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MAXWELL'S VISION: ELECTROMATNETISM WITH HAMILTON'S QUATERNIONS

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We present two results that we have not found in the literature and that we believe therefore to be new, and some of their consequences. First, the Maxwell equations and the Lorentz force are formulated a with strict use of Hamilton's quaternions (two quaternion field equations and one quaternion force equation). Second, formulas for the Lorentz transformation, in fact for the 15 parameter conformal group, are presented, again with strict use of Hamilton's quaternions.

The first result was expected by Maxwell, but he did not complete this program. He presented the theory as eight field equations in Cartesian coordinates and, of course, did not include the three components of the Lorentz equation of motion. The task of reaching the first of our results has been discussed extensively with the use of biquaternions ("complex quaternions")[3]. While this direction is interesting in itself, we insist in the present work on the strict use of Hamilton's quaternions and prove that they are fully adequate for the task.

The second result of our paper, the formulas for the Lorentz transformations, was attempted by Dirac[2]. Dirac's analysis shows the existence of the subgroup of the algebraic field of quaternions that corresponds to Lorentz transformations in abstract terms, but does not reach explicit formulas. Biquaternions have also been used to characterize the Lorentz group from the early days[1][4].

We give below explicit quaternionic formulas for 3-space rotations, for the proper Lorentz transformations (boosts), and for proper conformal transformations (accelerations). We thus provide one explicit quaternion representation for the 15 paramter conformal group $C_{15} \sim \mathrm{SO}(2,4) \sim \mathrm{SU}(2,2)$. Two observations are in order: (1) our formula for space rotations is not identical but equivalent to the Rodriguez-Hamilton half-angle similarity formula. (2) One can see the proper conformal transformation in Dirac's paper but no explicit mention of the subject is made by him. In 1945, the C_{15} group was not as prominent in the mind of theoretical physicists as it is now.

The basic idea of our analysis is that any expression that involves three dimensional scalars and/or vectors (Gibbs' "scalar" and "vectors") can be written strictly in terms of Hamilton's quaternions. This "translation rule" extends to 4-scalars and 4-vectors as well as to tensors and interestingly to spinors using on the left-hand part of the Rodriguez-Hamilton similarity. Imitating Hamilton's formulation of the complex numbers as ordered pairs of reals, we view quaternions as embedded pairs of a Gibbs scalar (= 3-scalar) and a Gibbs 3-vector (= 3 vector). The formulation of classical electrodynamics which is the current standard, for example in the text of Stratton and Jackson, is based on Gibbs' vector algebra, two 3-vector and two 3-scalar field equations plus one 3-vector equation of motion. Consequently, our solution to both problems, the formulation of electrodynamics and the formulation of the Lorentz transformations, in strict quaternion terms, consists of a strict "translation" of Gibbs' formulas into Hamilton's counterparts.

Immediate consequences of our results are quaternion formulas for charge and energy conservation. These two basic conservation laws are based on a quaternion current and on a Poynting's quaternion, respectively.

We now give the essential formulas.

1. Quaternions
   A quaternion
   $q = (s, \overrightarrow{V})$
   The conjugate
   $q^* = (s, -\overrightarrow{V})$
   The 3-scalar
   $s = Sc(q)$
   The 3-vector
   $\overrightarrow{V} = Vect(q)$
   The anti-commutator
   $\{q_1, q_2\} = q_1 q_2 + q_2 q_1$
   The commutator
   $[q_1, q_2] = q_1 q_2 - q_2 q_1$
2. Fields (c = 1)
   The grad
   $\Box = (\partial_t, \overrightarrow{\nabla})$
   The potential
   $A = (\phi, \overrightarrow{A})$
   The electric field
   $E = (0, \overrightarrow{E}) = \frac{1}{2} Vect(\{\Box^*, A^*\})$
   The magnetic field
   $B = (0, \overrightarrow{B}) = \frac{1}{2}[\Box^*, A^*]$
   The current density
   $J = (\rho, \overrightarrow{J})$
3. The Maxwell Equations
   The homogeneous Maxwell equations
   $\{\Box, B\} + [\Box, E] = 0$

The inhomogeneous Maxwell equations
$([\Box, B] - \{\Box, E\})/2 = 4\pi J$
Electric charge conservation
$Sc(\Box^* \frac{1}{2}([\Box, B] - \{\Box, E\})) = Sc(\Box^* 4\pi J)$
$0 = \partial_t \rho + \overrightarrow{\nabla} \cdot \overrightarrow{J}$
Poynting's quaternion
$Sc(E^* \frac{1}{2}([\Box, B] - \{\Box, E\})) = Sc(E^* 4\pi J)$
$\overrightarrow{\nabla} \cdot (\overrightarrow{B} x \overrightarrow{E}) - \frac{1}{2}\partial_t^2 \overrightarrow{E} - \frac{1}{2}\partial_t^2 \overrightarrow{B} = 4\pi \overrightarrow{E} \cdot \overrightarrow{J}$
The Lorentz force
let e = electric charge
β = relativistic velocity, v/c
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
$u = \gamma(1, \overrightarrow{\beta})$
$F = \frac{e}{2}(\{u^*, E\}^* + [u, B])$

4. Transformations
Space rotations
let $\widehat{e}$= unit rotational 3-vector
$R : q \to q' = q \cos\Theta + \frac{1}{4}\{\{\widehat{e}, q\}, \widehat{e}\}(\cos\Theta - 1) - \frac{1}{2}[\widehat{e}, q] \sin\Theta$
Lorentz transformations
let $\widehat{v}$ = unit velocity 3-vector
$\Lambda : q \to q' = q + \frac{1}{4}(\gamma - 1)\{\{\widehat{v}, q\}, \widehat{v}\} + \frac{1}{2}\gamma\{\overrightarrow{\beta}^*, q^*\}$
Note: this has an antilinear term
Proper conformal (accelerations)
Let b = the acceleration parameter
$N = \sqrt{1 + 2Sc(bq) + Sc(q^2)Sc(b^2)}$
$C : q \to q' = N(q + bSc(q^2))$

Among the many formulations of electrodynamics known, an obvious competitor to quaternions is the Minkowski tensors in terms of succinctness. Both formulations consist of two field equations and a single force equation. In addition, both utilize a single current and a single potential. Quaternions have a simpler version of Poynting's Theorem but Minkowski extends to n-dimensional manifolds. We do not wish to express a preference in applications, but we believe that the quaternion formulation opens the way to asking novel and interesting questions such as the meaning of the antilinear terms in the Lorentz transformation and in the conservation laws.

References

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