RF01: THE SIGNIFICANCE OF TASK DESIGN IN MATHEMATICS EDUCATION: EXAMPLES FROM PROPORTIONAL REASONING

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In the context of the overall focus of PME29 on *Learners and Learning Environments*, we have chosen the topic of pedagogical task design for this Research Forum. We see task design as a crucial element of the learning environment, and wish to explore further the role that it plays for learners. The overarching question for this Research Forum is: Why is task design significant?

To make progress on this question, we raise two issues: how does the task design impact on student learning? How does the agenda of the researcher or teacher shape the task design? More specifically we ask: how does the nature of the task influence the activity of students? What is important for mathematics educators in designing a task?

In order to work on these questions, both in the preparations for the Forum, and within the sessions at the conference, we have chosen to take a specific topic within the curriculum, that of *proportional reasoning*, and to invite the contributors to the Forum to work on designing tasks for the learning and teaching of proportion for pupils of around 11-12 years old.

The contributors

There are four groups of researchers contributing to this Forum, all of whom work on aspects of task design from different perspectives.

Dirk De Bock, Wim Van Dooren and Lieven Verschaffel explore features of the use of words problems in a number of mathematical areas, and have focussed on the ability to discriminate proportional and non-proportional situations.

Koeno Gravemeijer, Frans van Galen and Ronald Keijzer use design heuristics from Realistic Mathematics Education (guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling) in an approach which also draws on design research.

Alex Friedlander and Abraham Arcavi have many years experience within the Compumath project, which is developing a technology-based curriculum and studying the effects on pupils' learning.

Janet Ainley and Dave Pratt have developed an approach to task design based on creating tasks which are purposeful for pupils within the classroom environment.

We hope that our understanding of task design will be enhanced by making explicit reflections on these differing perspectives in the context of specific examples of tasks and their use by pupils.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 93-122. Melbourne: PME.

The design brief for the contributors

Each of the teams of contributors was asked to design a task which focussed on proportional reasoning. The task had be suitable for pupils aged about 11-12 years, and it also had to be a 'stand alone' task, which could be tackled within one lesson. This condition was a significant constraint for some of the contributors, who would normally design tasks as part of a sequence. Contributors were asked to prepare their task in a form that could be presented to pupils, and were also asked to provide teachers' notes.

Each of the tasks has been trialled with pairs of pupils and the papers by each of the contributing teams which follow this introduction draw on this data to illustrate the discussion of the principles which underpinned their task designs.

Dirk, Wim and Lieven's task

This task focuses on similarities and differences in a set of word problems, some of which require proportional reasoning, while others have a similar format, but are not, in fact, proportional problems.

Yesterday, Mrs. Jones made some word problems to use in the math lessons. But they got all mixed up! Can you help Mrs. Jones to put some order in the word problems? Look at the problems very carefully and try to make groups of problems that belong together.

A Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 rounds. When Ellen has run 30 rounds, how many has Kim run?

B Mama put 3 towels on the clothesline. After 12 hours they were dry. The neighbour put 6 towels on the clothesline. How long did it take them to dry?

C Mama buys 2 trays of apples. She then has 8 apples.Grandma buys 10 trays of apples. How many apples does she have?

D John runs a bakery. He uses 10 kg of flour to make 13 kg of bread. How much bread can he make if he uses 23 kg of flour?

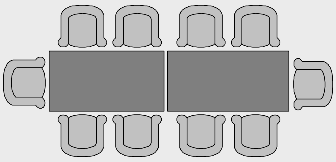
E The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. If there were 8 carriages connected to the locomotive, how long would the train be?

F Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?

G A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece. How long will it take this group to play it?

H Yesterday, a boat arrived at the port of Rotterdam, containing 326 "Nissan Patrol" cars. The total weight of these cars was 521 tons. Tomorrow, a new boat will arrive, containing 732 "Nissan Patrol" cars. What will be the total weight of these cars?

I In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now the teacher puts 6 tables in a line. How many chairs fit around these tables?



J In the shop, 4 packs of pencils cost 8 euro. The teacher wants to buy a pack for every pupil. He needs 24 packs. How much must he pay?

Now answer the following questions:

- Write here the different groups of problems. (Use the letters on the sheets)
- Why did you make the groups in that way?
- Can you think of a different way to put the problems in groups? Explain that as well.

Koeno, Frans and Ronald's task

This task is based around the story of Monica and Kim making a cycle trip from Corby to Cambridge. Various resources such as a map of the route, photographs and background information (the reason for the trip, the weather conditions) are provided.

After cycling for 1 hour 30 minutes, the girls reach a village called Catworth where there is a signpost showing 18 miles from Corby and 30 miles to Cambridge. "Okay", Monica says, "this is going well."

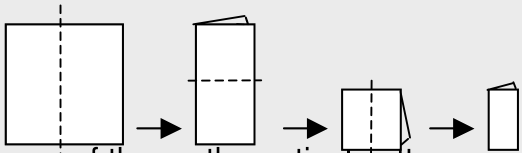
1. Could you tell why she might say this?
2. How much time has it taken them to get to Catworth? And what is the distance they have covered?
   So what can you say about the speed of Monica and Kim? You can use the table to judge their speed.

Cycling at a slow speed:	8 miles per hour
Cycling at a normal speed:	12 miles per hour
Cycling at a fast speed:	18 miles per hour

3. In the table, the speeds of various kinds of cyclist are given. However, if you want to compare the speeds of cyclist who are not riding the same road on the same day, conditions might be different.
   Could you mention the things that have to be taken into account, if we were to measure the speed of a cyclist.
4. After a short stop, Monica and Kim are moving on. They get on the road from Catworth to Cambridge, a distance of 30 miles. At about what time do you think they will arrive in Cambridge?
5. Of course, you cannot be absolutely sure about how long it will take them.
   Could you mention some reasons why you cannot be sure? Still, to make a sensible guess, it might be helpful to know how much time she would need if she were to keep up the same speed.
6. How much time would the ride to Cambridge take if they were to keep up the same average speed as before?

Alex and Abraham's task

This task is based around the practical activity of folding a 32x32 square piece of paper, as shown below. There are then a series of questions to address, some of which use a spreadsheet. In the pupils' materials some guidance for using the spreadsheet is included, which has been omitted here.



2. Describe some of the mathematical patterns you notice as you fold the shapes.
3. ***Predict***: What is the pattern of change in the perimeter, as you fold the shapes?

4.a. Write on the drawings the ***dimensions*** and the ***perimeter*** of the first four shapes in the sequence.
  b. ***Collect your data in a spreadsheet table*** that shows the dimensions and the perimeter of the first ten shapes in the sequence.

5. ***Draw a graph*** to show the perimeter of the first ten squares and rectangles in the sequence.
6. Look for ***patterns*** that describe the change in the perimeter, as the square is folded. Explain the connection between your patterns and the folding shapes.

7.a. The teacher asked: *By how many length units does the perimeter get shorter at each folding?* Daniel replied: *At each folding the perimeter gets shortened by the same length.* Do you agree with Daniel?
  b. ***Collect data*** that may help you to answer the teacher's question.
  c. Do you see any ***patterns*** in the collected data? Explain the connection between your patterns and the folding shapes.
  d. Did you change your initial opinion about Daniel's answer? Explain why you did or did not.

8.a. The teacher asked: *By what ration does the perimeter get smaller at each folding?* Daniel answered: *At each folding the perimeter of the new shape is half the perimeter of the previous one.* Do you agree with Daniel?
  (b, c and d as for question 7)

9.a. Find pairs of shapes that have a **perimeter ratio of one half**.
  b. Give a "rule of thumb" for finding such pairs.
  c. Convince a fried why your rule always works.

Janet and Dave's task

For this task pupils have measuring tapes, a spreadsheet. Each group also has a different item of dolls' house furniture.

Children in a primary school want to make a 'dolls' house classroom'. Use the piece of furniture you have been given to work out what size they should make some other objects for their classroom.

DIFFERENT PERSPECTIVES ON TASK DESIGN

The four tasks presented here offer significant differences in the kind of activity that pupils may be engaged in when working on them, but they also arise from different approaches to task design. These are explored and elaborated within the individual papers, but we also draw attention here to one issue which may be discussed within the Forum sessions: the role of the teacher.

Gravemeijer, van Galen and Keijzer emphasise the central role which they see the teacher as playing when a class is working on the task in guiding discussion to focus on mathematical issues and the development of tools to support proportional reasoning. De Bock, Van Dooren and Verschaffel have designed a task which it appears pupils may work on independently, but they also acknowledge the potential role of the teacher in encouraging whole class discussion around the task. Friedlander and Arcavi have constructed a task made up of a sequence of questions, which balances structured questions with more open invitations to make conjectures. Some of the questions are based on hypothetical conversations between the teacher and a pupil, and clearly offer support for pupils to work independently, or for an inexperienced teacher to use the materials. Ainley and Pratt’s task is stated very briefly. There is clearly a crucial role for the teacher, who would need an understanding of the approach, in leading discussion to explore and develop the task, but the authors also contrast the activity of pupils who need to rely on continuing support from the teacher, and those for whom the task itself determines the direction of their activity.

NOT EVERYTHING IS PROPORTIONAL: TASK DESIGN AND SMALL-SCALE EXPERIMENT

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INTRODUCTION

Proportional (or linear) reasoning is a major tool for human beings in many cultures to interpret real world phenomena (Post, Behr, & Lesh, 1988; Spinillo & Bryant, 1999), even when the phenomena are not linear ‘stricto sensu’. Therefore, not surprisingly, proportional reasoning constitutes one of the major topics in school mathematics from the lower grades of the elementary school to the lower grades of secondary school. From Grades 2 and 3 onwards children learn to multiply and divide and to apply these operations in simple word problems like “1 pineapple costs 2 euro.

How much do 4 pineapples cost?", which are predecessors of proportional reasoning tasks. During Grade 4 and afterwards, proportional reasoning skills are further developed. From this age on, students are frequently confronted with proportionality problems, most often stated in a so-called missing-value structure such as: "12 eggs cost 2 euro. What is the price of 60 eggs?", and are trained to set up and solve the corresponding proportion $12/60 = 2/x$ for the unknown value of x. However, in the last decade, mathematics educators formulated two main deficiencies of this current school practice for teaching and learning proportionality.

First, because almost all proportional tasks students encounter at school are formulated in a missing-value format – and at the same time, non-proportional tasks are very rarely stated in this format – students tend to develop a strong association between this problem format on the one hand and proportionality as a mathematical model on the other hand. Recently, De Bock (2002) provided empirical evidence for that claim. In a series of exploratory studies in one specific mathematical domain, namely, problems about the relations between the linear measurements and the area or volume of similarly enlarged or reduced geometrical figures (such as the dolls' house context in Janet and Dave's task), it was shown that 12-16-year old students have an almost irresistible tendency to improperly apply direct proportional reasoning to length-area or length-volume relationships, especially when the problems are stated in a missing-value format. Changing the problem formulation by transforming the problems into a "comparison format" proved to be a substantial help for many students to overcome the trap of inappropriate proportional reasoning in this domain. This study – together with analogous findings by other researchers – suggests that teachers should at least bring more variation in proportionality tasks and especially take care that these tasks are not *always* formulated in a missing-value format.

Second, as reflected in the *Standards 2000* (National Council of Teachers of Mathematics, 2000, p. 217), "facility with proportionality involves much more than setting two ratios equal and solving for the missing term. It involves recognising quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship". In the same respect, Schwartz and Moore (1998, p. 475) explicitly stated that "when proportions are placed in an empirical context, people do not only need to consider at least four distinct quantities and their potential relationships, they also need to decide which quantitative relationships are relevant." The example they gave relates to mixing 1 oz. of orange concentrate and 2 oz. of water, compared to mixing 2 oz. of orange concentrate and 4 oz. of water. If the question is which mixture will taste stronger, the ratios should indeed be compared, but if the question is which mixture will make more, a ratio comparison is of course inappropriate. The claim for the unwarranted application of proportionality was made even stronger by Cramer, Post and Currier (1993, p. 160). They argued that "we cannot define a proportional reasoner simply as one who knows how to set up and solve a proportion".

For the design of a task, we focussed on students' ability to discriminate between proportional and (different types of) non-proportional situations.

DESIGN OF A TASK

Inspiration for the task design was found in a recent study by Van Dooren, De Bock, Hessels, Janssens and Verschaffel (2005). These researchers studied how students' tendency to overgeneralise the proportional model develops in relation to their learning experiences and their emerging reasoning skills. For that purpose, they presented 1062 students from Grade 2 to 8 with a test containing 8 word problems: 2 proportional ones (for which a proportional solution was correct) and 6 non-proportional ones (2 additive, 2 affine and 2 constant). The following are examples of the non-proportional items:

- Additive problem: "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 round. When Ellen has run 30 rounds, how many has Kim run?" (correct answer: 40, proportional answer: 90)
- Affine problem: "The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. How long is the train if there are 8 carriages connected to the locomotive?" (correct answer: 92 m, proportional answer: 104 m)
- Constant problem: "Mama put 3 towels on the clothesline. After 12 hours they were dry. Grandma put 6 towels on the clothesline. How long did it take them to get dry?" (correct answer: 12 hours, proportional answer: 24 hours)

The results showed that many 2nd graders already could solve simple variants of proportional word problems, but the firm skills to conduct proportional calculations (i.e. to solve proportional word problems) were acquired between 3rd and 6th grade. With respect to the non-proportional items, more than one third of all answers contained an erroneous application of the proportional model. The tendency to over rely on proportionality developed in parallel with the ability to solve proportional word problems: it was noticeable already in 2nd grade, but increased considerably in subsequent years, with a peak in 5th grade where more than half of the answers to non-proportional items were proportional errors. After this peak, the number of proportional errors gradually decreased, but they did not disappear completely: in 8th grade still more than one fifth of the answers contained a proportional error. There were some remarkable differences according to the mathematical model underlying the non-proportional problems: One would expect that the word problems with a "constant" model (like the "clothesline" problem mentioned above) were the easiest ones in the test (since there was no need for calculations), but these problems got the highest rate of proportional errors (up to 80% in 5th grade). For some word problems (like the additive "runners" item), the performances even decreased (with 30%) from 2nd to 6th grade. The authors concluded that, throughout primary school, students not only acquire skills to calculate proportions and solve proportional problems. The proportionality scheme becomes so prominent in students' minds that they also begin to transfer it to settings where it is neither relevant nor valid.

For the task that we designed, we worked with the same kind of word problems (4 proportional ones, labelled with the letters C, D, H and J) and 6 non-proportional ones, namely 2 additive, 2 affine and 2 constant, respectively labelled with the letters A and F, E and I, and B and G). The exact formulation of the different problems is given in the introductory section of this research forum. To avoid confusion, we didn't include problems for which the proportional model gives a more or less good approximation, but one can discuss its accuracy on the basis of realistic constraints (such as it is the case in the task of Koeno, Frans and Ronald). Although all ten problems in our task have an exact numerical answer, the task that we gave the students was not to calculate a numerical answer, but to group the problems in at least two different categories and to explain the motivation for their grouping. To allow at least one other way of grouping than the one based on the underlying mathematical model, two of the proportional problems (D and H) were given with a non-integer internal ration, while all other problems were based on easy, natural ratios.

To clearly explain and illustrate the nature of the task (and, at the same time, to show its open-ended character), we first confronted the participants with 13 cardboard figures (stars, triangles and circles) in three different colours (grey, black and white). Two fictitious students, Tommy and Ann, were asked to help their teacher, Mrs. Jones, to classify these figures. Tommy suggested grouping all figures with the same shape (i.e., a grouping based on a "mathematical" criterion), while Ann proposed to bring together the figures with the same colour (i.e. a grouping based on a "non-mathematical" criterion). Then, it was stated that Mrs. Jones made a series of 10 word problems to use in the math lessons (labelled with the letters A to J), but again, they got all mixed up. Students were asked to do as Tommy and Ann had done and to help Mrs. Jones to classify the word problems. More concretely, they were invited to "look very carefully at the problems and to try to make groups of problems that belong together". After that, they had to answer the following questions:

- Why did you make the groups in that way?
- Ann and Tommy did something different when they made groups of the figures. Can you think of a different way to put the problems in groups? Explain that as well.

A SMALL-SCALE EXPERIMENT

The task was given to four students (aged 11 years): Alice, Freya, Hans and Jonas. The researcher first introduced the task and checked pupils' understanding of the instructions. Then, for about 20 minutes, the children were allowed to read the problems and sort them into groups. As each finished, the researcher directed the pupils to record their reasoning, and then to find other groupings.

Alice worked for about 14 minutes to find a first grouping in three categories: group 1 (A and F, the two additive problems) because “they sound similar”, group 2 (B and G, the two constant problems) because “it is all like ‘how long will it take this person to do this?’ and stuff like that”, and group 3 with the six remaining problems (the four proportional and the two affine problems). Alice’s grouping is based on the underlying mathematical model of the problem, although she was unable to articulate this criterion. In her grouping, she made no distinction between the “pure” proportional problems and the affine problems (which, in fact, ask for a combination of multiplication and addition). After the researcher insisted, Alice came with a second (rather superficial) grouping into two categories (discriminating the problems with “how” and the problems with “what” in the problem statement).

Freya needed about 14 minutes to find a first grouping into three categories: group 1 (H), group 2 (B, C, D, E, F, I and J) and group 3 (A and G). She explained her criterion as follows: “I made the groups due to the operation you have to do to work out the answer. E.g. in group 2, you have to do multiplication to find the answer, and in group 3, you have to divide to find the answer”. Clearly, Freya’s actual grouping was not based on the criterion she formulated. Being invited by the researcher to find other ways of grouping, Freya proposed a second grouping in three categories: group 1 (A, B, C and F), group 2 (D, E, G, I and J) and group 3 (H) and gave the explanation “I sorted my groups in this way by how easy, moderate or hard the questions were to work out”.

Hans who worked for about 19 minutes before coming up with a first grouping also proposed three categories: group 1 (C, D and I), group 2 (A, B and E) and group 3 (F, G, H and I), explaining the motivation for his grouping as follows: “because group 1 is ‘times question’, group 2 is questions you divide by and group 3 are add and multiply”. We cannot see any rationale in Hans’ grouping, nor a link between his actual grouping and the explanation he gave for it. After the researcher directed Hans to find a second set of groupings, Hans came with a categorization in four distinct groups: group 1 (A, E and H), group 2 (B and C), group 3 (I and J) and group 4 (D, F and G), but, once more, his justification remained unclear for the researcher.

John, who worked for about 17 minutes, found a classification into two different groups: group 1 (C, E, F, G, H and J) and group 2 (A, B, D and I). He rather superficially explained the motivation for his grouping as follows: “I made these groups because I think it was the most common way and I managed to make them into two groups without any left over”. After directed to find a second grouping, John proposed four categories: group 1 (E), group 2 (A, C, I and J), group 3 (B, G and F) and group 4 (D and H). He now explained: “I put them into groups of weight, time and number (respectively groups 2, 3 and 5) and I could not find a group for the ‘train and locomotive’ one (problem E)” (which is not in line with John’s actual grouping).

CONCLUDING REMARKS

The scale of the experiment was very small, so one can hardly infer definite conclusions from it. We observed that the four participating students showed great difficulties in making and motivating classifications of the ten word problems. They mainly looked for linguistic or other superficial differences between the problem formulations and not for an underlying mathematical structure. Possible explanations refer to the nature of the task and the type of problems that we used.

With respect to the task, one can argue that, these students were unfamiliar with classification tasks. Typically, students are expected to "solve" mathematical problems, i.e., to give numerical answers (most often based on the numbers given in the problem formulation), and not to classify problems. Moreover, in retrospect, we think the task was also rather difficult or too "abstract" for 11-year old students. A possible alternative approach meeting more or less the same goals would have been to ask students to combine different problem statements with correct and incorrect (proportional or non-proportional) solution strategies provided by the teacher or experimenter.

With respect to the problems we used, one can argue, in line with Ainley (2000) and several other authors, that the "word-problem" format is inadequate or insufficient to meaningfully contextualise mathematics in the mathematics classroom. Several authors (e.g. Reusser & Stebler, 1997) showed the beneficial effect of meaningful, authentic tasks also for problems where students inappropriately tend to apply linear methods. In this respect, Van Dooren, De Bock, Janssens and Verschaffel (2005) recently showed that students' problem-solving behavior strongly improves when non-linear problems are embedded in a meaningful, authentic context and students are invited to perform an authentic action with concrete materials (i.e. when students are invited to cover a dollhouse floor with "real" tiles instead of calculating this number of tiles in a word-problem context).

Notwithstanding these limitations and shortcomings and the rather disappointing results of our small-scale experiment, the various reactions of the four participating students also suggest that that this type of task design can be a rich starting point for significant classroom discussions on mathematical modelling: which operation is needed in a given problem situation?

DESIGNING INSTRUCTION ON PROPORTIONAL REASONING WITH AVERAGE SPEED

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Instructional design in Realistic Mathematics Education aims at both fostering student reasoning, and at putting instructional tasks in a perspective of long-term learning processes. We try to illustrate this with a task on reasoning about average speed.

TASK DESIGN

There is a long history of instructional design, within which instructional tasks were designed with a primary focus on behavioral objectives. Central instructional design strategies were task analysis and the construction of learning hierarchies. Lessons would be planned on the basis of well-defined prerequisites and precise lesson goals. Teachers were expected to evaluate each lesson by assessing whether those goals were reached at the end of the lesson.

Today this type of instruction is criticized as being ‘instructionist’ or as reflecting a ‘transmission model’ of teaching. In contrast to teachers *instructing*, the emphasis is now on students *constructing*. Following Cobb (1994) we may argue that constructivism—as an epistemology—does not have direct implications for teaching, as “the constructivist maxim about learning may be taken to imply that students construct their ways of knowing in even the most authoritarian of instructional situations” (Cobb, 1994, 4). Still, constructivism may inspire one to consider how we can influence the construction processes of the students. One of the results of such considerations is a shift in attention from behavioral objectives to the mental activities of the students. In this respect, we may refer to Simon’s (1995) notion of a hypothetical learning trajectory. We may notice the flexibility and the situatedness of this concept. A teacher will design a hypothetical learning trajectory for the students in his or her classroom, given where the students are at this moment, while taking into account goals and teaching practices. Moreover the teacher will adjust the hypothetical learning trajectory on the basis of his or her interpretation of how the students act and reason. This puts the notion of task design in a different perspective. What the task entails is not fixed, as tasks are interactively constituted in the classroom. When we expect teachers to orient themselves on the mental activities of the students, and consider those in relation to the intended end goals, we might argue that teachers should be supported in making these considerations.

In the Netherlands we constructed an instructional design strategy, which is aimed at developing prototypical instructional sequences and local instructional theories that are to offer teachers a framework of reference for constructing their own hypothetical learning trajectories. This strategy is based on what is called design research and on the use of three design heuristics from realistic mathematics education (RME), namely, guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling. In the following paragraphs we explain this in more detail.

Design research can be thought of as a combination of design and research aimed at developing both a sequence of instructional activities and a local instructional theory. A classroom teaching experiment forms the core element of this type of research (Gravemeijer, 1998). This consists of an interactive and cumulative process of designing and revising instructional activities. To this end, the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them. These instructional activities are tried out in the classroom. Then, new instructional activities are designed or redesigned on the basis of analyses of the actual learning processes. At the end of a cumulative process of designing and revising instructional activities, an improved version of the instructional sequence is constructed. After some design experiments, the rationale for the instructional sequence eventually acquires the status of a local instructional theory.

The other core element of our instructional design strategy is the use of the three design heuristics that characterize the domain-specific instruction theory of RME. This educational theory originated in the Netherlands inspired by Freudenthal's idea of mathematics as an activity of organizing or mathematizing. The first heuristic has to do with Freudenthal's (1973) idea that students should be given the opportunity to experience a process similar to the process by which mathematics was invented, and is called guided reinvention through progressive mathematization. According to this heuristic, the designer takes both the history of mathematics and the students' informal solution procedures as sources of inspiration (Streefland, 1990), and tries to formulate a provisional, potentially revisable learning route along which a process of collective reinvention (or progressive mathematization) might be supported.

The second heuristic concerns the phenomenology of mathematics, and asks for a didactical phenomenological analysis. The developer looks at present-day applications in order to find the phenomena and tasks that may create the need for students to develop the mathematical concept or tool we are aiming for. The goal of a phenomenological investigation is, in short, to find problem situations that may give rise to situation-specific solutions that can be taken as the basis for vertical mathematization.

In the instructional design we are reporting in this paper, the focus is on the emergent modeling heuristic (Gravemijer, 1999). Models in RME are related to the activity of modeling. This may involve making drawings, diagrams, or tables, or it can involve developing informal notations or using conventional mathematical notations. It is important that these notations have the context situation of the problem as starting point and are developed by the students as they attempt to come to grips with the problem and find ways to solve it. The conjecture is that the emergence of the model is reflexively related to the construction of some new mathematical reality by the students, which may be labeled as more formal mathematics. Initially, the models refer to concrete or paradigmatic situations, which are experientially real for the students, and are therefore to be understood as context-specific models. On this level, the model should allow for informal strategies that correspond with situated solution strategies. As the student gathers more experience with similar problems, the model gets a more object-like character, becoming gradually more important as a base for mathematical reasoning than as a way of representing a contextual problem. The *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning.

THE DESIGN TASK: (UN)JUSTIFIED PROPORTIONAL REASONING

In the context of the research forum, we were asked to design a single task on proportional reasoning, while also addressing the issue of unjustified proportional reasoning. We chose a task on speed. Reasoning about speed in everyday-life situations asks students to coordinate pure proportional reasoning with realistic considerations on what may distort the proportionality in actual reality. The task we designed was a problem about two girls who make a bicycle trip. After 1 1/2 hour they pass a signpost telling them that they have already cycled a distance of 30 kilometers, and they still have 45 kilometers to go. In the story one of them comments: 'This is going well', and the question the students have to answer is why she would say so. There were five more questions, but, in a sense, the first one covers them all; the other questions discuss the relevant points more explicitly. The remark 'This is going well' is expected to raise a discussion about questions like:

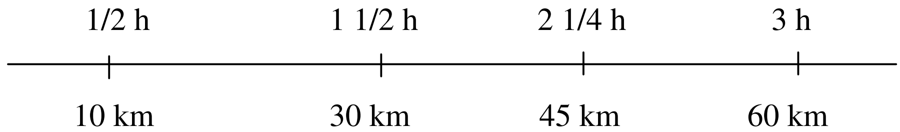
- Is 30 kilometers in one hour and a half an achievement one would be happy with? What would have been their speed, in terms of kilometers per hour, and would that be fast, or slow?
- The girl might be happy because she sees that they have done a big part of their trip already. So what is the relation between the 30 kilometers and the distance the girls still have to cycle? Would it be possible to estimate how much time they need for the rest of their trip?
- Will a calculation lead to an exact prediction, or are there other factors to take into account?

Note that the numbers were chosen carefully as to make easy computations. The task was tested both in the Netherlands and in the UK; the English version was about a trip from Corby to Cambridge, with 18 miles done and 30 miles still to go. Note also that there are various ways to calculate the time needed for the second part of the trip. Students can compare 30 km and 45 km and conclude that the second part will take 1 1/2 time as long, they might see that 30 km in 1 1/2 hour gives 10 km in half an hour and reason from this, or they might calculate the average speed in km per hour.

The student activities that we anticipate are threefold:

- The students will (start to) reason proportionally in the context of speed.
- The students' explanations will allow the teacher to start a discussion about how to record proportional reasoning on paper. This could be a lead in to a discussion about the use of models like the double number line or the ratio table.
- The students will realize that proportional reasoning does not predict the arrival time in a precise manner, but do realize that calculations are a useful tool in making estimations.

Models for proportional reasoning, and therefore also for reasoning with average speed, are the double number line and the ratio table. They both offer a systematic way of writing down the relation between distance and time. On the double number line the position of points is meaningful, whereas the columns of the ratio table can be in any order. Both models can function as a tool, allowing one to break down complicated calculations into intermediate steps.



time	1 1/2 h	1/2 h	2 hs	2 hs 15 min
distance	30 km	10 km	40 km	45 km

In our view students should be stimulated to reinvent these models; they should not be offered as a ready-made products. This does not mean that students are expected to reinvent the exact way numbers are written in rows and columns in the ratio table, but they should be stimulated to think about systematic forms of notations, and thereby learn to appreciate the 'official' ratio table as one of the possible forms.

Following the emergent modeling perspective, the students' activity with double number line and ratio table will be grounded initially in thinking about its contextual meaning. Doubling in the ratio table, for example, will be justified by thinking of traveling twice as long. Later the ratio table may be used for reasoning with linear relations. As we argued elsewhere, students may eventually start to use the ratio table in a semi-algorithmic manner to execute multiplications, without necessarily having to think of possible contextual meanings of the numbers involved (Gravemeijer, Boswinkel, Galen, & Heuvel-Panhuizen, 2004).

SOME FINDINGS

The task was tested twice, once with a small group of four students in England and once in a class with 10 to 12 year old students in the Netherlands. In the experiment in England the teacher introduces the problem by focusing heavily on exploring the situation and the circumstances that influence the time one needs to cycle from Corby to Cambridge. The situation is meaningful enough for the students to bring forward many aspects that could influence the cycling time. They mention that the time to travel the whole distance could be influenced by the weather, the hills alongside the route, the breaks the girls take, etcetera. In this setting the students developed ideas on how much time it takes to cycle the whole tour, but the numbers they bring forward are mostly guesses. They agree that it should take the children at least two hours to ride the 30 miles from the road sign to Cambridge. Only two students replace their guesses about the time needed to cycle from Corby to Cambridge by calculations and schemes.

The Dutch experiment also starts with an exploration of the context. As the students here are more familiar with a bike a means of transport, they easily bring forward what should be done if one undertakes a tour as mentioned in the task. When next the students receive the worksheet with the map and the road sign, they find little problem in interpreting the situation. The teacher here, like her English colleague, discusses one of the girls saying ‘This is going well’, when they arrive at the road sign.

In the Dutch version of the task in took the children one and a half hours to cover the first 30 kilometer. At that point there is still 45 kilometer to go. The students formulate several arguments why 30 kilometer in one and a half hour is quite a distance for such a short time.

The teacher frequently asks the students to explain their ideas. Therefore the discussion focuses more and more on mathematical arguments. One of the students for example claims that he cycles 3 kilometers in a quarter of an hour. He argues that in that speed it takes one and a half hours to cover 18 kilometers. 30 kilometer in one and a half hour therefore is fast cycling.

Unlike her English colleague, the Dutch teacher at certain points redirected the discussion to the use of mathematical arguments. The Dutch students therefore all reasoned in terms of ratios to calculate the arrival time. Moreover, the arrival time is next discussed in terms of the context, where the students decide to add about an hour for breaks, flat tires and weather conditions.

We were in the fortunate position to thus find two settings where the teachers both choose a different manner to guide the students. This enabled us to analyze the teacher’s role and to test (in this specific context) our ideas on this. We noticed that the Dutch students did not have any problem with putting their calculations into perspective. They could easily compute how much time would be needed for the next 45 km, but it was also obvious to them that such calculations only give you a first approximation. In the English experiment the students were aware that one could only estimate the arrival time, but the setting did not stimulate them to further mathematize the problem.

CONCLUSION

In Realistic Mathematics Education instructional design concerns series of tasks, embedded in a local instruction theory. This local instruction theory enables the teacher to adapt the task to the abilities and interests of the students, while maintaining the original end goals. The task we designed should be viewed from this perspective. In an educational setting it would not be an isolated task, but part of a longer learning route. Goals of such a learning route would be:

- Students learn to reason proportionally.
- They develop tools for proportional reasoning, tools that can also be used for calculations, like the double number line and the ratio table.
- At the same time, however, they learn to see the relativity of their calculations; when making predictions other factors in the context may have to be taken into consideration.

When our task was tested, the emphasis was on the third goal. Within a longer learning route, however, the challenge would be more to help students develop the right tools for proportional reasoning. Among other things, these tools would help children to discriminate between situations where proportional reasoning is, and is not justified. RME describes this process of developing mathematical tools as emergent modeling.

In the test situations there was no discussion, or only a limited discussion about tools like the double number line and the ratio table. Within design cycles of testing and revising this could lead to the decision to make certain changes, in this case, for example, to change the numbers in such a way that students would not be able to do the calculations in their heads. But even when an activity, after some revisions, has found its definite form, success cannot be guaranteed, of course. This underscores the central role of the teacher in supporting the learning process. The teacher should be capable to make changes, like asking certain questions, focusing the discussion on certain topics, and so on. An essential condition to establish this is, that the teacher knows and understands the local instruction theory behind the activities.

FOLDING PERIMETERS: DESIGNER CONCERNS AND STUDENT SOLUTIONS

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In this paper we first describe some of the concerns and approaches that have influenced the process of designing the *Folding Perimeters* activity. Then, we will present several selected episodes from the actual solutions produced by two pairs of

12-year-old, higher ability students, in view of the design concerns that were encountered in the development of this activity.

TASK CHARACTERISTICS

Folding Perimeters was designed as the last and most advanced activity in a learning series on ratio and proportion. This section describes the main characteristics of the activity, and some considerations that led to its present design.

Context. In this activity, students investigate the perimeters of an alternating sequence of squares and rectangles, during a process of repeated folding-in-two (Fig. 1). The use of context enables a constructivist path of learning (Hershkowitz et al., 2002). When students start with a problem situation such as the above, they can rely on their acquaintance with its non-mathematical components and on their ability to observe, to experiment and to act on the situation itself. As indicated by Ainley and Pratt in this collection of papers, the characteristics of a task may also contribute to provide a sense of purpose and ownership. Moreover, a problem situation can also contribute to students' understanding of the need for constructing appropriate tools and concepts, first investigating the problem at an intuitive level and later on, analysing the newly formed tools and concepts in a more extended and mathematically formal manner. Tourniaire and Pulos (1985), in reviewing the research on proportional reasoning, concluded that context plays a crucial role in student performance and that use of a wide variety of contexts is needed in the teaching of this domain. In our case, we considered the context of paper folding to be simple and familiar, on the one hand, and to be rich in mathematical opportunities on the other hand.

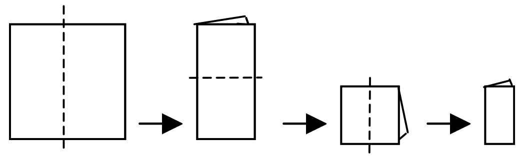

Figure 1. Context of *Folding Perimeters*

Mathematical content. The activity integrates various mathematical domains - for example, geometry (squares, rectangles, perimeters, opposite sides, measurement), arithmetic (numerical tables, operations, difference, ratio), and algebra (*Excel* formulas and pattern generalizations). The mathematical content is stated clearly throughout the activity, and is one of the factors that determine the sequence of tasks. The first three tasks in the activity require a more geometrical and visual investigation, there is a task that relates to the differences between the perimeters of two adjacent shapes, and the last two tasks focus respectively on the perimeter ratios of two adjacent, and of every other shape. However, some other tasks in the activity are less directive with regard to content or solution strategy open. More specifically, these tasks require students to find any patterns of perimeter change and justify them. Similarly to Dirk, Wim and Lieven's task, the patterns of change in our activity do not constitute a classical and straightforward application of the idea of proportionality, common in many textbooks.

Multiple representations. The presentation of mathematical concepts and operations in various representations is central in investigative activities (Friedlander & Tabach, 2001a). One of our reasons for using spreadsheets as a mathematical tool is their ability to simultaneously support work on various representations, and to present the algebraic representation as an efficient and meaningful means of constructing data. In our activity, students are specifically required to present perimeters and perimeter changes in actual paper, in drawing, in numerical tables, as algebraic formulas, in bar diagrams, and in verbal descriptions. Some of the tasks focus on the construction and use of a specific representation, whereas others leave this issue open to the students. Figure 2 presents a numerical and graphical representation of the data and some of the results obtained by the observed students, regarding the alternating sequence of shapes in the activity. Some of the algebraic formulas used by the observed students will be discussed in the next section.

	A	B	C	D	E	F
1	fold #	side	side	perim.	differ.	ratio
2	0	32	32	128	32	3/4
3	1	32	16	96	32	2/3
4	2	16	16	64	16	3/4
5	3	16	8	48	16	2/3
6	4	8	8	32	8	3/4
7	5	8	4	24	8	2/3
8	6	4	4	16	4	3/4
9	7	4	2	12	4	2/3
10	8	2	2	8	2	3/4
11	9	2	1	6	2	2/3
12	10	1	1	4		

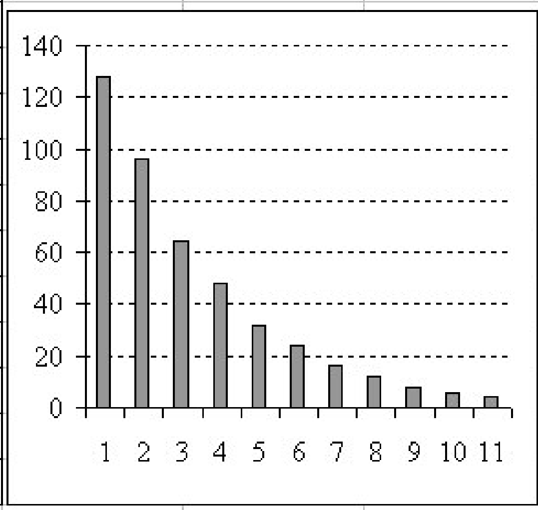

Figure 2. Spreadsheet representation of data and results in *Folding Perimeters*.

Task sequencing. Investigative activities (including *Folding Perimeters*) frequently follow a flow pattern that is in many ways similar to the PCAIC investigative cycle (*pose*, *collect*, *analyze*, *interpret, and communicate*) proposed by Kader & Perry (1994). This cycle is adapted from the domains of data investigation and scientific research, and is inductive in nature. First, specific cases are collected, organized, and analyzed, and then general patterns are formed and conclusions are drawn, interpreted and applied.

Generalization of patterns. Many activities associated with generalization – including ours, assume that the process of pattern generalization is inductive and based on a limited number of cases. In the next step, the discovered pattern is explained and justified (Friedlander et al., 1989). This flow pattern is frequently used in the design of generalization tasks. In our activity, this sequence of tasks is applied in several cycles, with regard to any patterns of perimeter change, then regarding the difference, and finally regarding the ratio of perimeters of two consecutive shapes.

Level of task openness. The process of task design is based on a constant state of tension that exists between the design of unstructured open tasks that do not require that the problem posed be solved by a specific method, a certain representation or an implicitly given sequence of steps, as opposed to a structured approach that poses specific requests with regard to the variables mentioned above. The open approach reflects the designers' striving to develop problem solving skills, to develop creative mathematical thinking, to provide opportunities for students to actually experience investigation, and to achieve a meaningful construction of knowledge. The structured approach enables students to pursue a more predictable and planned agenda in the domains of mathematical content and the processes of problem solving. The activity discussed here addresses this issue by presenting a sequence of tasks of both kinds. Open tasks require students to identify any properties of the presented sequence of shapes, make predictions, and then look for patterns that describe the change in perimeter. Tasks that are more directive require the student to collect data for the first ten shapes in the sequence, organize it in a spreadsheet table, present it as a diagram, investigate patterns of perimeter change by considering first the difference and then the ratio between pairs of adjacent shapes, and of shapes placed in the sequence at a distance of two steps. One may argue that leading students through a sequence of tasks, rather than presenting only a problem situation and a "big question", decreases in itself the extent of freedom in student work. We suggest, "walking a fine line" between opening and closing a task by directing students to some extent through a sequence of leading questions, within an open problem situation. This approach to task design supports a convergence towards a meaningful progress in the students' solution, without curtailing their sense of ownership of the task (in the sense of Ainley and Pratt in this collection of papers). Such a sense of ownership stems from the opportunity to observe, experiment and act on a "realistic" situation, and not necessarily from the task's degrees of freedom.

Verbalization. Requests for descriptions of patterns, explanations, discussions of another (fictitious) student's solutions and reports of results are included in this, as well as many other activities. These requests are the result of designers' desire to develop communication and documentation skills, to make students consider verbal descriptions as mathematical representation, and to change the stereotypic view of mathematics as the exclusive domain of numerical and algebraic symbols only.

Use of spreadsheets. Our experience of students working in a spreadsheet environment shows that spreadsheets can serve as a powerful tool, and allow for some of the design heuristics proposed by Gravemeijer and his colleagues in this collection of papers. They support students' processes of creating emergent models and their "vertical mathematization" of the problem situation. The use of this technological tool to support and promote processes of generalization and algebraic thinking has been amply discussed in terms of theory and investigated empirically (for design considerations in spreadsheet activities, see for example, Hershkowitz et al., 2002; Friedlander & Tabach, 2001b). Because of space limitations, we will only briefly list the following considerations that led the designers to use spreadsheets in this particular activity:

- they serve as a powerful tool for data collection, organization and representation,
- they provide continuous and non-judgmental feedback throughout the solution process,
- they present the concept of proportion dynamically, as a sequence of constant ratios obtained by applying the same rule to numerous pairs of numbers or quantities,
- they enable the analysis of an extended collection of data,
- they emphasize the meta-cognitive skills of monitoring and interpreting results,
- they promote algebraic thinking and present algebraic formulas as a useful and meaningful tool.

STUDENT SOLUTIONS

As previously mentioned, two pairs of students (referred here by the initials of their first names as MS and MG) were observed by one of the authors as they worked on the *Folding Perimeter* activity, during a period of about 80 minutes for each pair. For the purpose of this paper, we will not distinguish between the two members of a pair, and will refer to each pair as an entity. The students had previous experience in using *Excel* in mathematical investigations, but had not pursued the learning sequence of ratio and proportion that included our activity. The interviewer's interventions were minimal and limited to occasional requests to clarify answers or to start working on the next item. The latter case included dealing with "unproductive" paths of solution – defined by Sutherland et al. (2004) as cases of "construction of idiosyncratic knowledge that is at odds with intended learning", and require the teacher's intervention in regular classroom situations. A systematic analysis of student work, according to the eight designer concerns described in the previous section is not possible, because of the space limitation.

In general, the students followed the prescribed sequence of tasks and solved them in a mathematically rich and resourceful manner. However, we will focus here on some differences between the observed students' solution processes and the designers' plans and predictions.

Contrary to our expectations (see the comments on task sequencing and generalization of patterns in the previous section), both pairs reached, at the initial stage of predictions, generalizations that were "scheduled" by the designers to be reached only later on, and on the basis of the collected data. By examining their folded paper square and the drawing of the folding process (Fig. 1), the students considered visual and global aspects regarding the sides that were "lost" through folding, and made the following predictions:

MG: It [the perimeter] gets smaller by the length of the side that gets halved.

MS: In my opinion it [the perimeter] will be 3/4. The vertical lines will stay and the horizontal lines lose one half and one half – and that's a whole side. [After Interviewer asks "And what happens from the second to the third shape?"] It comes out 4/6 because we are left with 4 out of 6 halves [of the longer sides of the rectangle].

Both pairs produced general patterns at a very early stage of the activity - MG is reasoning additively, by looking at differences, whereas MS is thinking proportionally, by considering ratios. The issue of interest for designers and/or researchers is that the processes of pattern generalization can follow two routes:

- inductive generalization based on the collection and analysis of data (as followed by the sequence of tasks in this activity),
- deductive generalization based on a global analysis of the problem situation, and on general reasoning (as followed by the two pairs of students).

We assume that both the students' mathematical ability and task design (e.g., the representation used in the initial description of the problem situation) affect the choice of the route.

The use of spreadsheets was also a source of unexpected developments. The observed students did not encounter any technical difficulties with regard to the handling of the tool. They read, understood, and performed the computer-related instructions, and were familiar with the *Excel* syntax for writing formulas. However, the following three episodes observed during the students' work indicate that the spreadsheet's intrinsic properties can provide opportunities for higher-level thinking, and help both the student and the teacher detect and relate to conceptual difficulties.

a) MS: They construct the spreadsheet table for the first ten shapes (see Fig. 2). They write in the first line of the perimeter column (for the perimeter of the original square) the formula =4*B2 and in the next line (for the perimeter of the rectangle produced by the first folding) =2*B3+2*C3.

"But we can't drag down [two formulas]…Then let's change this [the first formula] into this [the second]". They rewrite the formula for the square as =2*B2+2*C2 and drag it down.

b) MG: They write for the length of sides (see Fig. 2) a formula (pattern) indicating the halving of the above-situated cell, and drag it down cell by cell – one cell at a time, hoping that this method would produce the desired sequence of pairs of identical numbers.

c) MG: They construct the column for the difference of adjacent perimeters (see Fig. 2) by writing in the first line the formula =D2-D3 and dragging it down to the last line. As a result, the last number shows an uncharacteristic increment in the difference sequence (….8, 8, 4, 4, 2, 2, 4) – a result of the difference of the last perimeter (4) and the next empty cell that is interpreted by Excel as zero. They notice the outcome, retype the same pattern (=D3-D4) in the second line and again drag it down to the last line –obtaining of course the same results as before.

In episode (a), work on *Excel* provided an opportunity to perform a higher-level analysis for students without any background in formal algebra: they compared two algebraic expressions and identified one (4B) as a particular case of the other (2B+2C, when B=C). However, episodes (b) and (c) showed that the observed pair of students thought that changing the place or the physical handling of a pattern expressed as an algebraic formula will change its essence.

CONCLUSIONS

The considerations related to the design of the *Folding Perimeter* activity are closely connected to a wide variety of theories and research findings on student cognition, and on the use of technological tools for teaching mathematics. Our experience in implementing many similarly structured investigative activities indicates that they provide opportunities for meaningful learning of mathematical concepts.

We also described here several episodes of student work on a particular activity to show that differences between a designer's planned actions and student work should be expected. Whether, and if so how, these differences should influence the design of this particular activity or the principles of task design remains an open question.

THE DOLLS' HOUSE CLASSROOM

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The design of our task uses the framework of *purpose and utility* (Ainley & Pratt, 2002, Ainley et al., forthcoming). *Purpose* reflects our concern to create tasks which are meaningful for pupils. One strand of research on which we draw is that of mathematics in out-of-school contexts (e.g., Nunes et al., 1993) which has highlighted the contrast between the levels of engagement of learners in mathematical activities in and out of school. In a PME plenary, Schliemann (1995) claimed '*we need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers. And such situations may be very different from everyday situations.*' (p. 57). We argue that setting school tasks in the context of 'real world' situations, for example through the use of word problems, is not sufficient to make them meaningful for pupils. Indeed, there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide a purpose that makes sense to pupils (see for example Ainley, 2000; Cooper & Dunne, 2000).

We see the *purposeful* nature of the activity as a key feature of out-of-school contexts which can be brought into the classroom through the creation of well designed tasks. Drawing partly on constructionism (Harel & Papert, 1991), we define a purposeful task as one which has a meaningful outcome for the learner in terms of an actual or virtual product, the solution of an engaging problem, or an argument or justification for a point of view (Ainley & Pratt, 2002; Ainley et al., forthcoming). This feature of purpose for the learner, *within the classroom environment*, is a key principle informing our pedagogic task design.

The purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. The purpose of a task is not the 'target knowledge' within a didactical situation in Brousseau's (1997) sense. Indeed it may be completely unconnected with the target knowledge. However, the purpose creates the necessity for the learner to use the target knowledge in order to complete the task, whether this involves using existing knowledge in a particular way, or constructing new meanings through working on the task. Movement towards satisfactory completion of the task provides feedback about the learner's progress, rather than this being judged solely by the teacher (Ainley et al., forthcoming). Harel (1998) proposes the 'necessity principle', which addresses the issue of creating the need to learn particular things in a different way. In Harel's terms an 'intellectual need' for a mathematical concept should be created before embarking on the teaching of the concept. However, intellectual need and purpose clearly differ, since intellectual need is related specifically to a mathematical concept, while the purpose of a task is not explicitly mathematical, but relates to the outcome of the specific task. The necessity principle perhaps relates more closely to the second construct within our framework: utility.

UTILITY

Understanding the *utility* of a mathematical idea is defined as knowing how, when and why that idea is useful. A purposeful task creates the need to use a particular mathematical idea in order reach the conclusion of the task. Because the mathematics is being used in a purposeful way, pupils have the opportunity not just to understand concepts and procedures, but also to appreciate how and why the mathematics is useful. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings. In contrast, within school mathematics ideas are frequently learnt in contexts where they are divorced from aspects of utility, which may lead to significantly impoverished learning. Utility thus has some similarity to Harel's 'intellectual need'. However, Harel sees intellectual need as providing the motivation for learning a concept, whereas utility, why and how the concept is useful, is seen as an intrinsic, but frequently unacknowledged, facet of the concept itself.

THE DOLLS' HOUSE CLASSROOM TASK

The Dolls' House Classroom task focuses on scaling, which is a key idea in proportional reasoning. The outcome of the task is a set of instructions for another group of children to make items for the dolls' house classroom. The purposeful nature of the task would, of course, be increased if the pupils were involved in the actual manufacture of the product. We developed the idea for this task from the work of a primary school class who used a similar approach to building scenery for a play based on the Nutcracker ballet. There was a need to make the scenery large enough for the people to appear the size of rats.

At the beginning of the task, each group of pupils is given an item from a dolls' house which corresponds to something they will have in their own classroom (e.g., a chair, a table, a door, a computer). The activity of comparing this with its full-size equivalent will involve measuring and discussion, as pupils decide on which are the most important measurements to use. For example, although the particular design of chairs may vary, the height of the seat above the ground remains fairly constant.

Once they have arrived at a pair of measurements for the full-size and dolls' house items, they enter the most crucial part of the task: deciding how the use these in order to scale other measurements. The role of the spreadsheet is important here in allowing pupils to experiment with different ways of using the measurements, and applying them to other items which they decide to include. It is important that there is an opportunity here for the pupils to make decisions about which other classroom items they will use, as this adds to their ownership of the task. We note here a close affinity with Friedlander and Arcavi, who set out in this collection of papers some of the reasons why they also adopted spreadsheets.

The above considerations reflect our practical research and teaching experience as well as our theoretical perspective. In order to illustrate some of the characteristic features of such a design approach in action, we gave the dolls' house task to two pairs of eleven year old students (one pairs of boys and one of girls). It turned out that the girls needed considerably more support than the boys from the teacher/researcher. Interestingly, this had the effect of closing down the task for the girls, who followed a much more one-dimensional route through the problem, staying close to the suggestions of the teacher. In contrast the boys were more adventurous in their approach and were able to exploit the opportunities that the task offered. This contrast acts as a useful reminder that the notions of purpose and utility are design imperatives, which act as potentials for the students but how those potentials are realised will vary according to a range of personal attributes (knowledge, confidence and so on) brought to the situation by the children and the structuring resources of the setting, including *inter alia* the approach of the teacher. (Indeed, we note that all authors in this collection of papers found to a greater or smaller extent that there were discrepancies between the learning trajectory that they had envisaged and that which ensued in practice. We make further comment on this at the end of this section.) As a result of this contrast between the boys and the girls, we focus below more on the activity of the boys, which better illustrates the implications of designing for purpose and utility.

PURPOSE AND UTILITY IN ACTION

We were struck by the relationship between the boys' construction of purpose and utility and how the interplay between the two evolved during the 40 minute session. Initially the boys tried to relate the task to their own experiences. One boy told the teacher about how his grandfather used to make dolls' furniture. The other talked about scaling in maps in response to the teacher's mentioning of the term *scale factor*. From an early stage, the boys questioned the nature of the task that they had been set. (Figures in brackets indicate time elapsed in minutes.)

> [6:06] Is this real? Are a Year 6 class really going to do this?

The researcher admitted that this was not actually going to happen.

> [6:35] Why can't they just buy the dolls' house?

What do we make of these questions? Are they challenges that suggest the boys are resisting the invitation of the teacher to engage with the problem? If so, it would be hard to explain the subsequent activity, which was marked by the boys' considerable intent and persistence. Rather, we believe that these questions indicate a process in which the boys were beginning to take ownership of the task, They were, in our opinion, delimiting the task, asking where are its boundaries with reality, recognising that is was important to appreciate the true nature of the task as this would later inform their strategies for its solution.

The task itself continued to act as the arbitrator of the activity (in contrast, the girls required the teacher to direct their activity throughout the session). At one point one of the boys encouraged his partner to move on.

> [17:14] You can't just keep doing the table; we've got to do something else.

The boys recognised that there was an implication in the task to build a range of artefacts. It was not necessary to ask the teacher what they should do next.

At times, the boys were even prepared to follow the path indicated to them by the task rather than that suggested by the teacher. Thus, at one point, the teacher asked how the boys would find the height of the little shelf for the dolls' house.

> [13:40] Before we do that, won't we have to do the width of this table first?

When students take ownership of a task, the levels of engagement can be very high; it is our belief that the opportunity to make choices is influential in helping students to make a problem their own. Furthermore, a well-designed task will also enable students to follow up their own personal conjectures when they try to make sense of the task. Such personal conjectures might be seen by other researchers as misconceptions but our stance recognises the need, from the design point of view, for students to be given the opportunity to test out for explanatory power their own meanings, in this case for proportion. Thus the boys' spreadsheet shows several different attempts at ratio. In one set of cells, they divided the height of the real table by that of the supplied dolls' table (68.5 / 4.3 = 15.93). But when it came to the width of the table, they divided the dolls' table by the real table (5.5 / 134.2 = 0.040983607). In another part of the spreadsheet, they divided the real shelf width by the real table width (75.5 / 134.2 = 0.562593). Each of these calculations has possible utility for their task but whether any particular approach has explanatory power depends on exactly how the boys wanted to use the result and what sense they could make of the feedback. The nature of the task allowed them to explore all three routes, rather than following a route defined prescriptively by the teacher.

Such explorations enabled the boys to construct meanings for the divisions being carried out on the spreadsheet. The spreadsheet handled the calculations, allowing the boys to focus on whether the ratio was actually useful to them in their task. Even so, the technical demands of deciding what to divide by what could become so absorbing that the context could be temporarily forgotten.

> [13:20] So, this table [pause] the height of this table divided by the height of that table [pause] I've forgotten how this is going to help!

Nevertheless, the boys recognised that there was a purpose to this technical effort and they were eventually able to reconstruct the reason behind that work. We see this statement and the subsequent activity as evidence that the boys were indeed linking the purpose of the task to a utility for comparing dimensions. The measurements enabled them to derive a scale factor, which could be used to calculate the dimensions of imaginary objects. The utility emphasises how the scale factor might be useful, admittedly in a situated narrative, rather than the technical aspects of how to calculate a scale factor.

This utility was planned. However, when we design for purpose and utility, there is a strong likelihood of other utilities emerging in unpredictable ways. In well-designed tasks there should be a richness of possibilities. When we listened to the recording of the boys working on this task, we were able to identify unplanned opportunities to focus on a utility for rounding. Thus, consider again the occasion when the boys divided the width of the dolls' table by the real table to obtain 0.040983607.

> [17:40] How do you shorten that down?

The boys intuitively knew that it would be useful to reduce the length of the decimal. However, they did not know the technicalities of how to do this. Had the teacher been available at that point, there may have been an opportunity to focus on rounding in the context of making numbers more manageable. In the event the boys moved away from this calculation and considered an alternative approach. Nearly ten minutes later [26:50], another rounding opportunity appeared. On this occasion the numbers were easier and so the boys were able to round manually 8.0665 to 8.1.

Another illustration of the richness of such tasks occurred when the boys were considering the area of the tables.

[13:50] We have to find the area of that *(referring to the dolls' table)* and then the area of one of these tables and then combine the area of…

One of the most difficult ideas in secondary level work on proportion is the notion of an area scale factor and how it relates to a linear scale factor. There was potential here for the students to explore the utility of area scale factors.

FINAL COMMENTS

We advocate stressing in task design how mathematical concepts might be useful in particular situations. Such utility does not imply real world relevance. The dolls' house task is somewhat contrived if judged against such a criterion. Nevertheless, the boys took ownership of the task, partly because they were able to make choices of their own and partly because they were able to construct their own narrative for the task. As the activity evolved, the emphasis on making sense of the task itself by relating it to personal experiences and testing its boundaries transformed into creating solution strategies, guided by the purpose of task. In their efforts to construct meanings for the feedback from the spreadsheet, the boys constructed a utility for scale factor. At the same time, there was a richness in the task that is typical in our experience of tasks designed according to the constructs of purpose and utility. This richness manifested itself in the way that the boys followed numerous paths and stumbled into situations that offered potential for engagement with other mathematical utilities.

We note with interest that all the authors in this collection of papers appear to have attempted to include some aspect of purpose or utility in their task designs, without of course seeing what they did in precisely those terms. Word problems in themselves can appear dry, even hackneyed, but in Dirk, Wim and Lieven's task, the problem was transformed. The children had to work on the word problems at a meta level, deciding which problems were like which others. As De Bock, Van Dooren and Verschaffel subsequently observed, the task proved to be rather challenging but we too have seen in the past that this type of transformation can imbue a sense of purpose to the task for many children. In Koeno, Frans and Ronald's task, there was an attempt to connect children's thinking to their experiences of journeys. The approach seemed to offer the children the opportunity to construct a utility for proportion in relation to planning such journeys. In Alex and Abraham's task, we saw the potential for practical activity, which might even have been opened up further by considering other aspects of paper folding that can lead to other interesting proportions.

Finally, and almost as a cautionary tale, we remind you (and ourselves) that the girls working on our own task went down a much narrower predictable pathway than did the boys. One level of response to this result is simply to argue that no task can offer rich pathways for all children. On the other hand, perhaps there are lessons to be learned, not just from the boys' work, but also from that of the girls. Gravemeijer, van Galen and Keijzer have explained how they see the demands of this research forum as at variance to some extent with their normal activity. The principle of progressive mathematization, utilised by designers in the Realistic Mathematics Education school, is not one that sits easily with designing a single task in one shot. We too see task design in terms of design research and, in that spirit, would interpret all these efforts at task design as "bootstrapping" or first exploratory attempts.

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