument. He claimed that the graph should be something like a decreasing curve, although the details were still unclear for him. In an attempt to better understand the details, he deployed a series of arguments and gestures that were intended not only for his group-mates but for him as well. In Fig. 2 there is an excerpt of the discussion.

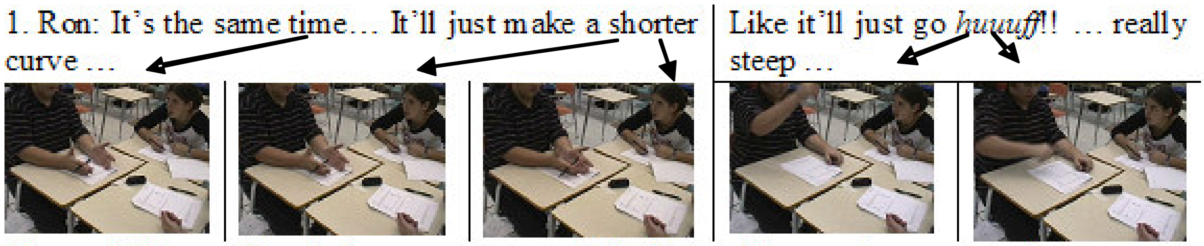

Figure 2. Pictures 1 to 5. Some gestures made by Ron while uttering sentence 1.

To objectify the relationship between distance and time, in the first picture, Ron put his hands one on each one of the students of the story-problem as drawn in the activity sheet. Insofar as the hands stand for something else, they become signs. But in opposition to written signs, which are unavoidably confined to the limits of the paper, hands can move in time and space. Capitalizing on this possibility, to make *apparent* the fact that the distance decreased, Ron moved his hands in opposite directions (pictures 2 and 3). In pictures 4 and 5 he made a vertical gesture sketching the graph time vs. distance, right after have finished the sentence. Three seconds later, remarking that Claudine was not convinced, he started his explanation again. Uttering the first sentence led him to better understand the mathematical relationship, so in the second attempt he was able to produce a more coherent discourse and to better co-ordinate gesture and word. Here, he reached a clearer layer of knowledge objectification.

Pictures 6 to 8 show gestures similar to those in Figure 2, except that now they are made in the air and Ron talks in the first person. In pictures 9 and 10 a familiar situation is invoked (the motion of two trucks). There is, however, another more fundamental aspect that has to be stressed. While in sentence 1, time remained essentially implicit (it was mentioned to emphasize the fact that the children started walking at the same time), in sentence 2, time became an explicit object of reference. Time, however, was not indicated through gestures. It was indicated with words. Even if both are semiotic means of objectification, gestures and words dealt with different aspects of the students' mathematical experience.

In each of the previous cases, the different co-ordination of words and gestures constitutes a distinct semiotic node reflecting different layers of knowledge objectification. One of the research problems that my collaborators and I are currently investigating is related to the theoretical and practical characterization of layers of knowledge objectification. As we saw, gestures play an important role therein. But this role, we suggest, can only be understood if gestures are examined in the larger context of the dialectical interplay of the diverse semiotic systems mobilized by teachers and students in the classroom.

Acknowledgment: This paper is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

GESTURES, SIGNS AND MATHEMATISATION

Julian Williams, School of Education, University of Manchester, U.K.

Where to start: To summarise, criticise, synthesise? The topic of 'gesture' seems so vast, and yet we know (especially with regard to mathematisation) so little. Reading these four papers for the first time, they seem like four ships crossing a huge ocean, moving in different directions, occasionally signalling each other using semaphore!

A SUMMARY: CONTEXT

None of these papers is about gesture alone. All see gesture as part of an integrated communication system with language and, in this case, mathematics. Edwards even defines gesture, after McNeill, in this way, i.e. the gesticulation accompanying speech. Two of these papers are about externalisation in the Vygotskyan sense (Arzarello et al. and Bussi & Maschietto are explicit about this reference) when children are involved in group problem solving. This is also true implicitly of Edwards' students' who gesture as they talk about their previous mathematical work, though her primary reference to theory is in that of Embodied Cognition.

But Ferrara & Nemirovsky's study situates gesture in a more complex setting where seeing (active 'interrogating' with the eye-brain-muscle) is integrated with externalising actions involving gestures, and actually graph-drawing (despite the others' papers' reference to Vygotsky's remark to the effect that gesture gives birth to writing/script, the quote seemed even more apt here!) I highlight the context of gesture, because it influences function and hence categorisation systems.

CLASSIFICATION OF GESTURES/GESTICULATIONS

There is a 2000-year history to the development of classifications of gestures (see Kendon, 2004). Edwards builds her corpus of gestures in the mathematics education context, and this inevitably extends and refines that of McNeill (1992, extended in 2000). Her recognition of context is important: the different functions of gesture in mathematics education imply the need for multiple corpora, each perhaps with its own, albeit related, classification systems.

McNeill's context of interest was mostly that of narrative/narrators, and he was particularly influenced by the significance of 'imagistic' functions of gesture in relation to the emergence of language in an utterance (the so-called growth point, where the gesture precedes the linguistic formulation).

Such an approach has obvious relevance for the emergence of mathematics in children's talk, such as when the child points to figures before articulating (Radford, 2003a, p 46, Episode 1,1, the video clip is not downloadable):

Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […]

McNeill's notion that gestures are associated with 'internal', intra-mental images, and their linguistic 'parallels' associated more with the external, inter-mental social/socio-cultural 'verbal' representation, is an interesting one for mathematisation (e.g., in Arzarello et al.). The idea here is that the 'sign' constituted by a gesture with its linguistic parallel constitutes a unity of internal with external elements, and that conflicts between these elements represent contradictions, and hence opportunities for realignment, or learning. The gesture-and-word unit offers a reflection of Vygotsky's thought-and-word (or thought-and-utterance) unit of analysis.

So Edwards takes, applies and extends McNeill 'imagistic' categories (iconic and metaphoric) to mathematics contexts. This is a good start, and I immediately want to extend this formulation to include McNeill's non-imagistic gesture categories: I think I see 'beats' (Radford speaks of 'rhythm') in the gestures used by children to indicate number patterns in 'factual generalisation', as in the rhythmic articulation and pointing-beating of the "1 plus 2", "2+3" etc.

In my own work, I have stressed the significance of deictics in mathematical communication: pointing and waving when associated, or better fused, with models signify mathematics (e.g., Williams & Wake, 2003; Misailidou & Williams, 2003).

In coordination with a model (such as a graph in Roth's original examples) deictic gestures can signify mathematical objects before they are named, and when the points/segments of a drawing, model or graph have multiple significations, we have an ambiguous moment in communication that can perhaps hold just the right tension in communication.

Beyond gesticulation, there are yet other categories of gesture that mathematics education should consider: 'Cohesives' and 'Butterworths' will perhaps emerge or even dominate corpora involving problem solving and proving for instance.

And, to extend further, do the students' graphing gestures, in Ferrara & Nemirovsky, belong to a different category system, somewhere near the 'conventional language' end of the gesturing spectrum (where Kendon and McNeill put sign-languages)?

SEMIOTICS, GENERALISATION AND GESTURE

Arzarello et al. and Bussi & Maschietto inscribe gesture, in part, within Radford's cultural semiotic theory of 'semiotic objectification'. Radford's classification of factual, contextual and symbolic generalisation draws on Peircean categories and conceptions of sign: the index, icon, and symbol, but these are not to be too superficially identified with deictic, iconic, and metaphoric or symbolic gestures.

When a gesture, possibly integrated with parallel action/utterance, is used to denote another object, it constitutes a sign (hence Radford's term: semiotic objectification). In such a case the gesture can be indexical, iconic, and/or symbolic in Peirce's (but not McNeill's) sense. (Peirce, 1955). This now provides a semiotic classification of gestures-in-context that Radford used to analyse significant differences in meanings, such as when the meaning of a formal algebraic expression is indexical for the children but symbolic for the teacher (marking a contradiction between contextual and symbolic generalisation).

I think this difference between McNeill and Peirce/Radford (Wartofsky is another story) explains my concerns with classification systems being equated in Bussi & Maschietto, table 1: a classification system works best if it associated with a particular theoretical scheme. The table thus begs us to examine the relation between the underlying frameworks: Embodied cognition/cognitive linguistics, linguistics, cultural semiotics, that the category systems 'indicate'. (And then there is Wartofsky.)

At this point I would like to consider the disjuncture between the imagistic gestures, or gesticulations in Arzarello et al., Bussi & Maschietto, and Edwards with the gestures and eye foci of the graph-drawing students of Ferrara & Nemirovsky. The gestures of a graph drawer are less strongly bound to the linguistic parallel; but they form a unit of signification with the graph itself, as when the gestures of an operator working a machine form an action because of the mediation of the machine.

In addition, graph drawing has more 'conventional' and 'symbolic' reference rather than iconic, and operate more at the conscious level (in this data anyway, these operations on the graph have not yet descended with practice into the subconscious). In the context of cultural semiotics, this distinction between conscious-unconscious in action-operation suggests an activity theory perspective (Leont'ev, 1981; Williams & Wake, under review) might provide an analytical framework for bringing the two elements together.

It seems there is plenty of empirical and theoretical work to be done still.

BUILDING INTELLECTUAL INFRASTRUCTURE TO EXPOSE AND UNDERSTAND EVER-INCREASING COMPLEXITY

James Kaput
University of Massachusetts, Dartmouth, USA

From the abstract brain-in-a-vat, to the brain neurologically instantiated in a head, to a brain interacting with symbolic tools, to a brain embodied in a walking, talking, gesturing body, to a brain situated in a culture-imbued crowd , … we confront ever increasing complexity in phenomena. Ever more of what was invisible or ignored becomes visible and subject to study, what was excluded becomes included. As so clearly pointed out by Nemirovsky, the subtle new phenomena of gesture, bodily action and perception, eye-movement, and so on, are inevitably and intimately connected with the larger phenomena of thinking, learning, acting and speaking. Indeed, these newly studied phenomena seem, in many cases, to be what the gross phenomena are made of.

With the increasingly complexity comes pressure to expand our repertoire of techniques, conceptual frameworks, and perspectives, our intellectual infrastructure.

Each Forum paper reflects a sophisticated response to the new phenomena being exposed, and each reflects the process of building new intellectual infrastructure intending to expose and make sense of these subtle new phenomena. To a significant extent, the value of the papers resides in the intellectual infrastructure that they are making available to the field of Mathematics Education, a contribution that extends well beyond the particulars of the specific studies reported.

DISTINGUISHING FORMS OF GENERALIZATION AND ASSOCIATED SEMIOSIS

Arzarello, Ferrara, Robutti, Paola, & Sabena develop two means of analysis of the processes of semiotically-based objectification, Parallel and Serial, and, most importantly for our purposes, a way of accounting for the *grounded genesis of a new sign*, which in turn includes Radford's notion of contextual generalization. This account is very similar to one developed by Kaput, et al. (in press). However, the latter make a distinction between contextual generalization and the lifting out of repeated actions as the following example illustrates.

Consider a situation where students have been working with open number sentences such as 8+_=13 or perhaps using a literal, $8 + x = 13$. After solving and discussing some number of these kinds of sentences, it is noticed that the answer always seems to be of the form 13 – 8, that is, in verbal terms, "you subtract the left-hand number from the right-hand number to get the answer." The students can be thought of as being in the process of building a rule, a generalization that applies to a parallel set of additive number sentences written in a number-sentence symbol system. This is an example of the grounded genesis of a new sign, where children's intermediate step could be in form of the verbal version of the rule as given. Mathematically, it is a generalization over a subset of the expressions writable in the number sentence system. At some point, as the result of a combination of discussion and perhaps the teacher-led cataloging and recording of cases, the rule gets extended to cover cases where the "unknown" is in the first position, as in "_ + 6 = 15." But now, in order to ensure that the rule covers all such cases and will extend to more cases in the future, the teacher suggests that they think of it as "subtracting the same number from both sides (of the equation)." While it need not be written in what we would recognize as algebraic form, this new verbally described operation on the number-sentence objects is another, and major, contribution to building a new symbol system which consists of expressions of generalizations about *actions on number sentences*. It is a distinct representation of general actions, and as such is part of a new *operative* symbol system being "lifted out" of in order to serve as a new, more general way of thinking about and operating on the number sentence objects.

This is a critically important kind of symbolization in mathematics, but it is a different *kind* of move, I believe, from contextual generalization. Whereas the previously described move involved expressing variation across *statements*, the new one expresses *actions* on the inscription-objects of the initial symbol system. Indeed, the number-sentence statements themselves are likely to be products of such a lifting-out-of-actions. Further, some of the lifted actions based in arithmetic can be represented directly in terms of the structure of the system, such as the distributive law of multiplication over addition in the usual number systems, which allows the substitution of a * (b + c) by a * b + a * c or vice-versa. The action is an equivalence-preserving substitution, which has parallels in the other basic properties of operations as well as substitution actions such as factoring and expanding polynomials that are built directly on them. I expect that the Research Forum will help us unify these different forms of semiosis.

GESTURE, SEMIOSIS AND DELIBERATE GENERALIZATION

I hope that we can jointly address the matter of those acts of communication and sense-making that are driven by deliberate generalization vs. those that are driven by more immediate acts of communication as described in the papers by Arzarello and the paper by Bartolini Bussi & Maschietto. A similar issue can be raised in the study by Ferrara & Nemirovsky, who examine a particular, highly concrete act of representation. Given the essential role of argument and expression in generalization, and the fact that younger learners need to use natural language and other naturally occurring forms of expression, my sense is that we have much to learn about generalization and hence the development of algebraic thinking, from studies of gesture and talk – including intonation.

My sense is that the purposively integrative style embodied in Radford's notion of *semiotic node* holds great promise in deepening our understanding of how speech, gesture and the many different systems of signs interact, particularly if we adopt his perspective that knowledge objectification is almost always, particularly in education, a multi-modal, semiotically mediated phenomenon. His prime example is of particular interest to me because we have used such tasks in a technological context, where the motions of two objects approaching each other, for example, can be created on a computer screen through almost-free-hand drawn graphs produced by students. The interaction between the particular and the general becomes even more pronounced. Indeed, our work also involves activities similar to that used by Ferrara & Nemirovsky, but where the students' graphs can be re-enacted dynamically. Furthermore, these kinds of constructions can be done in a wirelessly connected classroom where different students can systematically contribute different parts of the same graph in the context of a classroom discussion by sending to a shared public display a graph segment produced on their own hand-held device. Or they can import a physical motion that then, as it is relayed (and not merely graphed) interacts in specifiable ways on a public screen with someone else's imported and reenacted motion. In this case, the semiotic acts become highly public and social, and the need for theoretical constructs such as those offered by Radford becomes more acute than ever before.

THE ISSUE OF GENERALITY OF FINDINGS

Edwards' taxonomy of gestures reveals subtleties that any long-term account of gesture in mathematics education would seem to include. Clearly, we need to examine cases of all sorts, from people describing mathematics that they already know, to people learning mathematics, to people teaching mathematics, to people using mathematics in modeling and problem solving, and, most importantly, we need to vary the kinds of mathematics involved, including mathematics centered on generalization vs. mathematics centered on visualization or computation. Taxonomy, of course, helps generate theory, which informs the structuring of the taxonomy. Of particular interest is the use of gesture in the context of technology use, especially because certain actions in a technological environment amount to tracing gestures – as when one drags a hotspot in a dynamic mathematics system, especially a geometric one such as Cabri or Sketchpad. All such actions amount to gestures captured within a mathematically defined system, so the design and use of such systems is an arena for the immediate application of research in gesture.

The eye-tracking microanalytic work by Ferrara and Nemirovsky, pioneering as it is, raises all sorts of questions and tempts all sorts of hypotheses. While more intrusive eye tracking work has been used for many years in areas that involve traditional character-string symbol systems, including arithmetic and algebra, as well as geometry as they cite, the contexts that Ferrara and Nemirovsky investigate are extremely rich, both visually and in mathematical content. In keeping with an underlying theme of the Forum, the authors stress the functional unity of eye motion, kinesthetic experience, and thought. It will be especially interesting to see how differences in eye-tracking patterns relate to prior experience. For example, how would a novice learner of motion-graph interpretation differ from one who is very experienced, or how would the patterns change if the motion were more regular and perhaps algebraically definable? In this case, the graph might, in fact be seen in a more gestalt-like manner.

I will close by briefly offering yet another perspective on the core issues being explored, the perspective of evolutionary psychology, in particular, the highly integrative, culturally oriented approach developed by Merlin Donald (1991, 2001). Donald's analysis of the physical, "mimetic" roots of reference helps explain the intricately intertwined role of physical gesture in thought and communication and, more broadly, the physical-social embodiment of thought and language. Space limitations prevent further exploration of Donald's more recent work on the co-evolution of human consciousness and culture (2001) that helps provide a rationale for Radford's strongly cultural approach that deliberately takes into account layers of objectification that integrate the many forms of symbolic expression and the major modalities (action, speech, writing/drawing) in which they can be instantiated.

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