DEGREES OF FREEDOM IN MODELING: TAKING CERTAINTY OUT OF PROPORTION

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In its empirical part this paper establishes a general weak understanding of the process of applying a mathematical model. This is also evident in the way teachers regard the application of alternative sharing in their own problem solving and in relating to children's answers. The theoretical part analyses problems that are considered as applications of proportional reasoning. It suggests that the rationale for applying a proportion model varies and includes, for example, cases with a scientific rationale and others with a social one. In some problems there are no degrees of freedom in applying proportion, but in other cases this model should not be taken as "engraved in stone". This analysis is supported by examples of alternative sharing in Talmudic laws or bankruptcy interpreted by game theoretic models.

THEORETICAL BACKGROUND

Modeling

This research expands the existing knowledge of the nature of mathematical modeling by offering an analysis of application rationale. The term modeling refers here to applying a mathematical model in a problem solving situation. As a less "automatic" act, modeling can be defined as the process of organizing and describing a situation or a phenomenon by using a mathematical model (or models) or "mathematizing" the situation by perceiving it through mathematical lenses (Greer, 1993).

Following the re-thinking of math education goals the interest in modeling processes increased, recognizing the importance of modeling expertise as a goal, and noticing that a good modeling activity, in turn, adds meaning to the applied mathematical model, increasing its power and enriching children's mathematical concepts. This meant that the nature of school problems and classroom practice had to change.

Some researchers showed that classroom norms were responsible for the fact that children do not use realistic considerations in problem solving (Reusser & Stebler, 1997; Greer, 1997; Verschaffel, De Corte & Borghart, 1997). Children and pre-service teachers in several different countries were given problems that called for use of everyday knowledge, such as the fact that even a very fast runner cannot keep up his hundred meter speed when running a whole kilometer. In conventional classroom conditions, and even when children were given some hints on the special nature of these problems, children did not use realistic considerations. However, when the didactical contract (Brousseau, 1997) was changed, children changed their problem

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solving habits (Verschaffel & De Corte, 1997; Verschaffel, Greer, & De Corte, 2002).

Modeling standard problems

Our earlier research (Peled & Hershkovitz, 2004) suggests that a more inquisitive modeling attitude should be used not only in specially designed problems of the type composed by Verschaffel et al. (2002), but become a common practice even in standard problems. Peled and Hershkovitz (ibid) asked teachers and students to solve a conventional proportional reasoning problem. Most of the teachers applied proportional reasoning. A few of them made some drawings and gave a different answer. Class discussion revealed that teachers who solved the problem (correctly) using proportional reasoning engaged in an almost automatic application of proportion without deliberation on the reason why this model fit the given situation. The discussion and comparison of alternative teachers' and children's solutions made teachers analyze the situation and enrich their understanding of ratio and proportion.

This research triggered our further investigation of modeling in standard school problems. We noted that teachers and textbook writers create sets of problem types they consider should be solved by using a certain mathematical model leaving no option for alternative solutions and leading students to view the connection between mathematical models and situations as an undisputed truth.

Doubting certainty in model application

Our concern in this study goes beyond the gap that exists between school mathematics and everyday mathematics. We focus on the assumptions behind applying a certain mathematical structure and analyse their nature. The following example will demonstrate what we mean by that.

The Lottery Problem:

> Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they share the money?

A problem of this type appears in textbooks in the ratio and proportion chapter as an example of a situation in which an amount should be shared using a given ratio. In this case, the $40 sum is expected be split into two amounts using a 3:2 ratio.

Ron, a seventh grader, suggested 3 different solutions to the Lottery Problem (converted here from the original IS to $):

Solution 1: 40:2=20 Each child gets $20.

Solution 2: One child (the one who paid $2) gets $19 and the other (the one who paid $3) will get $21 (although the difference is $2 and not $1).

Solution 3: One will get $16 and the other 24 because $40:\overset{2\;\;3}{\overset{\vee}{5}}=8$

3x8=24, 24 to the child who paid $3

2x8=16, 16 to the child who paid $2

Ron: In my opinion, the first solution is the most fair, but the third is most right because of the ratio.

Ron, aware of classroom norms, knows that the teacher expects him to give the third solution, even if it doesn't feel so right to him. But is proportional sharing really the "right" (and unique) model? Why?

It should be noted that similar answers were given by pre-service teachers in a more complex money-sharing situation described by Koirala (1999) in a problem involving the purchase of shoes in a "3 for 2" sale. Rather than figuring the cost for each of the two friends who are making one purchase by taking care of giving each of them the same percent off, some pre-service teachers suggest different kinds of splits. For example, some (the author identifies them as students with good mathematical understanding) think that dividing the saving evenly is fair. Koirala (ibid) seems to think that there is one correct answer, using the same-percent-off split. In fact, as the title of his article implies, he is worried that academic mathematics might be lost by legitimizing alternative solutions. We do not agree with his point of view.

What is the basis for using proportional sharing? Is it inherent in the situation? Can we use another mathematical model? In the theoretical analysis we will contrast this situation with other situations in showing that the fitting of proportion in this problem is done on a relatively weak basis. Our purpose is to develop and then encourage a meta-analysis of the modeling process that deals with the modeling assumptions, their nature, and the degree of certainty with which we apply a mathematical model.

FINDINGS

Although we split the findings report into a theoretical part and an empirical part, the two parts were conducted simultaneously. The empirical findings start with data that establishes alternative answers in a money sharing situation, and presents attitudes towards these different ways of mathematizing the situation. It continues with a short description of class discussion that was conducted after an initial theoretical analysis was done. Our analysis of this discussion and additional workshop discussions resulted in a refinement of the theoretical analysis.

Empirical findings

We detail here a part of the data collected including children's answers and teachers' reactions, and summarize one of the discussions we conducted.

A group of 24 seventh graders and a group of 43 elementary school teachers were given the original version of the Lottery Problem. They were asked to solve the problem and then react to the following children's answers:

Aviv's answer: 40:2=20 Each one should get 20 IS.

Danit's answer: Anne should get 21½ IS and John should get 19½ IS, because Anne invested 3 IS and John invested 2 IS, the difference is 1 IS therefore the difference in their winning shares should also be 1 IS.

Yaron's answer: Anne should get 24 IS and John should get 16 IS, because 40:5=8 and 3x8=24 and 2x8=16.

The reaction distribution for each of the two groups is depicted in Table 1.

	Aviv (equal)			Danit (diff.)			Yaron (prop)	
*	+	-	+/-	+	-	+/-	+	-
Teachers n=43	7	23	13	1	36	6	43	0
Students n=24	12	12		14	10		16	8

* + regard answer as correct – as incorrect +/- correct and incorrect

Table 1: Teacher and student reactions to ways of money sharing.

As can be seen in Table 1, some of the teachers said that Aviv's answer or Danit's answer were both correct and incorrect. In their explanation they argued that the given solution might be correct socially or morally but incorrect from a mathematical point of view. Some typical answers: "It is their right to share the money anyway they choose, but in principle they should share their winnings using the 3:2 ratio" or "From a moral point of view equal sharing is great, but from a mathematical perspective the sharing ratio should be equal to the investment ratio". There were also some comments such as: "On a second thought, nowhere in the problem does it say that they will receive [money] according to their investment ratio, so it is possible to accept the equal share option".

These (and additional) findings motivated our efforts to develop an analytical tool for modeling. Following an initial theoretical analysis, we conducted several student and teacher workshops where we brought up the issue of modeling rationale.

In one of our first discussions with mathematics education graduate students the Lottery Problem was presented in 3 different versions (the effect of different story conditions is discussed in the theoretical analysis):

1. The original version with a $40 win. 2. A million dollars (actually IS) win.

3. As in version #2 with the additional information that Anne says: *I only have $3,* implying that she could not have bought a ticket if it were not for John's $2 contribution.

Each student was given one of the three versions and then asked to react to a variety of children's solutions. Then the Mixture Problem (shown in the theoretical analysis) was presented and the students were asked to compare the problems. Although students exhibited different reactions as a result of getting the 3 problem versions, and although they expressed positive attitude towards children's non-proportional distributions, the instructor (the first author) felt that the students did not fully understand the differences between the Lottery Problem and the Mixture Problem. As a result, she introduced a third situation that does not involve any chemical reaction.

The new situation involved a car assembly line, where each single car needs N parts type A and K parts type B. In this situation a constant ratio, N:K, exists between the number of parts independently of any given quantity of cars. As a result of this discussion, the Assembly Problem (detailed in the following section) was composed.

Theoretical analysis

One way to highlight and identify the nature of a process is by comparing it in different cases. We did that by composing a problem, the Mixture Problem, that is different in context from the Lottery Problem and yet supposedly (we will refer to the use of this word later) has the same structure. At a later point in the study, following class discussions, we composed a third problem, the Assembly Problem.

The Mixture Problem:

> Ron started painting his garden fence in green that he got by mixing 3 cans of yellow paint with 2 cans of blue paint. When he ran out of paint, he calculated that he needed 40 more cans to finish the fence. He also decided that he would like to mix yellow and blue and get the same shade of green that he had had before. How many of the 40 cans should be yellow and how many should be blue?

The Assembly Problem:

> In a certain car assembly line each car has to be equipped with large cushions for the 2 front seats and smaller ones for the 3 back seats. A load of parts arrived for a certain amount of cars. It included a total of 40 cushions which were indeed used in assembling the cars. How many of them were large cushions?

Although we took care of composing the problems to be analogical in structure (the need to revise the definition of analogical structure will be raised in our work), we claim that they are very different. In the Lottery Problem proportional sharing is a result of the assumption that it is fair to have the same profit for each dollar invested. However, in the Mixture Problem proportion is used because this is how colors behave chemically when they are mixed. The Assembly Problem does not require a moral or a scientific excuse and as will be further analysed, the use of a mathematical model in this case is not only straightforward but also very stable.

Resistance to change in problem conditions

The differences between the three problems become apparent by looking at the solution’s "resistance" to variations in story details. If, for example, John and Anne win a million dollars, will we still expect them to use a 3:2 sharing ratio? And what if Anne only had \$3 and would not have been able to buy the \$5 ticket without John's contribution?

Several questions are elicited by these problem variations: Who decides how money should be shared? Is there some normative social agreement that it would be fair to distribute the money proportionally? What if John wants more than his proportional share and goes to court, what does the law say about such cases?

The resistance to change criterion applies to the mixture problem in a different way: Sometimes in mixing very large amounts the chemical behavior does not follow the same pattern as in smaller amounts. Greer (1997) refers this phenomenon in cooking, where doubling the amount to be cooked does not necessarily mean that all receipt elements preserve the original ratio. The mathematical model for mixing different amounts depends on the chemical and physical mechanisms that are involved in the process. Some questions may prove pertinent here as well: How was a certain mixture formulae created in the first place? Was it an outcome of experimental observations resulting in a phenomenological connection? Or was it perhaps the result of a theoretical analysis of some chemical relations?

While the mathematical model for the Lottery Problem and the Mixture Problem might depend on problem conditions, this is not the case for the Assembly Problem. The ratio between the total number of large cushions and small ones that are used in the process is constant and independent of the amount of assembled cars. (As a matter of fact even this situation is not completely "clean"… To avoid a quality control issue that would involve unfit parts, the number of which does depend on sample size, the problem refers to the used parts only).

Thus, at this point, we have three different cases (mixture, sharing and assembly) at different locations on the “strength of application” axis (a temporary description): Assembly situations are located on the “very certain” side, moral-social situations on the “less certain” side and scientific situations somewhere in between, not too far from assembly situations.

To use or not to use proportion: Learning from other disciplines

Focusing on the specific mathematical model of proportion, our theoretical analysis takes some of its ideas from other disciplines in several ways: We interviewed specialists such as lawyers and scientist in industrial plants in an effort to understand the actual fitting of a mathematical model.

It is interesting to note that when we asked a lawyer to solve the Lottery Problem, her first reaction was surprisingly similar to Ron’s answers and she suggested a proportional distribution of the money. On further prompting she admitted that this is not the way it would work in real practice. She explained that her first answer was based on identifying the problem as a school problem that should be answered as taught in school.

We also looked into ways people solve similar situations in everyday cases, and into solution procedures suggested by disciplines that deal with such problems. One of the cases involves Talmudic laws. As is shown in the following example, it offers a different solution.

Mishnah 3 in chapter 10 of the volume Ketubot (a Ketuba is a document signed by the groom listing what he will pay his bride in case of death or devorce) deals with a case where a man dies leaving 3 widows. In their Ketubot he had promised to give the first woman 100 gold coins, give the second 200 coins, and the third 300 coins. Unfortunately, what he left is smaller than the sum of the promised amounts. Rather than splitting it using a 100:200:300=1:2:3 ratio, the Mishna rules that money distribution (in our terms: the mathematical model that is applied) depends on the given amounts. For example, if the whole inheritance is 100 coins, it is equally distributed. If it is 200 coins, then the second and third wives get 75 coins and the first gets 50 coins.

For years these laws seemed inconsistent and their rationale was not known, until Aumann and Maschler (1985) developed an explanation based on game theory and the distribution of gains suggested by Shapley Value (satisfying the properties of efficiency, fairness and consistency) (Castrillo & Wettstein, 2004). This rationale can be applied in different cases (as in bankruptcy) where existing assets are smaller than the total claims.

DISCUSSION

This study followed our earlier realization (Peled & Hershkovitz, 2004) that the application of proportion in a standard problem is done automatically, with hardly any motivation to explore the situation or the reason a specific mathematical model should be applied.

The empirical findings show that in problems that look like conventional proportion problems most teachers apply a proportion model even in cases that would have called for alternative solutions in reality. Some of the teachers reluctantly accept children's alternative solutions saying that they might be morally fair but mathematically wrong.

Following these results we concluded that teachers need an analytical tool that would make them aware of the differences between situations with regard to the reasons for applying a model. A tool that would identify the modeling rational, establish the degree of certainty for applying a mathematical model, and help indicate where alternative solutions can be legitimate even in the eyes of math educators such as Koirala (1999) who do not want to loose academic mathematics.

Our theoretical analysis describes the direction we take in constructing this tool. We show examples of problems that look analogical in structure but use different contexts. These problems stand for different types of application rationale and can be located at different places on a scale that represents the amount of certainty in using the relevant mathematical model. The Assembly Problem represents a straight case of proportion while the Mixture Problem is a case for a scientific investigation and the Lottery Problem is a case for social norms and existing social laws.

We also found that game theory and Talmudic laws support our claim that the status of proportional distribution of assets (as depicted in the Lottery Problem) is different from the status of corresponding modeling in a scientific problem (as in the Mixture Problem). Several Talmudic laws suggest a non-proportional solution in cases that would probably have been solved in textbooks by applying proportional reasoning.

In our continued research we intend to refine the theoretical analysis, apply and validate it in our work with teachers in an effort to improve their understanding (and subsequently their students' understanding) of the modeling process.

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