STUDENT THINKING STRATEGIES IN RECONSTRUCTING THEOREMS

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A mathematics university student as a future mathematician should have the ability to find "new" mathematics structures or construct theorems based on particular axioms. That ability can be created by using problem posing tasks. To do the tasks, students with different abilities will use different thinking strategies. To understand them exactly, we conducted descriptive research. The high group initiated the process of reconstructing theorems with identifying and understanding axioms, making a visual diagram or making a conjecture and constructing the "new" theorem. The modest group began with understanding information (making a definition), drawing diagrams and calculating the number of lines and parallel lines, then constructing theorems. That pattern is similar for the low group.

INTRODUCTION

The fundamental changing of mathematics structure especially geometry occurred when axioms (i.e., the parallel axiom (postulate)) of Euclid geometry were modified. It fostered the developmental of non Euclid geometry, which was developed by Janos Bolyai, N. Lobachevsky and Rieman. Bolyai and Lobachevsky constructed a new geometry structure by changing the Euclid parallel axiom with the statement: "There exists a line l and a point **A** such that at least two distinct lines pass through A that are parallel to l". The statement of this geometry seems of questionable truth, but in the end it was useful and important in application when modern physics developed rapidly. This parallel postulate is known as the Hyperbolic Parallel Postulate and its Geometry is called Hyperbolic Geometry. In the mathematics deductive structure, this creatively invented axioms of Bolyai and Lobachevsky generated consistent theorems and no contradiction each other. Another creation by changing the parallel axiom of Euclid was done by the Germany mathematician Bernhard Riemann. He changed the parallel axiom of Euclid to the statement: "Given any line l and any point P not on l, there is no line through P that is parallel to l". This axiom is known as the Elliptic parallel axiom and its geometry is called Elliptic Geometry.

Based on that axiomatic system, they derived different theorems such as the following:

Hyperbolic Geometry:	the angle sum of triangle is less than 180°.
Euclid Geometry:	the angle sum of triangle is equal to 180°.
Elliptic Geometry:	the angle sum of triangle is more than 180°.

(Wallace and West, 1992).

The innovation of such a mathematics structure is initially a creation of a mathematician who is challenged to explore a new structure or to prove a truth with a different method. Such a creation may lead to a structure that becomes its own right. Bell (1981) pointed out that the first most important activity of research mathematicians is creating new mathematics and discovering relationships within and among mathematical structures. The second most important work of mathematicians is to demonstrate theorem proving to the satisfaction of the mathematical community. Occasionally, mathematicians are curious to prove theorems with different methods. An example is the Pythagoras' theorem which is already proved with more than 100 methods. Modification of Pythagoras' theorem developed with attempting to discover some positive integers such that $x^n + y^n = z^n$, for $n > 2$. This theorem is known as Fermat's last theorem.

The activity to discover and construct axioms or theorems is very important to students who study mathematics in order to understand and involve them in constructing new mathematical structures. Otherwise, they will ask questions such as: "how did mathematicians discover this theorem or theory? Where does it come from? Is there anything to encourage them constructing a conjecture? What is the next developing?" Our students must make a final project which one alternative is to construct some theorems based on particular axioms or initial theorems. Thereby, when the lecturer teaches about a mathematics structure, they should not just introduce and prove theorems and let student's asking questions go unsatisfied. If it happens, their motivation will decrease and their attention becomes weak. Of course, this situation would disadvantage the development of mathematics. A lecturer should teach students how to discover and generate theorems or mathematics structures. The question is how to train them to develop such skill?

Villiers (1995) suggested problem posing activities or constructing conjectures at regular intervals in classrooms and encouraged students to formulate their own questions and to investigate them. At the heart of making conjectures and problem posing lies the ability to look and ask questions from different perspectives. For example, a good habit is to ask questions such as "what happen if it is changed?", "what happen if…?", "what if not", which will direct the students to form conjectures or new theorems. Lecturers should explain that an intelligent mathematician is not just as a good problem solver but also a creative problem poser. A mathematician's task is never stopped, they continually look back to the original problem or its solution and pose questions related with original or initial problems. Based on Villers's experience, this activity motivated students because they were involved in constructing and proving theorems. Beside that, problem posing gives other benefits, such as to increase problem solving ability, making students be active, and enriching fundamental concepts.

Silver and Cai (1996:292) has noted that the term "problem posing" is generally applied to three quite distinct forms of mathematical cognitive activity:

1. Pre-solution posing, in which one generates original problems from a presented stimulus situation;

2. Within-solution posing, in which one reformulates a problem as it is being solved;

3. Post solution posing, in which one modifies the goals or conditions of an already solved problem to generate new problems.

This investigation uses a form of pre-solution posing. Students pose or generate a problem (theorems) from a stimulus situation (an axiomatic system).

Considering the above argument, I saw that problem posing can be used as an alternative to train student's skill in reconstructing theorems. This activity can be given mainly in the Mathematics Foundation Course, such as logic or set theory. The description of this course is to give understanding and training to set up deductive reasoning which is systematically and regularly based on language and principles of logic and set theory (UNESA Handbooks, 2000). In such a course, mathematical content is divided into 3 parts that are axiomatics, logic and set theory. In axiomatics, students can be taught to construct theorems based on some given axioms and try to prove the truth of these theorems, so that is compatible with deductive-axiomatic structure in mathematics. To understand the result of implementation of the problem posing task, I conducted research which explored the students' ability after they are trained using that task and the difficulties which are experienced by students. The result of that research (Siswono, 2004) pointed out that students encountered some difficulties, such as (a) to determine a number of lines which are constructed, (b) to understand a definition, (c) to understand axioms (stimulus situation/information), (d) to implement a constructed sketch or distinguish between their concepts and sketches, (e) to prove or describe their own theorems and (f) to set up the language of theorems.

I also interviewed some students to grasp their thinking strategy when they were posing a theorem. Thinking is a process by which a new mental representation is formed through the transformation of information with complex interaction of mental attributes of judging, abstracting, reasoning, imagining and problem solving (Solso, 1995). When students face a problem posing task, they are encouraged to think aloud to propose a theorem. They use their skill of judging, abstracting, reasoning, imagining and solving the task as a problem. Occasionally, students use different strategies to transform or interpret information. Krutetskii (1976) explained the term "mathematical cast of mind" which refers to a tendency to interpret the world mathematically. He identified three basic types of mathematical cast of mind: the analytic type (who tends to think in verbal-logical terms), the geometric type (who tends to think in visual-pictorial terms) and the harmonic type (who combines characteristic of the other two). These types can be used as a basis to look at students' thinking strategy. Students' thinking strategy in this research is defined as a process or stage when they are constructing a theorem; how they are understanding the axiom system and what their considerations are to decide on a theorem. To identify their

strategies clearly, I focused on three students group: a high, modest, and low group. This classification is based on the score of the pre-test.

METHOD

Subject of this research is 34 mathematics students at The Department of Mathematics, Faculty of Mathematics and Natural Sciences, the Surabaya State University in academic year 2003-2004. This is a descriptive research which tries to describe students' strategies in constructing theorems when they finished the problem posing task. Procedures of this research are as the following:

1. Teaching students to construct theorems based on the presented axiom system. The axioms are about Finite Geometry, known as Four-Point Geometry (Wallace and West, 1992). Because this system has not been taught to them, their ability is authentic.

2. Presenting a problem posing task to students. This task is to understand and identify students' ability in constructing theorems and their difficulties.

3. Analysing the students' problem posing task with the descriptive-qualitative method. Choose some students to be interviewed about their thinking strategy in constructing theorems from high, modest, and low group.

4. Writing the research report

Instruments of this research are a task of problem posing and the interview guidelines. The task of problem posing is as described below.

Consider this axiom system below.

1. There exist exactly four points and no three points in one line.

2. Any two distinct points have exactly one line on both of them.

3. Construct at least two theorems by deriving from the axiom system above. You can determine a definition first about a particular concept.

RESULT

I determined to interview in depth two students of the high group, three students of the modest group and three students of the low group. All groups said that they never have done the task in other mathematics course. However, the task of proving theorems frequently is given by lecturers. Therefore, it is a new model of mathematical activity.

All of the subjects just posed theorems about a number of lines and a number of paired parallel lines. Almost all students just made two theorems. There are 14 students producing true theorems without making a mistake, ten students making one mistake and ten students making two mistakes. Examples of true theorems are the following.
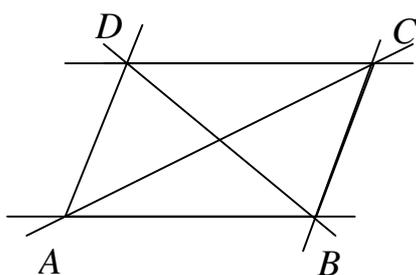
Example 1

Through 4 distinct points, just 6 straight lines exactly can be constructed.

Proof:

$$_4P_2 = \frac{4!}{2!(4-2)!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = \frac{12}{2} = 6$$

Example 2

Definition: Two straight lines are parallel if they don't have a common point whatever they are extended.

AB//CD, AD//BC, AC//BD.

Thus there are only 3 pairs of parallel lines.

An example of a mistaken theorem is the following.

Example 3

If there exist exactly four points and no three points in one line and any two distinct points have exactly one line on both of them, there are just 4 straight lines.

Proof:

$$_4K_3 = \frac{4!}{3!(4-3)!} \quad \text{(also known in some countries as } _4C_3 \text{)}$$

$$= \frac{4!}{3!}$$

$$= \frac{4 \times 3!}{3!}$$

$$= 4$$

That mistake may be caused by difficulties in drawing a sketch or understanding their sketch. Another possibility is that they can create a sketch although they didn't trust the number of lines shown by a sketch. They believe more in analytical procedures actually, but they don't understand the combinatorial concept ($_4K_3$).

The high group referred to a sketch to bridge them in deriving the theorem. They initiated the process of constructing theorems by identifying and understanding axioms, making a visual diagram (sketch) or making a conjecture and constructing the "new" theorem. When they thought they were making a mistake or a wrong theorem or the statement is not suitable with their sketch, they examined and kept some attention to the theorem and revised or regenerated it without changing a visual diagram. While they constructed a theorem, they never thought about a definition

because the task just asked them to construct a theorem. Another student gave the reason that her theorem didn't need a definition. They realized the important role of a definition although they did not write it down. They prefer a task to find a solution of problem or prove a theorem rather than construct theorems because it is difficult to understand axioms. The part of the interview of Iva is the following.

Interviewer: So what did you think when you constructed a theorem?

Iva: Well, Firstly I examined two axioms. Then I tried to draw a sketch by connecting some points and counting a number of straight lines.

Interviewer: When your conjecture was wrong, what did you do then?

Iva: I identify my theorem [conjecture] again and revise it.

Interviewer: Do you not change your sketch?

Iva: No I don't.

Interviewer: Why?

Iva: Because a sketch can be of many different forms, but actually they represent just one structure.

The modest group began by understanding information (making a definition), drawing diagrams and calculating the number of lines and parallel lines, then constructing theorems. When they thought they were making a mistake or a wrong theorem or the statement was not compatible with their sketch, they examine a visual diagram (a sketch) and they changed a diagram, then revised the theorem or regenerated a new one. While they were constructing a theorem, they never thought about a definition because the task just asked them to construct a theorem. A definition comes from axioms. They prefer a task to find a solution of problem rather than to prove a theorem or construct theorems. However, there were some students who thought that constructing theorems is easier than proving theorems. Sonie is one of those students.

Interviewer: When your conjecture is wrong, what do you do then?

Sonie: Change the diagram and see again all the axioms, and make a new theorem.

Interviewer: Do you prove it?

Sonie: No I don't. If it's wrong, I change the sketch.

Interviewer: Do you make a definition? What for?

Sonie: A definition is needed and a sketch is to help in constructing a theorem.

Interviewer: How do you make a definition?

Sonie: Based on two axioms, I combined some words and change with other word. A definition can be based on a sketch but a theorem not.

The low group initiated the process of constructing theorems by identifying and understanding axioms, making a visual diagram (sketch) or making a conjecture and constructing the "new" theorem. The stages seem similar with the high group. Actually, the quality of each step was different. They tried to understand the information even though sometimes they experienced difficulties. When they thought they were making a mistake or a wrong theorem or the statement was not in agreement with their sketch, some students used a different strategy. One student

directly revised or constructed a theorem without changing the visual sketch. Some students did not change theorems but they changed the visual diagram in order to make it agree with the theorem statement. While constructing a theorem, they never thought about a definition because they didn't understand the role of a definition. They prefer a task to find a solution of problem to the task of constructing theorems or proving theorems because it was difficult to understand axioms. This can be seen in a part of the transcribed interview of Bayu (the low group):

Interviewer: When your theorem was wrong, what did you do then?
Bayu: Changed and revised a diagram; the theorem is not changed.
Interviewer: What do you think about a definition?
Bayu: A definition is needed by a theorem.
Interviewer: But, you don't make it?
Bayu: Yes.
Interviewer: Well. What would you do to formulate a definition?
Bayu: I make a statement from terms of some axioms which is the meaning unclear.

DISCUSSION

This activity actually pointed out student representations of understanding a theorem and learning how to prove theorems. There are different strategies in the three groups, although they can be classified as the visual strategy or geometric type. The high group tends to think systematically. They did not explicitly say that they wrote down a definition before constructing theorems, but they realized the importance of the role of definition. Based on the theorems which were posed, we saw that they used the definitions of Euclidean Geometry about parallel lines, intersections and triangles. The modest group also tends to work systematically, even mentioning about the stage of making a definition. However, they never write down the definition. The actual difference with the high group, is that they sometimes change a sketch to match with the original theorem when they are not sure or are in doubt. This situation causes the possibility of mistakes.

The low group also followed similar stages to other groups. They do not make definitions because they don't understand the role of definitions. When they make a mistake, they frequently change the statement of the theorems or the sketch. These situations created very high potential for them to make other mistakes.

The classification of the students does not show their abilities in constructing theorems. The high group still makes some mistakes, such as formulating a sentence for a theorem. The low group also has some mistakes, basically caused by the weakness of their abstracting or understanding ability.

In terms of the thinking strategy, students tend to use the geometry type. This happened because the task directed them towards this type. However, if we look back at their proving theorems in determining a number of lines, there were two students from the high group using different strategies. One student used analytic type

thinking with the combinatorial concept. Another student determined it by calculating the number of lines based on a sketch or diagram, so she used geometric type. The modest groups contain two students using geometric and one student using analytic thinking. All students of the low group apply geometric thinking. To prove parallel lines, all students implement geometric thinking.

According to the students, this task was more difficult than proving theorems or finding a solution of a problem. The reasons are: it needs higher order thinking skill and requires understanding of information and axiom system. However, the problem posing task can still be used as an alternative to teaching thinking mathematically and creative thinking. As noted by Dunlop (2001), problem posing is a valid tool for teaching of mathematically thinking and it can foster a creative thinking.

SUMMARY

Students in the three groups have different strategies for constructing theorems. The difference will impact on their mistakes. Their thinking process leads to the following steps: identifying and understanding axioms, making a visual diagram or making a conjecture and constructing the "new" theorem.

This research has not been understood effectively in the teaching and learning process yet, so it needs further research. The task of problem posing should not only be the straight to geometry type, but also analytic type and harmonic type.

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