CASE STUDIES OF CHILDREN'S DEVELOPMENT OF STRUCTURE IN EARLY MATHEMATICS: A TWO–YEAR LONGITUDINAL STUDY

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Two-year longitudinal case studies of 16 Sydney children extended a study of 103 first graders' use of structure across a range of mathematical tasks. We describe how individual's representations change through five stages of structural development. Children at the pre-structural stage showed inconsistent development presenting disorganised representations and incoherent mathematical ideas. High achievers progressed to a more advanced stage of structural development depicted by an increased level of abstraction.

INTRODUCTION

In our PME 28 report (Mulligan, Prescott & Mitchelmore, 2004) we described an analysis of structure present in 103 first graders' representations as they solved 30 tasks across a range of mathematical content domains such as counting, partitioning, patterning, measurement and space. We found that:

- Children's perception and representation of mathematical structure generalised across a range of mathematical content domains and contexts.
- Early school mathematics achievement was strongly linked with the child's development and perception of mathematical structure.

Individual profiles of responses were reliably coded as one of four broad stages of structural development:

1. *Pre-structural stage*: representations lacked any evidence of mathematical or spatial structure; most examples showed idiosyncratic features.
2. *Emergent (inventive-semiotic) stage*: representations showed some elements of structure such as use of units; characters or configurations were first given meaning in relation to previously constructed representations.
3. *Partial structural stage:* some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays were found.
4. *Stage of structural development*: representations clearly integrated mathematical and spatial structural features.

We build further upon previous analyses (De Windt-King & Goldin, 2001; Goldin, 2002; Gray, Pitta & Tall, 2000; Mulligan, 2002; Thomas, Mulligan & Goldin, 2002), by providing longitudinal case study data with the aim of making as explicit as possible the bases for our identification of developmental stages of mathematical structure. We focus particularly on cases representing extremes in mathematical ability.

THEORETICAL BACKGROUND

Our interest in children's development of structure in early mathematical concepts has been highlighted in our studies of number concepts, multiplicative reasoning (Mulligan, 2002; Mulligan & Mitchelmore, 1997) and measurement concepts (Outhred & Mitchelmore, 2000; Outhred & Mitchelmore, 2004). Related studies have identified that mathematically gifted children's representations show recognisable structure and dynamic imagery, whereas low achievers' representations showed no signs of underlying structure, and the use of static imagery (Thomas et al., 2002). Our findings support the hypothesis that the more that a child's internal representational system has developed structurally, the more coherent, well-organised, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be.

Our theoretical framework is based essentially on Goldin's model of cognitive representational systems (Goldin, 2002) where we examine our data for evidence of structural development of internal cognitive mathematical ideas and representations. Current analyses have also been influenced from two other perspectives: the study of spatial structuring in two and three dimensional situations (Battista, Clements, Arnoff, Battista & Borrow, 1998); and the role of imagery in the cognitive development of elementary arithmetic (Gray, Pitta & Tall, 2000). We consider 'spatial structuring' a critical feature of developing structure because it involves the process of constructing an organization or form. This includes identifying spatial features and establishing relationships between these features. Pitta-Pantizi, Gray & Christou (2004) discuss qualitative differences between high and low achievers' imagery. Children with lower levels of numerical achievement elicit descriptive and idiosyncratic images; they focus on non-mathematical aspects and surface characteristics of visual cues.

Goldin (2002) emphasises that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within representational systems. Such systems of representation, and sub-systems within them develop in the individual through three broad stages of construction:

1. An inventive/semiotic stage, in which characters or configurations in a new system are first given meaning in relation to previously-constructed representations;
2. An extended stage of structural development, during which the new system is "driven" in its development by a previously existing system (built, as it were on a sort of pre-existing template); and
3. An autonomous stage, where the new system of representation can function flexibly in new contexts, independently of its precursor.

Our analysis of developmental stages of structure was initially framed by Goldin's three broad stages of construction. From our data with young children we have identified an initial pre-structural stage and two sub-stages (partial structure and structure) preceding Goldin's stage 2 (extended stage). We seek to extend Goldin's model based on longitudinal evidence from young children.

Our analyses have not yet tracked our proposed stages of structural development for individuals over time. Thus, we pose further research questions:

- Do young children continue to develop and use structure consistently across different mathematical content domains and contexts over time?
- Do all young children progress through these identified stages similarly?

METHOD

The sample comprised 16 first grade children, 7 girls and 9 boys, ranging from 6.5 to 7.8 years of age, drawn from the initial 103 subjects. Four children representing each stage of structural development were tracked as case studies in the second year. Selection of a representative sub-sample of children of low or high mathematical ability was supported by clinical assessment data such as IQ tests, and system-based assessments. Four low ability children were classified at the pre-structural stage; one low ability child at the emergent stage; and four high ability children at the stage of structural development. The case study sample was drawn from five state schools in Sydney and represents children of diverse cultural, linguistic and socio-economic backgrounds.

Cases representing extremes in mathematical ability were subject to in-depth study and supporting evidence compiled from classroom assessment data. The same researchers conducted videotaped task-based interviews at approximately three intervals: March and October in the first year and August/September in the second year, including a second phase of interviews.

Thirty tasks, developed for the first year of the study were refined and/or extended to explore common elements of children's use of mathematical and spatial structure within number, measurement, space and graphs. Tasks focused on the use of patterning and more advanced fraction concepts were included. Each task required children to use elements of mathematical structure such as equal groups or units, spatial structure such as rows or columns, or numerical and geometrical patterns. Number tasks included subitizing, counting in multiples, fractions and partitioning, combinations and sharing. Space and data tasks included a triangular pattern, visualising and filling a box, and completing a picture graph. Measurement tasks investigated units of length, area, volume, mass and time. Children were required to explain their strategies for solving tasks such as reconstructing from memory a triangular pattern and to visualise, then draw and explain their mental images (see Figure 1). Operational definitions and a refined coding system were formulated from the range of responses elicited in the first year of interviews and compared with analysis of new videotaped data; a high level of inter-rater reliability was obtained (92%).

Analysis focused on the reliable coding of responses for correct/incorrect strategies and the presence of structural features to obtain a developmental sequence. The coding scheme developed for the first stage of interviews was extended to classify strategies for several new tasks. A fifth stage, an *advanced stage of structural development* was identified, where the child's structural 'system' was developed or extended by using features of the previously existing system. We examined whether this structural development was consistent for individuals across tasks and over a two–year period. Responses to all 30 tasks were coded for all 16 children and the matrix examined for patterns. Achievement scores were compared with individuals' types of representations. It was found that the children could be unambiguously classified as operating at one of five stages of structural development at each interview point.

DISCUSSION OF RESULTS

These results support our initial findings indicating consistency in structural features of individual children's representations across tasks at each interview point. Our report at PME 28 (Mulligan et al. 2004) represents Interview 1 data.

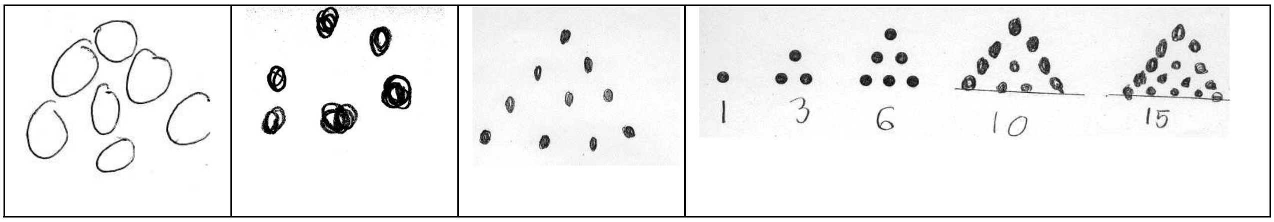
Case Study No.	Interview 1 March 2002	Interview 2 Oct 2002	Interview 3 Sept 2003	Code
1	PRS	PRS	PRS	Pre-structural Stage (PRS)
2	PRS	PRS	ES	Emergent structural stage (ES)
3	PRS	PRS	ES	Stage of partial structural development (PS)
4	PRS	ES	ES	Stage of structural development (S)
5	ES	PRS	PRS	Advanced stage of structural development (AS)
6	ES	ES	PS	
7	ES	PS	S	
8	ES	PS	S	
9	PS	PS	PS	
10	PS	PS	S	
11	PS	PS	S	
12	PS	PS	S	
13	PS	S	AS	
14	S	S	AS	
15	S	S	AS	
16	S	S	AS	

Table 1. Classification of cases by interview by stage of structural development

Table 1 summarises patterns of structural development for the 16 case studies at three interview points across the two-year period. Cases 1 to 5 represent children identified as low ability; cases 12 to 16 as high ability. For most cases there was clearly some developmental progression by at least one stage; cases 7, 8 and 13 progressed by two stages. Cases 1 and 9 showed no observable development of structure in representations or in achievement scores at interviews 2, and 3. For all high ability children there was progression to an advanced stage of structural development encouraged by the inclusion of more advanced tasks. It is not possible to ascertain whether these children may have been operating at this advanced stage at interviews 1 and 2. Cases 1, 4 and 5 showed inconsistencies in their development. Although the low ability children (cases 1 to 5) made some progress, there was more dissimilarity than similarity in their responses, within and between cases.

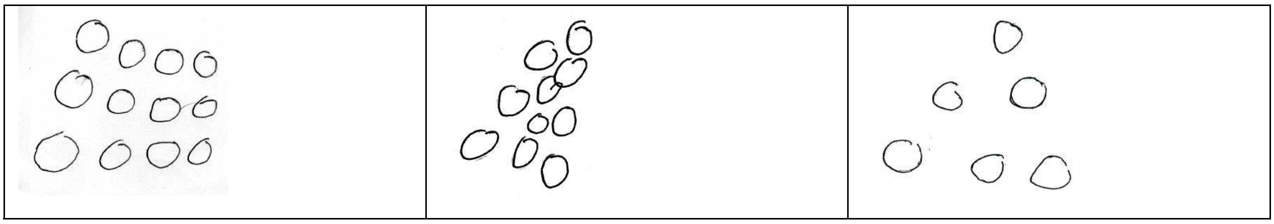
In order to illustrate developmental levels of structure, we discuss representative examples below of children's responses to the triangular pattern task (where the pattern was reconstructed from memory and extended). We selected examples from each stage of structural development identified at the first interview and some exceptions of developmental patterns. The analysis centres on how representations conform to structural features such as numerical quantity, use of formal notation, spatial organization and shape, and construction of pattern.

Figure 1 compares responses given by a high ability child showing the extension to a spatial and numerical pattern of triangular numbers. There is clear development from the stage of partial structure to an advanced stage of structural development. She was able to construct and explain the triangular pattern by repeating the previous row and adding one more circle. Her response indicated that she recognised the pattern, both structurally and numerically, and was therefore, in the early stages of being able to generalise pattern. This ability was also found in her other responses, for example, where she was able to discuss the pattern of digits in a multiple pattern of threes from 3 to 60.



Interview 1 Partial Structure	Interview 2 Structure	Interview 3 Advanced Structure	Interview 3 (second phase) Advanced Structure

Figure 1: Case No. 13. Triangular Pattern Task: Structural Stages

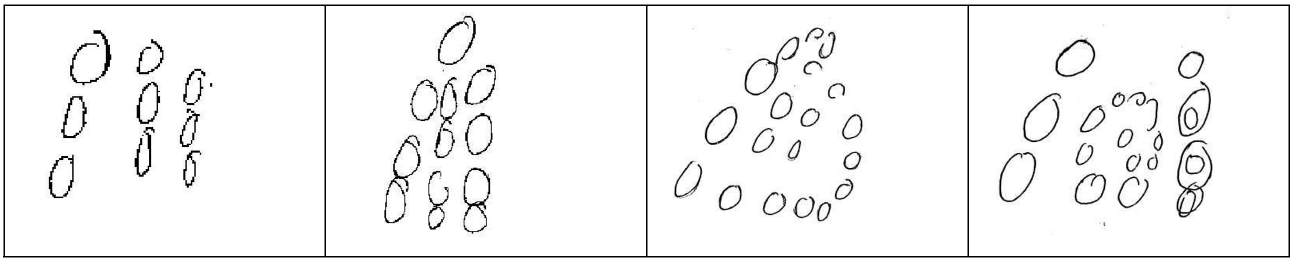


Interview 1	Interview 2	Interview 3
Partial Structure	Partial Structure	Structure

Figure 2: Case No. 10. Triangular Pattern Task: Structural Stages

In Figure 2 the child's first interview shows evidence of some structure in the organization of circles. This becomes more clearly defined as a triangular pattern by interview 3 where superfluous features are excluded.

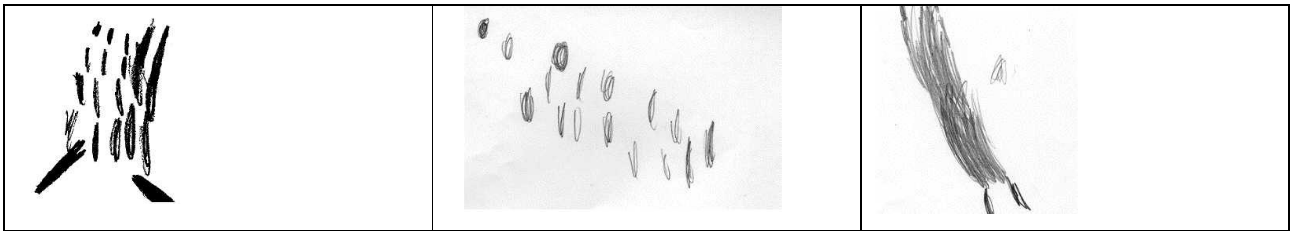
In contrast, Figure 3 shows a child's awareness of a pattern of circles with partial structure. This becomes transformed into triangular form at interview 2, but by interview 3 the image becomes more complex and there is no awareness of the numerical pattern. At a second attempt the image is replicated in a less coherent manner. The images become more disorganised and it can be inferred that the child's internal representational system becomes more 'crowded' with unnecessary icons. It appears that the child loses sight of the initial, clearer numerical and spatial structure that he produced at interview 1. His profile of responses showed no improvement across tasks from interviews 1 to 3.



Interview 1	Interview 2	Interview 3	Interview 3
Partial Structure	Partial Structure	Partial Structure	(2nd phase) Partial Structure

Figure 3: Case Study No. 9. Triangular Pattern Task: Structural Stages

Figure 4 shows an initial idiosyncratic image depicting emergent structure; the child draws a triangular form as a 'Christmas tree' and attempts to draw a pattern as vertical rows of five circles. There is little awareness of the structure or number of items in the pattern; there is some indication of spatial structure with equally spaced marks. Interestingly the child produces a completely different image of circles drawn in a diagonal form at interview 2. She could not provide any explanation for an emerging numerical or spatial pattern. At interview 3 the child produced some elements of her initial image but it had fewer structural features. In responses to other tasks she was unable to use multiple counting, partitioning, equal grouping and equal units of measure.



Interview 1	Interview 2	Interview 3
Emergent Structure	Pre-structural	Pre-structural

Figure 4: Case Study No. 5. Triangular Pattern Task: Structural Stages

CONCLUSIONS & IMPLICATIONS

Longitudinal data supported our earlier findings that mathematical structure generalises across a wide variety of mathematical tasks and that mathematics achievement is strongly correlated with the child's development and perception of mathematical structure. This study, however, advances our understanding by showing that stages of structural development can be described for individuals over time. We extend Goldin's model to include two substages of developing structure and an advanced stage of structural development for young children.

There was wide diversity in developmental stages shown for children of the same age range, and some progress shown for most children in their achievement scores across tasks and in their representations. However developmental patterns for low ability cases were inconsistent; the transition from pre-structural to an emergent stage was somewhat haphazard and some children revert to earlier, more primitive images after a year of schooling. There was evidence that some children may not progress because they complicate or 'crowd' their images with superficial aspects. Our data supports the findings of Pitta-Pantazi, Gray & Christou (2004) in that different kinds of mental representations can be identified for low and high achievers. Low achievers focus on superficial characteristics; in our examples they do not attend to the mathematical or spatial structure of the items or situations. High achievers are able to draw out and extend structural features, and demonstrate strong relational understanding in their responses. It was not possible to identify consistently, common features impeding the development of structure in the examples presented by low ability children.

An important new finding gleaned from the cases is the phenomenon of increasingly 'chaotic' responses over time. Representations over time became more complex with configurations and characters of the child's earlier 'system' used inappropriately. In terms of Goldin's theory, we infer that these children fail to perceive structure initially and continue to rely on reformulating superficial and/or idiosyncratic, non-mathematical features in their responses. It appears that these children may benefit from a program that assists them in visual memory and recognising basic mathematical and spatial structure in objects, representations and contexts.

However, our findings are still limited to a sample of 16 cases at three 'snapshots' of development. We plan to undertake longitudinal investigations (using multiple case studies) to track the structural development of low achievers from school entry, and to evaluate effects of an intervention program focused on pattern and structure. In 2003, a school-based numeracy initiative, including 683 students and 27 teachers, was successfully trialled using our research instrument. This initiative implemented a professional development program aimed at developing teachers' pedagogical knowledge and children's use of pattern and structure in key mathematical concepts.

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