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THE MAXIMAL G-COMPACTIFICATIONS OF G-SPACES WITH SPECIAL ACTIONS

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Abstract. An action on a G-space induces uniformities on the phase space. It is shown when the maximal G-compactification of a G-space can be obtained as a completion of the phase space with respect to one of these uniformities. Structure of G-spaces with special actions is investigated.

This paper is the continuation of the previous work of the authors [2] and is partially supported by Kungliga Vetenskapademian, project 12529. Besides old results which are now proved using another techniques the new ones are presented.

All spaces are assumed to be Tychonoff and mappings are continuous mappings of spaces. Let R denote the real numbers, and nbd is an abridged notation for neighbourhood.

Let G be a topological group. By a G-space X we mean a Tychonoff space X (phase space) with a continuous action of group G. If for a G-space X there exist a compact G-space bX and an equivariant dense embedding of X into bX then we call bX a G-compactification (see, for example, [3]). If a G-space has a G-compactification then (see, for example, [1]) there is the largest element $\beta_G X$ among all G-compactifications which is called the maximal G-compactification.

Uniform structures are introduced by coverings [4], and we say that the uniformity U_1 is finer than the uniformity U_2 if $U_2 \subset U_1$. If the topology of a topological space and the topology induced by a uniformity on it are the same then we say that the uniformity is compatible with the topology of the space. If X is a G-space then the uniformity U on X is called *invariant* if for any $g \in G$ and $\gamma \in U$ $g\gamma \in U$.

In 1975 J. de Vries [3] introduced the notion of a *bounded action* (an action on a space X is bounded if there exists a uniformity U on X compatible with its topology such that for any $u \in U$ there is a nbd O of identity in G such that the pair of points x and gx belong to one element of u for any $x \in X$

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and any $g \in O$) and proved that a G-space X has a G-compactification iff the action is bounded.

In 1984 M. G. Megrelishvili [5] introduced the concept of an *equiuniformity* (uniformity is equiuniformity if it is compatible with the topology of the phase space, invariant and the action is bounded by it) and proved the following theorem.

Theorem A. (M. G. Megrelishvili [5]) If U is an equiuniformity on a G space X then its completion $\tilde{X}$ with respect to U is a G space. Besides if $f : X \to Y$ is an equivariant uniformly continuous mapping to a complete uniform space Y then there exists the unique equivariant continuous mapping $\tilde{f} : \tilde{X} \to Y$ such that $\tilde{f} \circ i = f$ where i is a natural embedding of X into $\tilde{X}$.

The following statements are evident.

Proposition 1. *Let $U_\alpha, \alpha \in A$, be the family of equiuniformities on a G space X. Then its least upper bound is an equiuniformity.*

Corollary 1. *Among all equiuniformities there is a maximal one.*

Proposition 2. *If U is an equiuniformity on a G space X then the set of all coverings which can be refined by a finite covering from U is an equiuniformity.*

Corollary 2. *Among all equiuniformities there is a maximal totally bounded one.*

Let A be the family of all open nbds of the identity in G. Every $O \in A$ sets two coverings of a G-space X:

$$\gamma_O = \{Ox : x \in X\} \text{ and } \bar{\gamma}_O = \{\mathrm{cl}(Ox) : x \in X\}.$$

Denote by U_G ($\bar{U}_G$) the family of all coverings of X which have a refinement of the form γ_O ($\bar{\gamma}_O$), $O \in A$. It may be easily checked that the family U_G is a uniformity on X (not nessesary compatible with the topology of X).

Remark 1. If $\bar{U}_G$ is a uniformity on X then the uniformity U_G is finer than $\bar{U}_G$, but they may not be compatible with the topology of the phase space.

Now we shall reformulate J. de Vries's criterion mentioned above.

Theorem B. (J. de Vries [3]) A G-space X has a G-compactification iff there is a uniformity U on X compatible with its topology such that U_G is finer than U.

Let U^* be the totally bounded uniformity on the space X compatible with its topology such that any bounded continuous function on X is uniformly continuous with respect to it. It is the maximal totally bounded uniformity on X.

Theorem 1. *Let X be a G-space. If the uniformity U_G is finer than U^* then*

$$\beta_G X = \beta X.$$

Proof. It is easy to see that U^* is an equiuniformity and the rest follows from Corollary 2 and Theorem A. □

Lemma 1. *Let U be a uniformity on a G-space X compatible with its topology. If $\bar{U}_G$ is a uniformity on X then the following conditions are equivalent:*

(1) *U_G is finer than U;*
(2) *$\bar{U}_G$ is finer than U.*

Moreover, if the uniformity U_G is compatible with the topology of X then uniformities U_G and $\bar{U}_G$ are the same.

Proof. Let the uniformity U be generated by the family of coverings. In order to show (1) $\Rightarrow$ (2) for any $v \in U$ take $v' \in U$ such that v' is a star refinement of v. Then the covering $[v'] \in U$ which consists of closures of elements of v' is a refinement of v. Take $\gamma_O \in U_G$ such that γ_O is a refinement of v'. Then $\bar{\gamma}_O \in \bar{U}_G$ is a refinement of $[v']$. From this it follows that $\bar{\gamma}_O$ is a refinement of v and hence $\bar{U}_G$ is finer than U.

The implication (2) $\Rightarrow$ (1) follows from Remark 1.

From Remark 1 it follows that U_G is finer than $\bar{U}_G$. If U_G is compatible with the topology of X then instead of U we can take U_G in our lemma. Then $\bar{U}_G$ is finer U_G also. Hence U_G and $\bar{U}_G$ are the same. □

Proposition 3. *If $\bar{U}_G$ is a uniformity compatible with the topology of X then it is a maximal equiuniformity.*

Proof. Since the uniformity U_G is finer than $\bar{U}_G$, it follows from Theorem B that the action is bounded by the uniformity $\bar{U}_G$.

In order to prove that the uniformity $\bar{U}_G$ is invariant it is sufficient to show that for any $\bar{\gamma}_O = \{\mathrm{cl}(Ox) : x \in X\}$, $O \in A$, $g\bar{\gamma}_O \in \bar{U}_G$. Since for any $g \in G$ the mapping $g : X \to X, x \to gx$ is a homeomorphism it follows that $g(\mathrm{cl}(Ox)) = \mathrm{cl}((gO)x)$. Take $U = gOg^{-1}$. Then $U \in A$ and $(gO)x = (Ug)x$ for any $x \in X$. Thus $g\bar{\gamma}_O = \bar{\gamma}_U$.

Hence $\bar{U}_G$ is an equiuniformity. Its maximality follows from Lemma 1. □

The proof of the following theorem immediately follows from Proposition 3, Corollary 2 and Theorem A.

Theorem 2. *Let X be a G-space. If $\bar{U}_G$ is a uniformity compatible with the topology of X then*

$\beta_G X$ is the Samuel compactification of X with respect to $\bar{U}_G$.

The next example shows that the usage of uniformity $\bar{U}_G$ gives us more opportunities in finding maximal G-compactifications.

Example 1. Let $S = \{z \in C : |z| = 1\}$ be a unit circle on the complex plain, and a be such an element of S that $a^n \neq 1, n \in N$. Let us put $G = \{a^n : n \in Z\}$ (it is a group with a natural multiplication), $X = S \setminus G$ and the action of G on X is induced by multiplication in C.

The uniformity U_G is not compatible with the topology of X because the group G is countable and the cardinality of each nonempty open set of X

is uncountable and the uniformity $\bar{U}_G$ is compatible because G is a dense subset of S.

Remark 2. Earlier Theorems 1 and 2 were proved in another way in [2] using results of J. de Vries [3] and Yu. M. Smirnov [1].

We can characterize the case when $\bar{U}_G$ is the uniformity compatible with the topology of the phase space.

Theorem 3. *Let X be a G-space. The family $\bar{U}_G$ is a uniformity compatible with the topology of X iff the action has the property:*

(a) *for any $x \in X$ and any nbd $O \in A$ there exists $y \in X$ such that $x \in \operatorname{int}\operatorname{cl}(Oy)$.*

Proof. First of all let us notice that property (a) is equivalent to the following one:

the family $\{\operatorname{int}\operatorname{cl}(Ox) : x \in X\}$ is a covering of X for any nbd $O \in A$.

Since the universal uniformity is finer than $\bar{U}_G$ an open covering of X may be refined in any covering $\{\operatorname{cl}(Ox) : x \in X\}$ from $\bar{U}_G$. From this necessity immediately follows.

In order to prove sufficiency we must first of all check that $\bar{U}_G$ is the uniformity (see, for example, [4, page 524]). Recall that A is the family of all open nbds of identity in G.

1. Right from the definition of $\bar{U}_G$ it follows that if $\gamma \in \bar{U}_G$ and γ is a refinement of a covering β of X then $\beta \in \bar{U}_G$.
2. It is evident that if β_1 and $\beta_2 \in \bar{U}_G$ be such that $\bar{\gamma}_V$ and $\bar{\gamma}_W$ are refined in β_1 and β_2 for some $V, W \in A$ respectively, then for $O \in A$ such that $O \subset V \cap W$ we have that $\bar{\gamma}_O$ is refined both in $\bar{\gamma}_V$ and $\bar{\gamma}_W$ and hence in β_1 and β_2.
3. For $\beta \in \bar{U}_G$ let $V \in A$ be such that $\bar{\gamma}_V$ is refined in β. Take $O \in A$ such that $O = O^{-1}$ and $O^3 \subset V$. We shall prove that $\bar{\gamma}_O$ is a barycentric refinement of $\bar{\gamma}_V$.

   Let us show that $O\operatorname{cl}(Wx) \subset \operatorname{cl}(OWx)$ for any nbds O and W of identity in G. If $a \in O\operatorname{cl}(Wx)$ then $a = ht$, where $h \in O$ and $t \in \operatorname{cl}(Wx)$. Since the action is continuous for any nbd V_a of a there are a nbd V_t of t such that $hV_t \subset V_a$. Thus there is $t' \in V_t \cap Wx$ such that $ht' \in V_a$. Hence, $a \in \operatorname{cl}(OWx)$.

   For any $x \in X$ there exists $z \in X$ such that $x \in \operatorname{int}\operatorname{cl}(Oz)$. Now if $x \in \operatorname{cl}(Oy)$ then $\operatorname{cl}(Oz) \cap Oy \neq \emptyset$ since $\operatorname{int}\operatorname{cl}(Oz)$ is a nbd of x. From this it follows that

$$y \in O^{-1}\operatorname{cl}(Oz) \subset \operatorname{cl}(O^2 z) \text{ and } Oy \subset O\operatorname{cl}(O^2) \subset \operatorname{cl}(O^3 z) \subset \operatorname{cl}(Vz).$$

   Thus $\operatorname{cl}(Oy) \subset \operatorname{cl}(Vz)$ and so $\operatorname{st}(x, \bar{\gamma}_O) \subset \operatorname{cl}(V_z)$. Hence, $\bar{\gamma}_O$ is a barycentric refinement of $\bar{\gamma}_V$.

   Using the same process, we can find a barycentric refinement of $\bar{\gamma}_O$ which would be the star refinement of $\bar{\gamma}_V$ [4, Lemma 5.1.15].

4. Let x, y be a pair of distinct points of X. Since the action is continuous and X is a Tychonoff space there are nbd $O \in A, O^{-1} = O$ and nbds W_x, W_y of x and y respectively, such that $\mathrm{cl}(OW_x) \cap \mathrm{cl}(OW_y) = \emptyset$. Let us show that no element of the cover $\bar{\gamma}_O$ contains both x and y. Indeed, if $x \in \mathrm{cl}(Oz)$ and $y \in \mathrm{cl}(Oz)$ for some $z \in X$ then $W_x \cap Oz \neq \emptyset$ and $W_y \cap Oz \neq \emptyset$. From this it follows that $z \in O^{-1}W_x$ and $z \in O^{-1}W_y$ and, hence, $OW_x \cap OW_y \neq \emptyset$. This is a contradiction with the choice of nbds O, W_x and W_y.

So all conditions for the uniformity $\bar{U}_G$ are fulfilled.

Since for any $x \in X$ and any nbd O of identity in G there exists $z \in X$ such that $x \in \mathrm{int}\,\mathrm{cl}(Oz)$ then an open covering can be refined in any covering from $\bar{U}_G$. So every open set in topology induced by uniformity $\bar{U}_G$ is open in X. If W is open in X and $x \in W$ then there exist $O \in A$ and a nbd V of x such that $O = O^{-1}$ and $\mathrm{cl}(O^2V) \subset W$. If $x \in \mathrm{cl}(Oy)$ and $z \in \mathrm{cl}(Oy)$ then there exist $x_1 \in V$ and $h \in O$ such that $x_1 = hy$. From this it follows that $y \in Ox_1$ and $z \in \mathrm{cl}(O^2x_1) \subset \mathrm{cl}(O^2V) \subset W$. Hence, $\mathrm{st}(x, \bar{\gamma}_O) \subset W$ and so W is open in the topology induced be the uniformity. □

Proposition 4. *Consider the following properties for a G-space X.*

(a) *for any $x \in X$ and any nbd $O \in A$ there exists $y \in X$ such that $x \in \mathrm{int}\,\mathrm{cl}(Oy)$.*

(b) *for any $x \in X$ and any nbd $O \in A$ $x \in \mathrm{int}\,\mathrm{cl}(Ox)$,*

(c) *for any $x \in X$ and any nbd $O \in A$ $x \in \mathrm{int}(Ox)$,*

Then (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) *and the inverse implications are not valid.*

Proof. The implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) are evident.

(c) $\not\Longleftarrow$ (b). Consider the following example. Let Q be the set of rational numbers of the interval $I = (0, 1)$. There is a natural linear order on Q. Let G be a group of all order preserving homeomorphisms of Q [4, page 18] with the topology of uniform convergence [4, page 329]. Using Theorem 2 one may show that $\beta_G Q = [0, 1]$ and $X = I$ is an invariant subset of $\beta_G Q$. Now it is easy to see that a G-space X satisfies (b) but not (c).

(b) $\not\Longleftarrow$ (a). Consider the following example. Put $X = \beta_G Q$ where Q and G as above. It is easy to see that a G-space X satisfies (a) but not (b) because the action has fixed points. □

Remark 3. G-spaces with property (b) were examined by V. V. Uspenskiĭ in [6].

Below we shall describe G-spaces with properties listed above.

Lemma 2. *If a G-space X satisfies property* (a) *then for any points $x, y \in X$ we have either*

$$\mathrm{int}\,\mathrm{cl}(Gx) = \mathrm{int}\,\mathrm{cl}(Gy) \ \textit{or}\ \mathrm{int}\,\mathrm{cl}(Gx) \cap \mathrm{int}\,\mathrm{cl}(Gy) = \emptyset.$$

Proof. For the proof it is sufficient to show that

$$\text{if } \mathrm{int}\,\mathrm{cl}(Gx) \cap \mathrm{int}\,\mathrm{cl}(Gy) \neq \emptyset \text{ then } \mathrm{int}\,\mathrm{cl}(Gx) \subset \mathrm{cl}(Gy).$$

Let $z \in \operatorname{int}\operatorname{cl}(Gx)$ and O_z is an arbitrary nbd of z. We may take $z' \in O_z \cap Gx$, $z'' \in \operatorname{int}\operatorname{cl}(Gy) \cap Gx$ and $g \in G$ such that $gz'' = z'$. From the continuity of the action it follows that there exists a nbd $O_{z''}$ such that $gO_{z''} \subset O_z$. If we take $y' \in O_{z''} \cap Gy$ then $gy' \in O_z$ and hence $Gy \cap O_z \neq \emptyset$. Since O_z is an arbitrary nbd of z it follows that $z \in \operatorname{cl}(Gy)$ and so $\operatorname{int}\operatorname{cl}(Gx) \subset \operatorname{cl}(Gy)$. □

Corollary 3. *If a G-space X satisfies property* (a) *then either*

$$\operatorname{int}\operatorname{cl}(Gx) = \operatorname{cl}(Gx) \text{ or } \operatorname{int}\operatorname{cl}(Gx) = \emptyset$$

for any point $x \in X$.

Theorem 4.

A) *If a G-space X satisfies property* (a) *then X is a disjoint union of clopen sets and each clopen set from this union is the closure of an orbit of some point and thus it contains the continuous one-to-one image of some quotient space of group G.*

B) *If a G-space X satisfies property* (b) *then in addition to* A) *each such clopen set is the closure of orbit of its any point. But orbits of different points from the common clopen set may not be homeomorphic.*

C) *If a G-space X satisfies property* (c) *then each such clopen set is homeomorphic to the orbit of its any point.*

Proof. Proof of the statements A) and B) follows from Lemma 2 and Corollary 3. The second example from Proposition 4 shows that in case of action with property (a) not orbit of any point may be taken (there may be fixed points). In case of action with property (b) the first example from Proposition 4 shows that different orbits (rationals and irrationals) may not be homeomorphic.

In case of action with property (c) we have for any $x \in X$ an open mapping $g \to gx$ of G into X and hence a homeomorphism of some quotient space of G onto its orbit. □

The following questions are not yet known to authors.

Question 1. When does the family $\bar{U}_G$ generate uniformity on X (not nessesary compatible with the topology of X)?

Question 2. Let a G-space X satisfy property (a). Does there exist a dense invariant subspace X' of X such that the restriction of action on it has property (b)?

Question 3. Let a G-space X satisfy property (b). Does there exist a dense invariant subspace X' of X such that the restriction of action on it has property (c)?

Question 4. Can every compactification of a Tychonoff space be obtained as a G-compactification for some acting group G on X?

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