REFERENTIAL AND SYNTACTIC APPROACHES TO PROOF: CASE STUDIES FROM A TRANSITION COURSE

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This paper aims to increase our understanding of different approaches to proving. We present two case studies from an interview-based project in which students were asked to attempt proof-related tasks. The first student consistently took a referential approach, instantiating referents of the mathematical statements and using these to guide his reasoning. The second consistently took a syntactic approach, working with definitions and proof structures without reference to instantiations. Both made good progress on the tasks, but they exhibited different strengths and experienced different difficulties, which we consider in detail.

INTRODUCTION

Writing proofs in advanced mathematics requires the correct use of formal definitions and logical reasoning. However, both mathematicians and mathematics educators have argued that intuitive representations are also necessary for reasoning to be effective (Fischbein, 1982; Thurston, 1994; Weber & Alcock, 2004). This paper highlights the fact that definitions and formal statements can be treated as strings of symbols that may be manipulated according to well-defined rules, or as formal characterizations of meaningful objects and relationships between these, and that either treatment can be the basis for productive reasoning. It is related to the work of Pinto and Tall (1999), who argue that one can extract meaning from a definition by logical deduction, or give meaning to it by refining existing mental images. We say a proof attempt is *referential* if the prover uses (particular or generic) instantiation(s) of the referent object(s) of the statement to guide his or her formal inferences. We will speak of a proof attempt as *syntactic* if it is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way.

We report two case studies from a project designed to investigate whether students think about the referents of mathematical statements while attempting proofs. In one case the student produces proofs referentially and in the other, syntactically. The specific purposes of examining the case studies are: 1) to show that students in transition-to-proof courses can take two qualitatively different approaches to proof writing, 2) to demonstrate that students taking each approach can be at least somewhat successful in writing proofs, and 3) to highlight what particular difficulties students have when using each approach.

RESEARCH CONTEXT

In this exploratory study, eleven students were interviewed individually at the end of a course entitled "Introduction to Mathematical Reasoning", the aim of which is to

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facilitate students' transition from calculation-oriented mathematics to more abstract, proof-based mathematics. It is designed to provide exposure to techniques of mathematical proof, as well as to content on logic, sets, relations, functions, and some elementary group, number and graph theory. The study aimed to 1) investigate the degree to which students at this level tended to instantiate mathematical objects while working on proof-oriented tasks, 2) discern any possible correlation between such a tendency and success at this level, and 3) identify purposes for which students used their instantiations. The participants were asked to complete three tasks, two of which involved producing proofs and one of which involved explaining and illustrating a provided proof. They were then asked to reflect upon their usual practices when trying to produce and read proofs.

This paper will exhibit data from the proof production tasks. These were presented to the students in written form, and are reproduced below.

Relation task

> Let D be a set. Define a relation ~ on functions with domain D as follows.
>
> $f\sim g$ if and only if there exists x in D such that $f(x) = g(x)$.

Function task

> Definitions: A function f:$\mathbf{R}\rightarrow\mathbf{R}$ is said to be **increasing** if and only if for all $x, y\in\mathbf{R}$, ($x>y$ implies $f(x)>f(y)$). A function f:$\mathbf{R}\rightarrow\mathbf{R}$ is said to **have a global maximum at a real number c** if and only if, for all $x\in\mathbf{R}$($x\neq c$ implies $f(x) < f(c)$.)
>
> Suppose f is an increasing function. Prove that there is no real number c that is a global maximum for f.

The participants were presented with these tasks one at a time on separate sheets of paper, and were asked to describe what they were thinking about as they attempted to answer. They worked without assistance from the interviewer until they either completed the task to their own satisfaction or became stuck. At this point the interviewer asked them about why they had taken specific actions and/or about why they now found it difficult to proceed. These questions focused on the student's choice of actions and conceptions of their own difficulties rather than on conceptual understanding or logical reasoning *per se*.

The interviews were transcribed, and the authors independently identified episodes in which the student used an instantiation and characterized the purpose for which this was used. It became clear that some students took a consistently referential approach, always instantiating in response to a question, and other students took a consistently syntactic approach, almost never instantiating. This was particularly evident in the more successful students. This paper will focus on two students, Brad and Carla, both of whom obtained A's on their midterm examination, and made substantial progress on the tasks in these interviews. Brad instantiated in response to all of the interview tasks. In contrast, Carla never did so. Since they were both articulate in reflecting upon their own strategies, they provide good material for us to see how each approach has distinct advantages and disadvantages. It is worth noting that Brad and Carla had attended the same class and so had been exposed to the same lectures, the same homework assignments etc.

REFERENTIAL APPROACH: BRAD

Response to relation task

Brad read the relation task and, after an initial comment that he was "trying to think what the question's asking", he announced,

> B: Alright, I'm just going to like write out some examples. To try and…like, set a *D*. And then…yes, write out a function or two. I don't know if that's going to help me.

He wrote the following on his paper:

$D=\{1,3,5\}$ $\quad f(x)=x^2$ $\quad g(x)=x$

He then said,

> B: Would this be an example? Like where *f* of *x* is equal to 1, and *g* of *x* is equal to 1…and since *x* is 1, like 1 is in the domain, *f* is related to *g*?

He went on to recall that an equivalence relation should be reflexive, symmetric and transitive. In reasoning about reflexivity and symmetry he spoke about *f* and *g* as though these stood for general functions, but referred back to his instantiation in which $f(1)=g(1)$ as if to confirm his thinking.

> B: So…so okay if it's reflexive, then…*f* of *x* should be equal to *f* of *x*. Or there should be *x* in *D* with, so that *f* of *x* is equal to *f* of *x*. Okay. That's all I'm going to say! *Laughs.* And…that's true. Because 1 is equal to 1. Symmetric, is um…*x* – *f* implies – *f* is related to *g* implies *g* is related to *f*. So…so this is really the *g*, there's an *x* in D such that *f* of *x* is equal to *g* of *x*. *g* is related to *x* – ah, *f*, when there's an *x* in D such that *g* of *x* is related to *f* of *x*. *Pause*. So…*writing*…implies that *g* of…*writing*…yes. Because if…because *x* one, *f* of *x* is equal to *g* of *x*, then the same *x* in D that *g* of *x* must be equal to *f* of *x*.

In reasoning about transitivity, he no longer referred to his instantiation, and made an error based on implicitly assuming that the value of *x* for which $f(x)=g(x)$ is the same as the value for which $g=h$.

> B: And then transitive. *f*, *g*, and *g* is related to some *h*, then *f* is related to *h*. So *f* is related to *g* is…*x* in *D* such that *f* of *x* is equal to *g* of *x*. And *g* related to *h* is there's an *x* in *D* such that *g* of *x* is related to…is equal to *h* of *x*. So then…*x* is in *D* in both cases. And if *x* is equal to *g* of *x* and *g* of *x* is equal to *h* of *x*, *f* of *x* must be equal to *h* of *x*.

The interviewer did not attempt to correct Brad's answer, but instead asked him what role his example had played for him. Brad said,

> B: Um, I guess it just…gives you something concrete […] because this is really general. And you can't really put your hands on this. You know I can't like, get a grasp of it.

It appears important to Brad to feel that he can "grasp" the concepts in the question, and he seemed to achieve this to his own satisfaction. However, he did not maintain explicit links between this example and the general argument, and did not spot his own error in this case.

Response to function task

Brad's response to the function task began in a similar way. He again commented that he was trying to understand the question, and stated:

> B: And I'm going to take an example to make sure I'm doing it right.

He wrote the following, along with a small sketch graph of $f(x)=x$:

$f(x)=x$ $\quad$ $x=2$ $f(2)=2$ $\quad$ $y=3$ $f(3)=3$

After overcoming some confusion caused by the fact that the notation was not used in the standard $y=f(x)$ format, Brad suggested a proof tactic.

> B: …I think we can do this by contradiction. Assume that…assume that um…if f is an increasing function then c…ah…then there is…a c? For which there is a max. And then prove that that can't happen. And then, so that'll prove it.

He began to work on this idea, but without a very good command of how the variables could be set up to make an argument on this basis.

> B: Alright so, if there is…a global max…writing, mumbling…f of c is greater than both f of x and f of y.

After some struggle, he considered a graphical instantiation:

> B: I'm just trying to see it by looking at the graph. How I can relate it. Like, the two terms interrelate. Why…because I can't even see – I want to know why, there can't be one […] like know why it can't be and then try to prove.

When the interviewer asked him to talk through his thinking, he said,

> B: Alright. I'm thinking that in the definition of increasing, there's never going to be one number that's the greatest. There's always going to be like, a number greater than x. Because it's, because it's increasing. So there's always going to be some number greater than the last. So if x is greater than – that's what I assumed here. x is greater than y, then there's going to be some x plus 1, that is going to be greater than y plus 1, so that f of x plus 1 is going to be greater than f of y plus 1. Or something like that. Where like, it's just going to change…. So then, there can't be some number, you know that…if it's increasing there can't be some number that's greater than all of them. Or, some f of c.

In our view Brad seemed to have a reasonable idea that for any number in the domain, one can always take a greater number, whose image under the function will be greater than that of the original. However, he did not have good control over the way in which the definitions, and in particular the variables x,y and c, could be used to express this argument.

In reflecting upon the function task, the interviewer commented that Brad had spent quite a lot of time thinking at the beginning before writing anything, and asked him what he was thinking in that time. Brad once again indicated that he was using examples to grasp the concepts.

> B: …I didn't, I never heard of a global maximum. I don't think we learned about increasing, but I'm not sure. I don't remember learning about it. So I wanted to teach it to myself first. And, I want to teach myself by examples, you know. And I was kind of starting to understand a bit more when I was trying to, in trying to grasp – I grasped increasing, it seemed like, okay. But then I was trying to grasp the global max.

The interviewer then asked what happened when Brad stopped thinking about examples and wrote "if f is an increasing function". Brad replied,

> B: …it doesn't tell you, proof by induction or proof by contradiction, and so…I'm just trying to think of a way that I *can* prove it. Like, take what's here and then prove it. So then, and then I was just going to write down what, a claim or like what we knew.

Summary

It seems that Brad used examples at the following junctures in his work:

1. To initially understand or grasp the concepts in a given question.
2. To decide on a type of proof to use.
3. To fall back to for more ideas when stuck.

This referential approach served him reasonably well in these respects, affording him a sense of understanding and an ability to decide how to proceed. What it did not seem to afford him was the ability to use this insight to write a full and correct general argument. He did not seem to use his examples to effectively guide his manipulation of the symbolic notation at the detailed level. In fact, his reflective comments on his proof-writing strategies suggest that he was not trying to do this, relying on his knowledge of standard types of proof to provide this structure:

> B: …I start out by forming an example to, you know, get a strong grasp of what they're asking me. And then, ah, probably play around with like, maybe do a few examples, so I can see what it's – actually maybe how I could prove it, which method of proof I should use. And then once I find a method, proceed from there […] because it seems like in all the different types of proofs we've done, there's always some kind of structure. […] Then you can structure it the way you've normally done it before.

SYNTACTIC APPROACH: CARLA

Response to relation task

Carla responded quickly to the relation task. She listed the properties of an equivalence relation, and went on to draw a conclusion.

C: Oh…okay. It's transitive, symmetric, and reflexive. *Writing*. So to prove that it's transitive…um…*pause*…if x is in D, f of x is…equal to g of x. f of x is equal to f of x, so f is related to g. So it's reflexive…um…symmetric is…if f is related to g, then…f of x is equal to g of x, so g is related to f as well…so…symmetric. And transitive is…f is related to g, that means f of x is equal to g of x, and g is related to…a I guess…so g of x is equal to a of x. So it's transitive as well. So…yes. It's an equivalence relation.

She made an error similar to Brad's by not giving due consideration to the existential quantifier. The interviewer then asked whether she would write anything else if she were going to hand this in for homework. Carla said yes and elected to provide an answer for symmetric. She wrote:

<u>Symmetric</u> YES if f~g, then $f(x)=g(x)$

if $f(x)=g(x)$, then $g(x)=f(x)$

thus g~f, so if f~g, then g~f thus it is symmetric.

As in Brad's case, Carla did not spot her own error.

Response to function task

Carla's response to the function task began in a similar way, with reading of the question followed by immediate writing.

C: So…I'm thinking the way to prove this is using contradiction. So, I would start out by assuming…there exists…a c…for which…f of x is less than f of c, when x is not equal to c. Okay. *Pause*. So now I'm trying to use the definition of increasing function to prove that, this cannot be. Um…so there exists a real number for which f of x is less than f of c, for all x…and there's…f…is an increasing function…for…all x…y in R, x greater than y implies f of x greater than f of y. Mm…*pause*…I guess what I'm trying to show is if x is in reals, and they are infinite…for all x…there will be…some function f of c greater than f of x. *Long pause*. So…there exists…an element…in R…greater than c. Um…for x…because…f is an increasing function…f of x will be greater than f of c. Um…a contradiction…so that…there is no c for which f of c is greater than f of x…for all x.

Despite successfully producing a proof, she commented that "it seems a bit flaky". When asked why, it seemed she lacked a sense of meaning.

C: I don't know, it just doesn't make sense for me. It, it feels like, I just, it's just proved systematically, without being able to imagine what's going on. So that's why it feels flaky.

When asked what made her decide to prove by contradiction, Carla answered that she had used the form of statement to decide upon an appropriate proof structure.

C: Because, in class, whenever we have some statement which says… "there is…no such number", or "there exists no such number", then we assume there is, such number. And then we go on to prove that that would cause a contradiction, thus, it doesn't exist. So it was just, something…automatically ingrained, when I see those couple of words, I think contradiction.

She found it rather difficult to describe how she moved from this point toward finding a link between what was given and what should be proven. What is interesting is that she was *not* referring to instantiations as she did this, as revealed by her later definite negative answer to a leading question from the interviewer.

I: Did you have any sort of picture in your head for this one?

C: No, no…not really. I mean I know what a global maximum is from calculus…I mean I've done these sort of things so many times. But I didn't imagine any, any sort of function. Something that would have a maximum. […] Really…I guess I did it very systematically and theoretically, because I just stepped – this is the rule, and do it through.

Summary

Overall it seems that Carla takes a syntactic approach to proving, beginning by writing down assumptions and using knowledge about standard forms of words to decide upon a structure for the proof. This is confirmed by her later reflective comments. When asked about any general strategies she had for writing proofs, she said,

C: Um, I just start with a claim…I usually don't have anything in my head beforehand. I start off with what I know, and then I assume, what they're talking about, that I should use, in that case. And then I just try to work off of there. And I try to imagine what my goal is, and kind of work from both sides, to the center.

When asked more specifically about the first things she does, she stated that she "thinks of a method to use" and went on to explain how she identifies an appropriate one:

C: If it's something that has to be proven for all…numbers in such a set, then I use induction. And…for instance, if uniqueness is supposed to be proven, I always assume there's two different numbers that produce the same result. Or something to that extent. And use contradiction. Or, for there exists no number such that, I say yes, assume there is and then use contradiction.

This basic strategy still stands when she does not immediately know which technique to use.

C: I would try out just different ones and see which one gets me the farthest. […] We don't really know many methods, so it's not that difficult, to get one right.

This last comment indicates that this syntactic approach affords Carla the ability to answer most of the questions she encounters in this transition course. What it does not appear to afford her is a sense of meaningful understanding of her answers, unlike that which Brad appears to obtain by reference to examples. Indeed, Carla expressed a discomfort with the use examples in proving, both as counterexamples and as a basis for constructing general arguments (the latter at least in graph theory).

C: I could never grasp the, just concept of giving a simple counterexample, any old thing. And those were usually the easiest problems on the exam. And I would always get zeros on them. Because I tried to disprove it in a general manner. And, I guess I'm just not, I don't trust examples, but…

C: …even if I have convinced myself that that proof would be true, and it would happen in certain examples, it wouldn't help me in writing out the proof itself. Because it has to hold for all graphs, and…I don't know how to explain it. I have trouble…generalizing graphs.

It is not clear whether she has over-adopted the maxim "you can't prove by example" or is simply unable to generate a proof based on examining an example.

DISCUSSION

Compared with the majority of the interview participants, both Brad and Carla were doing well in the class, and made good progress on the interview tasks. However, they worked differently: Brad took a consistently referential approach, and Carla a consistently syntactic approach. The referential approach afforded Brad a strong sense of meaningful understanding and a way to decide on an appropriate proof framework, but left him sometimes lacking an ability to coordinate the details of a general argument. The syntactic approach afforded Carla a systematic way of beginning a proof attempt and deciding on an appropriate proof framework, and pursuing this at the detailed level. However, it left her sometimes lacking a sense of meaning as well as confidence in situations in which examples could be useful.

We suggest that these different approaches deserve attention if we wish to help similar students build on their strengths. It would probably be more productive to help Brad describe his examples formally than to reject the examples in favor of a rigid approach to formal work; likewise, to allow Carla to keep using her syntactic strategy as a first approach, but to increasingly recognize situations in which examining examples can be useful. However, we also note that both students seem to have an underdeveloped notion of how to use examples and syntax *together* to construct a proof. Hence we suggest that those taking either approach could benefit from instruction that emphasizes the detail of links between formal statements and proofs and their referent objects and relationships.

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