GENERALIZATION STRATEGIES OF BEGINNING HIGH SCHOOL ALGEBRA STUDENTS *

Joanne Rossi Becker and Ferdinand Rivera

San José State University, USA

This is a qualitative study of 22 9th graders performing generalizations on a task involving linear patterns. Our research questions were: What enables/hinders students' abilities to generalize a linear pattern? What strategies do successful students use to develop an explicit generalization? How do students make use of visual and numerical cues in developing a generalization? Do students use different representations equally? Can students connect different representations of a pattern with fluency? Twenty-three different strategies were identified falling into three types, numerical, figural, and pragmatic, based on students' exhibited strategies, understanding of variables, and representational fluency.

BACKGROUND

In 1999, with a grant from the Noyce Foundation, San Jose State University and 30 school districts formed a Mathematics Assessment Collaborative (MAC) in an effort to balance state-sponsored multiple-choice tests and to provide multiple measures to evaluate students. The MAC exams are summative performance assessments in grades 3-10. The exams are hand scored using a point rubric and audited for reliability. Student papers are returned to teachers for further instruction and programmatic review. In developing this model system of performance assessment, the MAC spent a year writing Core Ideas for each grade level tested, adapting the National Council of Teachers of Mathematics *Standards* (NCTM, 2000). The assessments are written to match these Core Ideas. MARS results are correlated to state test results and analyzed by various demographic characteristics of students. In 2003, over 60,000 students were tested by the MAC.

At the eighth and ninth grades, one of the Core Ideas tested is that of patterns, relations and functions. Students are asked: to generalize patterns using explicitly defined functions; and, understand relations and functions and select, convert flexibly among, and use various representations for them. Over the five years of MARS data collections, we have found a similar pattern; while students are quite successful in dealing with particular cases of patterns in visual and tabular form, they have considerable difficulty in using algebra to express relationships or to generalize to an explicit, closed formula for a linear pattern. Summary data are shown in Table 1. To gain more insights, we embarked on an in-depth study of a small number of 9th grade students to pinpoint more specifically why they have difficulties in forming

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	1999	2000	2001		2002		2003	
	9th	8th	8th	9th	8th	9th	8th	9th
Ability to deal with particular	80%	87%	72%	80%	74%	71%	56%	54%
Ability to generalize	15%	21%	22%	12%	21%	17%	5%	9%

Table 1: 8th and 9th Grade Results on Patterns and Functions Items

generalizations so we could help teachers find ways to ameliorate deficiencies in this critical area. Specifically, our research questions were: 1) What hinders students' abilities to generalize a linear pattern? 2) What strategies do successful students use to develop an explicit generalization? 3) How do students make use of visual and numerical cues in developing a generalization? Do students use different representations equally? 4) Can students connect different representations of a pattern with fluency? 5) What can we glean from student work that will inform and improve instruction?

THEORETICAL FRAMEWORK

In everyday situations, children are naturally predisposed to performing generalizations. As bricoleurs, children use whatever is available to them to induce patterns from objects despite developmental physiological constraints and their limited social knowledge, experiences, and expertise (DeLoache, Miller, & Pierroutsakos, 1998). Contrary to either Piaget's (1951) or Bruner's (1966) view that children need powerful hypothetical analytic skills or that they must attain a certain level of conceptual and abstract development prior to being able to induce patterns from objects, developmental psychologists show that children certainly could on the basis of similarity. Medin, Goldstone, and Gentner (1993) perceive similarity as an initial organizing principle, and that similarity is not known a priori and it is not static (Smith & Heise, 1992). It is, however, variable as it is based on children's ability to compare objects and determine what counts as meaningful and relevant features. Medin and Schaffer (1978) claim the significance of context in induction (i.e., a particular sample is a member of a pattern if it resembles some or all of the previously known samples in the pattern). Rosch (1978) demonstrates the role of typicality in assessing for similarity (i.e, a specific instance is a member of a class of objects if it appears to the observer as a typical example and if it resembles the known prototype examples of the class). What is significant for us in this study is Gentner's (1989) classification of three kinds of similarity, namely: analogy, literal similarity, and mere-appearance matches. They differ from one another in terms of the role attributes and relations play in similarity. Attributes "describe properties of entities," while relations "describe events, comparisons, or states applying to two or more entities" (p. 209). Analogical similarity focuses on relations and is not object-dependent; mere-appearance matches focus on object attributes and descriptions;

literal similarity is an overlap between analogy and mere-appearance matches as it utilizes commonalities that exist between attributes and relations. Gentner (1989) claims that young children and novices rely on mere-appearance matches and literal similarity. Also, a relational shift has been documented whereby young children would perform similarity on objects while older children and adults would induce relations with minimal need for surface support.

Küchemann's (1981) study highlights the ease with which beginning algebra students could associate letters as representing particular values versus letters as representing relationships: while these students could correctly deal with particular instances in a table of values that implicitly describe some mathematical relationship involving two quantities, they are unable to easily deal with the additional tasks of generalizing by way of pattern recognition and predicting by way of determining values for the larger cases. A related study by Stacey and Macgregor (2000) provides us with a characterization of the mathematical thinking employed by beginning algebra students on tasks involving pattern formulation: 1) Beginning algebra students could see valid patterns emerging from a given table of values; however, some of those patterns could not be easily translated symbolically. 2) Beginning algebra students perceive patterns as being generated by procedural rules for combining and obtaining numbers in either sequence of dependent and independent values, and not functional relationships. 3) Beginning algebra students have difficulty assigning correct representational meanings to the variables. 4) Beginning algebra students' verbal and algebraic solutions are correlated in such a way that those who could clearly articulate their patterns tend to have greater success at writing the correct rules in symbolic forms. Stacey and Macgregor further insist that students' facility with the properties of numbers and operations could assist them in obtaining a correct description of rules and relationships. Also, students need to know the structural nature of rules such as having only one simple rule for a given table of values.

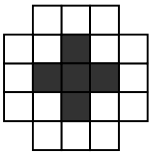
METHODS

Twenty-two ninth grade students (11 males, 11 females) in a beginning algebra course in a public school in an urban setting participated in the study. The students had completed the task (see Figure 1) in December 2002 and were given the same task in May 2003 during an individual interview by the second author that was audio-taped; interviews lasted about 20-30 minutes. Each participant was asked to read the problem and asked to think aloud as they solved the problem. The tapes were then transcribed by a graduate student and analyzed by both authors. The first level of analysis involved several individual readings of each transcript to identify patterns in strategies used for each of the six questions on the item. Then several follow-up discussions and cross-checking followed.

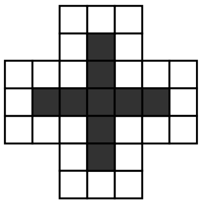
RESULTS

Twenty-three different strategies were used, as shown in Table 2 . The most common strategies are described more fully with portions of transcripts. Of course students used more than one strategy as they solved different portions of the task. Ten of the strategies in the table are primarily visual in nature: 1, 4, 5, 9, 13, 14, 18, and 20-22.

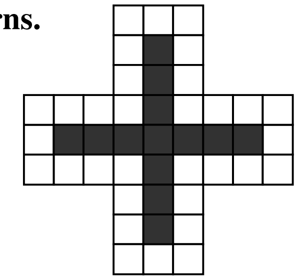
Marcia is using black and white square tiles to make patterns.



Pattern 1



Pattern 2



Pattern 3

1. How many black tiles are needed to make Pattern 4? ________________

Marcia begins to make a table to show the number of black and white tiles she is using.

Pattern Number	1	2	3	4
Number of White Tiles	16	24		
Number of Black Tiles	5	9		
Total	21	33		

2. Fill in the missing numbers in Marcia's table.

3. Marcia wants to know how many white tiles and black tiles there will be in the tenth pattern, but she does not want to draw all the patterns and count the squares.

Explain or show another way she could find her answer.

4., 5., & 6. Using W for the number of white tiles and P for the pattern number, write down a rule or formula linking W with P.

Using B for the number of black tiles and P for the pattern number, write down a rule or formula linking B with P.

Now, using T for the total number of tiles and P for the pattern number, write down a rule or formula linking T with P.

Figure 1: Tiling Squares Problem

Visual Grouping Strategy (S1). Edward provided a prime example of a strategy of counting each "arm" of the pattern and then multiplying to get the total.

> I looked at pattern 3 and I saw the three pattern, three tiles, that are on each side so I thought I looked at the pattern two and it just added one so I multiplied four times four with all the sides and just added one in the middle [for pattern 4].

Visual Growth of Each Arm Strategy (S4). This strategy is similar to #1 except students used an additive rather than multiplicative approach to get the total number of tiles.

Counting ELLs and Adding 4 Center Squares (S14). To find the number of white tiles in pattern #1, Alajandro saw four groups of three white tiles forming an L shape around the center black cross with an additional 4 center white squares on each side.

> You can see here, like, it's three, three, three, three, plus twelve, and four, and the same here [referring to the next pattern]. The thing is you just add four more and if you are doing a table you just add 8, that's 16, 24, 32, 40.

Strategies	No.	Strategy Description	Strategies	No.	Strategy Description
Numerical	2	Numerical use of finite differences in table	Visual	1	Visual grouping by counting each arm; multiplicative relationship
	6	Random trial and error		4	Visual growth of each arm; additive method of counting
	6’	Systematic trial and error		5	Visual symmetry
	7	Numerical finite differences to generalize to closed formula		9	Figural proportioning into pillars; add 4 for external and 4 for internal squares
	8	Implicit recursion		13	Concentric visual counting
	10	Confusing dependent and independent variables		14	Counting Ell shapes and adding 4 center squares
	11	Extending the table		18	Chunking
	12	Missing independent variable		20	Counting by one
	15	Adding two formulas for black and white		21	Visual finite differences after random count
	16	Incorrect use of proportionality		22	Visual finite differences after systematic count by 3s
	17	Get a formula and substitute to get 10^{th} term	DG*	3	Unable to generalize
	19	T in terms of B and W	*Disjunctive Generalization		

Table 2: All Strategies Identified in Solving Tiling Squares Task

Numerical Use of Finite Differences in Table Strategy (S2). Even some of the students who could not generalize, such as Rosendo, were adept at using finite differences in the table. This was obviously a strategy they had been taught. Rosendo showed her work on the paper by drawing a loop connecting the 5 and 9 in the table, then the 9 and the blank, which she filled in with 13.

> Marcia is using black and white square tiles to make patterns. How many black tiles are needed to make Pattern 4? Um, you keep adding 4, 4 plus [5] I think with the pattern.

Trial and Error Strategy (S6/S6’). If one combines the systematic and unsystematic trial and error approach, this was a common strategy. Interestingly, there were two students who used Strategy #2, Finite Differences, yet did not transfer that information into their attempt to generalize to a formula. A third, Jennifer, did not use the table, but was able to get a formula through a guess and check strategy.

S: Black is $4n + 1$.

Interviewer: How did you decide $4n + 1$?

S: I started off with more like 2 and that didn’t work so then I tried to make 5 work and I did the same thing with 2, 3, and 2 and then when I tried it with 4, and I tried to figure a number to make 5 so I add 1, and I tried it on 2 and it still gave me the number.

Individual Patterns. We next graphed each student's strategies on parts 1-6 of the task in Excel so we could examine trends over the course of the task as it ramped up from specific to general. Two examples of graphs are shown here for illustration. Katrina (Figure 2) began the problem with visual strategies, then changed to numerical use of finite differences to get a general formula, which she used to find the values for the 10^{th} pattern. For part 6, the total number of tiles, she indicated to add the number of black and white tiles but did not produce a closed formula.

Rani (Figure 3) also began with a visual strategy, then transitioned into using finite differences and extended the table to answer part 3. However, Rani had to construct the table all the way out to the 10^{th} pattern number in order to correctly answer part 3. Thereafter, he used trial and error to try to get to a generalization but was unable to do so. For example, in part 4:

I: What makes it difficult to figure out that formula?

S: Because I can't find what links them to like equal 16 and then 24 or add up to make it. Number of white tiles goes up by 8. I don't know how I would link to the number of patterns.

Group Patterns. Because of our particular interest in students' ability or inability to generalize, we focused our attention on part 3 of the task, which is the transition point between the specific and the general. In fact, 12 students used Strategy #17, in addition to other strategies, for this part of the task: they tried to get a formula that they could use to find the number of white and black tiles in the 10^{th} pattern. Of those 12, four students were unsuccessful in forming a generalization; one used purely numerical strategies, while the other three used visual or combined visual/numerical strategies. The other eight students were successful in generalizing; of those, three used purely numerical strategies to lead them to a generalization, while the other five used visual or visual/numerical.

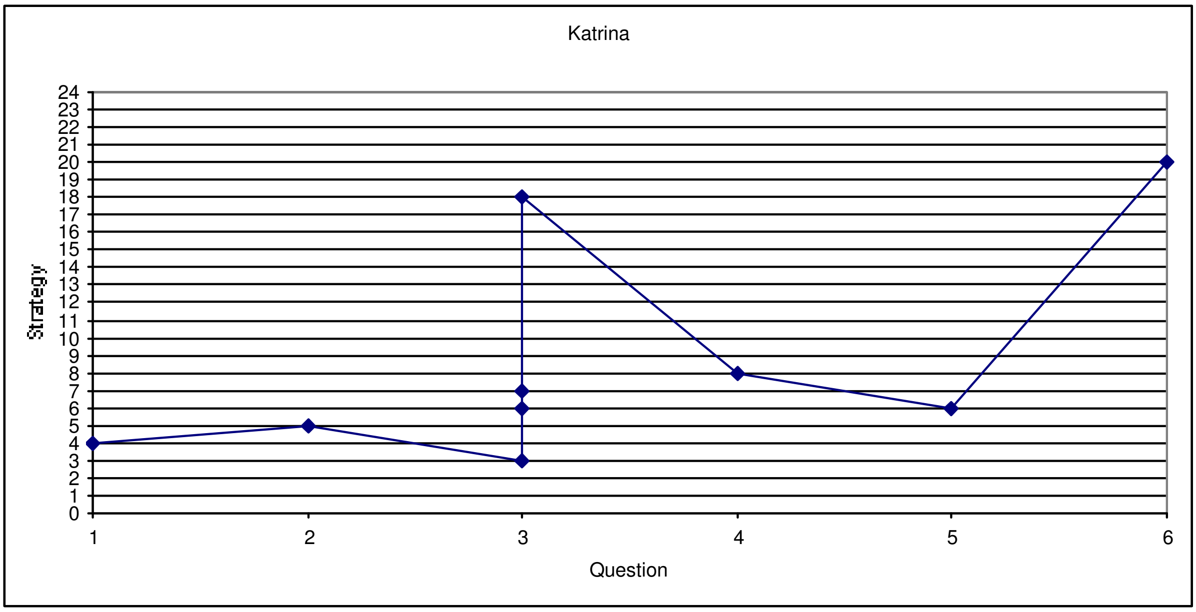


Figure 2: Katrina's Solution Path

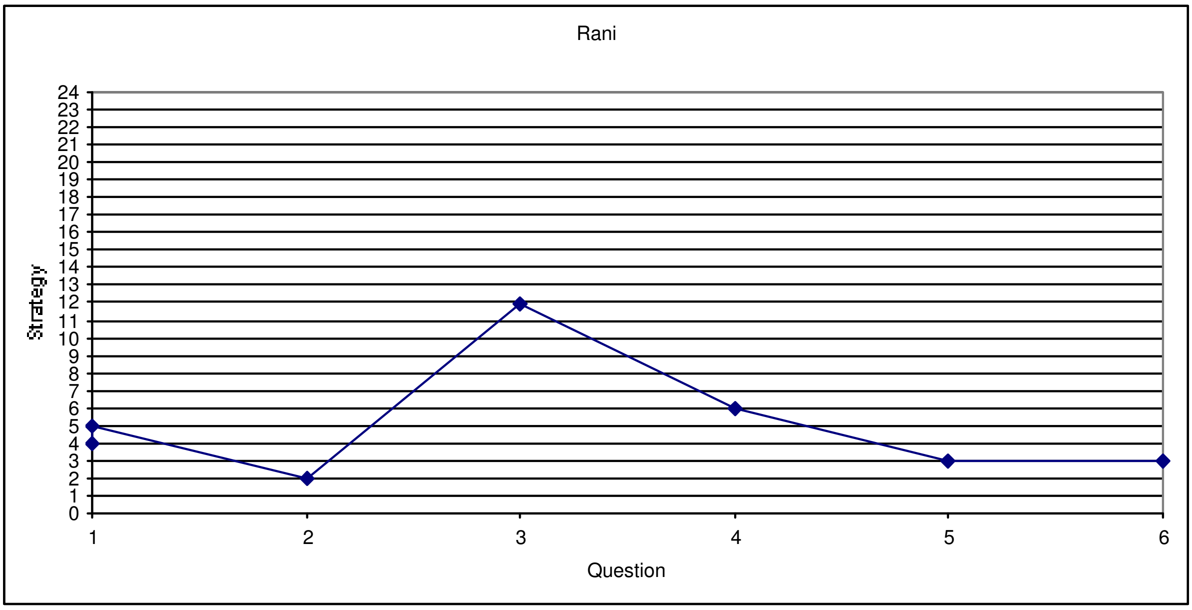


Figure 3: Rani's Solution Path

Inability to Generalize (S3). Table 3 below shows the results on generalization of the 22 students. Two of the 13 classified as unable to generalize had no success on any part of the problem, while the rest were able to do the first three parts of the task.

Category	Number
Able to generalize all parts	5
Able to generalize partially	4
Unable to generalize	13

Table 3: Summary of Results on Generalization

Of the remaining 11 who were unable to generalize, all but one started with a visual strategy but transitioned to a numerical one. At that point, they generally did not return to the visual cues at all, even when they got stuck using their numerical strategies. The most common numerical strategy was to extend the table. One student confused the roles of the independent and dependent variables, and another left out the independent variable. The four students who were able to generalize partially did parts 4 and 5 correctly but completed part 6 by indicating in words or symbols (e.g., B+W) to add the number of black and white tiles; that is, they did not find an explicit formula for the total number of tiles in terms of the pattern number as asked for in the task.

DISCUSSION

This study is consistent with findings from an earlier study we conducted with preservice elementary teachers (Rivera & Becker, 2003) as well as work done by Küchemann (1981) and Stacey & Macgregor (2000). Overall, students' strategies appeared to be predominantly numerical. In this study we identify three types of generalization based on similarity (numerical, figural. and pragmatic) in accord with findings by Gentner (1989) in which children were shown to exhibit different similarity strategies when making inductions involving everyday objects. Students who use numerical generalization employ trial and error as a similarity strategy with no sense of what the coefficients in the linear pattern represent. The variables are used merely as placeholders with no meaning except as a generator for linear sequences of numbers, with lack of representational fluency. Students who use figural generalization employ perceptual similarity strategies in which the focus is on relationships among numbers in the linear sequence. Variables are seen as not only placeholders but within the context of a functional relationship. Students who use pragmatic generalization employ both numerical and figural strategies and are representation-ally fluent; that is, they see sequences of numbers as consisting of both properties and relationships. We see that figural generalizers tend to be pragmatic eventually. Finally, students who fail to generalize (disjunctive generalizers) tend to start out with numerical strategies; however, they lack the flexibility to try other approaches and see possible connections between different forms of representation and generalization strategies.

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