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ON LEBESGUE THEOREM FOR MULTIVALUED FUNCTIONS OF TWO VARIABLES

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Abstract. In the paper we investigate Borel classes of multivalued functions of two variables. In particular we generalize a result of Marczewski and Ryll-Nardzewski [6] concerning of real function whose ones of its sections are right-continuous and other ones are of Borel class α, into the case of multivalued functions.

1. Introduction

Many results were publisched about the Borel classification of multivalued functions depending on the one variable (see [5, 3, 1, 4, 7, 8]). In the case of multivalued function of two variables we have the possibility of formulation of hypotheses concerning of its sectionwise properties.

Lebesgue has shown that any real function f of two variables with continuous ones of its sections and of Borel class α the other ones is of Borel class $\alpha + 1$. Marczewski and Ryll-Nardzewski have shown (see [6]) that the condition of continuity in this theorem may be replaced by right-continuity (or left-continuity). In this paper we generalize these results into the case of multivalued functions in possible general abstract spaces.

2. Preliminaries

Let T and Z be two nonempty sets and let $\Phi : T \to Z$ be a multivalued function, i.e. Φ denotes a mapping such that $\Phi(t)$ is a nonempty subset of Z for $t \in T$. Then two inverse images of a subset $G \subset Z$ may be defined:

$$\Phi^+(G) = \{t \in T : \Phi(t) \subset G\}$$

and

$$\Phi^-(G) = \{t \in T : \Phi(t) \cap G \neq \emptyset\}.$$

The following relations hold betwen these inverse images:

$$\Phi^-(G) = T \setminus \Phi^+(Z \setminus G) \text{ and } \Phi^+(G) = T \setminus \Phi^-(Z \setminus G). \tag{1}$$

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Let $(T, \mathcal{T}(T))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. The notations $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ will be used to denote, respectively, the interior and the closure of a set A.

Definition 1. A multivalued function $\Phi : T \to Z$ is said to be $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous at a point $t \in T$ if

$$\forall G \in \mathcal{T}(Z)\ (\Phi(t) \subset G \Rightarrow t \in \mathrm{Int}\Phi^{+}(G))$$

(resp. $\forall G \in \mathcal{T}(Z)\ (\Phi(t) \cap G \neq \emptyset \Rightarrow t \in \mathrm{Int}\Phi^{-}(G))$).

F is called $\mathcal{T}(T)$-continuous at the point t if it is simultaneously $\mathcal{T}(T)$-upper and $\mathcal{T}(T)$-lower semicontinuous at t.

A multivalued function Φ being $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous at each point $t \in T$ is said to be $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous.

It is clear that a multivalued function Φ is $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous if and only if $\Phi^{+}(G) \in \mathcal{T}(T)$ (resp. $\Phi^{-}(G) \in \mathcal{T}(T)$), whenever $G \in \mathcal{T}(Z)$.

Given any countable ordinal number α, let $\sum_{\alpha}(T)$ and $\Pi_{\alpha}(T)$ denote the additive and multiplicative class α, respectively, in the Borel hierarchy of subsets of the topological space $(T, \mathcal{T}(T))$.

We shall always assume α to be an arbitrary countable ordinal number.

In perfect spaces the following inclusions hold:

$$\sum_{\alpha}(T) \subset \Pi_{\alpha+1}(T) \subset \sum_{\alpha+1}(T). \tag{2}$$

Definition 2. A multivalued function $\Phi : T \to Z$ will be said to be of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) Borel class α if

$$\Phi^{-}(G) \in \sum_{\alpha}(T)$$

(resp. $\Phi^{+}(G) \in \sum_{\alpha}(T)$), whenever $G \in \mathcal{T}(Z)$.

Let us note that a multivalued function of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) class 0 is $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) semicontinuous.

Let $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ be point-valued functions. Then a multivalued function $\Phi : T \to \mathbb{R}$ defined by formula

$$\Phi(t) = [f(t), g(t)] \subset \mathbb{R} \tag{3}$$

is of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) Borel class α if and only if f is of $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) and g is of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) class α in the Young classification.

In fact, for $a < b$ we have

$$\Phi^{-}((a,b)) = \{t \in T : f(t) < b\} \cap \{t \in T : g(t) > a\}$$

and

$$\Phi^{+}((a,b)) = \{t \in T : f(t) > a\} \cap \{t \in T : g(t) < b\}.$$

3. Main results

Let $F : X \times Y \to Z$ be a multivalued function and $(x_0, y_0) \in X \times Y$. Then a multivalued function $F_{x_0} : Y \to Z$ such that $F_{x_0}(y) = F(x_0, y)$ is called x_0-section of F. Similarly a multivalued function $F^{y_0} : X \to Z$ such that $F^{y_0}(x) = F(x, y_0)$ is called y_0-section of F.

Theorem 1. *Let (Y, d) be a metric space and $(X, \mathcal{T}(X))$, $(Z, \mathcal{T}(Z))$ two perfectly normal topological spaces. Let $\mathcal{T}(Y)$ be a topology on Y which is finer than the metric one and such that $(Y, \mathcal{T}(Y))$ is separable. Let S be a countable $\mathcal{T}(Y)$-dense subset of Y. Suppose that to every point $v \in Y$ there corresponds a subset $U(v) \in \mathcal{T}(Y)$ such that*

$$\forall y \in S \ B(y) = \{v : y \in U(v)\} \in \sum_{\alpha}(Y, d)$$

and

$$\forall v \in Y \ \mathcal{N}(v) = \{U(v) \cap B(v, 2^{-n}) : n = 1, 2, \ldots\},$$

where $B(v, 2^{-n})$ denotes the open ball centered in v with radius 2^{-n}, forms a filterbase of $\mathcal{T}(Y)$-neighbourghoods of the point v.

Assume that $F : X \times Y \to Z$ is a multivalued function whose all y-sections are of upper class α and all x-sections are $\mathcal{T}(Y)$-continuous. Then F is of lower class $\alpha + 1$ on the product $(X, \mathcal{T}(X)) \otimes (Y, d)$.

Proof. Let D be an arbitrary $\mathcal{T}(Z)$-closed subset of Z. By (1) it is enough to show that

$$F^+(D) \in \prod_{\alpha+1}((X, \mathcal{T}(X)) \otimes (Y, d)).$$

Since Z is perfectly normal, there is a sequence $\{G_n\}_{n\in\mathbb{N}}$ of $\mathcal{T}(Z)$-open sets such that

$$D = \bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(G_n) \tag{4}$$

and

$$\mathrm{Cl}(G_{n+1}) \subset G_n \text{ for } n \in \mathbb{N}. \tag{5}$$

Let $S = \{y_k : k \in \mathbb{N}\}$. We will prove that

$$F^+(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \subset G_n\} \times V_n(y_k)), \tag{6}$$

where

$$V_n(y_k) = \{v \in Y : y_k \in U(v)\} \cap B(v, 2^{-n}). \tag{7}$$

Let

$$(u, v) \in F^+(D) = \{(x, y) \in X \times Y : F(x, y) \subset D\}.$$

Then $F(u, v) \subset G_n$ for each $n \in \mathbb{N}$, by (4). Let n be fixed. By the $\mathcal{T}(Y)$-upper semicontinuity of the u-section of F at the point $v \in Y$ there is a $\mathcal{T}(Y)$-open neighbourhood $U(v) \in \mathcal{N}(v)$ of v such that $F(u, y) \subset G_n$ for any $y \in U(v)$.

Let

$$K = \{m \in \mathbb{N} : y_m \in U(v) \cap S\}$$

and let

$$k = \min\{m \in K : v \in V_n(y_m)\}.$$

Then

$$(u, v) \in [F^{y_k}]^+(G_n) \times V_n(y_k)$$

and the inclusion

(8) $$F^+(D) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \subset G_n\} \times V_n(y_k))$$

is proved.

Conversely, let (u, v) belongs to the right-hand side of (6). Suppose that $(u, v) \notin F^+(D)$. Then by (4) we must have

(9) $$F(u, v) \cap (Z \setminus \mathrm{Cl}(G_m) \neq \emptyset \text{ for some } m \in \mathbb{N}.$$

By $\mathcal{T}(Y)$-lower semicontinuity of the u-section of F at the point $v \in Y$ there is a $\mathcal{T}(Y)$-open neighbourhood $W(v) \in \mathcal{N}(v)$ of v such that

(10) $$F(u, y) \cap (Z \setminus \mathrm{Cl}(G_m) \neq \emptyset \text{ for any } y \in W(v).$$

We have supposed that

$$(u, v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \subset G_n\} \times V_n(y_k)).$$

Therefore we conclude from (4) that to each n there corresponds an index $k = k(n)$ such that

(11) $$F(u, y_{k(n)}) \subset G_n.$$

For $v \in V_n(y_{k(n)}) \subset B(v, 2^{-n})$ we obtain $\lim_{n\to\infty} d(v, y_{k(n)}) = 0$. Since $y_{k(n)}$ tends to v in (Y, d) as n tends to infinity, (10) and (11) show that there is an index n_0 such that

(12) $$F(u, y_{k(n)}) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset \text{ for any } n > n_0.$$

By (5) and (11) we have

$$F(u, y_{k(n)}) \subset G_n \subset G_{n-1} \subset \dots$$

for $n \in \mathbb{N}$.

In particular,

$$F(u, y_{k(n+j)}) \subset G_{n+j} \subset G_n$$

for any $j \in \mathbb{N}$. Fixing now $n = m$ (see (9)) we obtain $F(u, y_{k(m+j)} \subset G_m$ for any $j \in \mathbb{N}$, which contradicts (12). We must have

$$\exists n \in \mathbb{N}\ \forall y \in S\ v \notin V_n(y) \vee F(u, y) \not\subset G_n.$$

This formula means that

$$(u, v) \notin \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} ([F^{y_k}]^+(G_n) \times V_n(y_k))$$

and the inclusion

$$\bigcap_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}(\{x : F(x,y_k)\subset G_n\}\times V_n(y_k))\subset F^+(D) \tag{13}$$

holds. By (8) and (13) the equality (6) is proved.

Observe that

$$\{x : F(x,y_k)\subset G_n\}\in\sum_\alpha(X,\mathcal{T}(X))$$

since y_k-section of F is of upper class α. Furthermore it is assumed that $V_n(y_k)\in\sum_\alpha(Y,d)$. Therefore by (6) $F^+(D)$ is a countable intersection of countable unions of the sets of the class

$$\sum_\alpha(X,\mathcal{T}(X))\otimes\sum_\alpha(Y,d)\subset\sum_\alpha(X\times Y),$$

where $X\times Y$ is the product of topological spaces $(X,\mathcal{T}(X))$ and (Y,d). This completes the proof of Theorem 1. □

We give below two examples of topology $\mathcal{T}(Y)$ on Y fulfilling requirements of Theorem 1. From these examples it will be clear, that the x-sections of a multivalued function F in Theorem 1 may be either all right-continuous or all left-continuous in some meaning.

Example 1. Let $(Y,\diamond,d)$ be a topological group, whose topology is induced by an invariant distance function d (i.e. $d(\theta,y)=d(v,y\diamond v)$), where θ denotes a neutral element of Y. Assume furthermore that (Y,d) is separable.

Let $U\subset Y$ be an open set such that θ is an accumulation point of U. Let

$$U_n=(B(\theta,2^{-n})\cap U)\cup\{\theta\}\text{ and } V_n(y)=y\diamond U_n=\{y\diamond v : v\in U_n\}$$

for $n\in\mathbb{N}$. Then $\{V_n(y)\}_{n\in\mathbb{N}}$ forms a filterbase of neighbourhoods of a point $y\in Y$ and the topology $\mathcal{T}(Y)$ in Y generated by this base fulfils all requirements of Theorem 1.

Indeed, it suffices to prove that $\{U_n\}_{n\in\mathbb{N}}$ forms a base of neighborhoods of θ. We have

$$U_n\cap U_m=U_{\min(n,m)}.$$

Let $n\in\mathbb{N}$ and $v\in U_n$. Then there is $k\in\mathbb{N}$ such that

$$B(v,2^{-k})=v\diamond B(\theta,2^{-k})\subset U_n.$$

Therefore

$$\forall n\in\mathbb{N}\ \forall v\in U_n\ \exists k\in\mathbb{N}\ V_k(v)\subset U_n.$$

A countable dense subset of (Y,d) is also $\mathcal{T}(Y)$-dense. It remains to show that $V_n(y)$ is a Borel set in (Y,d) for any $n\in\mathbb{N}$. Let $n\in\mathbb{N}$ and let $\Phi : Y\to Y$ be a multivalued function defined by formula $\Phi(y)=V_n(y)$. Then Φ is continuous and and its graph

$$\mathrm{Gr}(\Phi)=\{(y,v) : v\in\Phi(y)\}$$

is homeomorphic to the set

$$Y \times U_n \in \sum_1 (Y,d) \otimes (Y,d) \cap \prod_1 (Y,d) \otimes (Y,d).$$

Finally $V_n(y) \in \sum_1(Y,d) \cap \prod_1(Y,d)$ for each $n \in \mathbb{N}$.

Example 2. Let $(Y,d,\leq)$ be a linearly ordered metric space. We follow Dravecky and Neubrunn (see [2]) in assuming that the space $(Y,d,\leq)$ has the property $\mathcal{U}$, i.e. $(Y,\leq)$ is linearly ordered and there is a countable dense set S in $(Y,d,)$ such that for any $y \in Y$ we have $y = \lim_{n\to\infty} y_n$, where $y_n \in S$ and $y \leq y_n$ for $n \in \mathbb{N}$. Then the topology $\mathcal{T}(Y)$ on Y generated by all open sets in (Y,d) and also by all intervals $I_a = \{y \in Y : y \leq a\}$, $a \in Y$, fulfills the assumptions of Theorem 1. Indeed, let $y \in Y$ and $r > 0$. Then

$$U_r(y) = B(y,r) \cap I_y = \{x \in Y : d(x,y) < r \wedge x \leq y\}$$

is a $\mathcal{T}(Y)$-neighbourhood of the point y.

Let $x \in U_r(y)$. Then $x \in B(y,r)$ and $x \leq y$, and then there is $r_1 > 0$ such that $d(x,y) = r - r_1$. Let $\delta < \min(r,r_1)$. Then $B(x,\delta) \subset B(y,r)$. Let $n \in \mathbb{N}$ be such a number that $2^{-n} < \delta$. Then $U_{2^{-n}}(x) \subset U_r(y)$ and we see that $\{U_{2^{-n}}(y)\}_{n\in\mathbb{N}}$ forms a filterbase of $\mathcal{T}(Y)$-neighbourhoods of the point y.

The set S is also $\mathcal{T}(Y)$-dense. It remains to show that the set

$$V_r(y) = \{z \in Y : y \in U_r(Z)\}$$

is a Borel set in (Y,d). First we will show that

$$\text{(14)} \qquad \begin{array}{l} \text{If } y_0 \neq y \text{ and } y_0 \in V_r(y), \text{ then there exists } 0 < r_1 < r \\ \text{such that } U_{r_1}(y_0) \subset V_r(y) \end{array}$$

Suppose, contrary to our claim, that $U_{r_1}(y_0) \not\subset V_r(y)$ for any $r_1 < r$. Now let $n \in \mathbb{N}$ be such that $\frac{1}{n} < r$. Then there is y_n such that $y \leq y_n$ and $y_n \in U_{\frac{1}{n}}(y_0) \setminus V_r(y)$, and then

$$y \leq y_n \quad \wedge \quad d(y_n,y_0) < \tfrac{1}{n} \quad \wedge \quad y_n \leq y_0 \quad \wedge \quad (y_n \leq y \vee d(y_n,y) \geq r)$$

for $n > \frac{1}{n}$. If it were true that $d(y_n,y_0) < \frac{1}{n}$ and $y \leq y_n \leq y_0$ and $y_n \leq y$, we would have

$$\lim_{n\to\infty} y_n = y_0 = y,$$

in contradiction with $y \neq y_0$. Let $d(y_0,y) = \varepsilon$. If it were true that $d(y_n,y_0) < \frac{1}{n}$ and $d(y_n,y) \geq r$ we would have

$$r \leq d(y_n,y) \leq d(y_n,y_0) + d(y_0,y) < \frac{1}{n} + \varepsilon.$$

Then we would have $\frac{1}{n} > r - \varepsilon > 0$ for almost every $n \in \mathbb{N}$, which is impossible. This establishes (14).

Our next claim is that

$$\text{(15)} \qquad \begin{array}{l} \text{If } y_0 \neq y \text{ and } y_0 \in V_r(y), \text{then there is } \delta > 0 \\ \text{such that } B(y_0,\delta) \subset V_r(y). \end{array}$$

Indeed, according to (14) there is $r_1 \in (0, r)$ such that $U_{r_1}(y_0) \subset V_r(y)$. Let $\varepsilon = d(y_0, y) < r$ and let $\delta < \min(\varepsilon, r - \varepsilon, r_1)$. Let $z \in B(y_0, \delta)$. Then either $d(y_0, z) < \delta$ and $z \le y_0$ or $d(y_0, z) < \delta$ and $y_0 \le z$. In the first case $z \in U_\delta(y_0) \subset V_r(y)$. In the second one

$$d(z, y) \le d(z, y_0) + d(y_0, y) < \delta + \varepsilon < r - \varepsilon + \varepsilon = r$$

and $y \le z$ show that $z \in V_r(y)$. Combining these both results we conclude that $B(y_0, \delta) \subset V_r(y)$ and (15) is proved.

By (15) we see that the set

$$\{z \in Y : d(z, y) < r \wedge y \le z \wedge y \ne z\}$$

is open in (Y, d). Therefore

$$V_r(y) = \{y\} \cup \{z \in Y : d(z, y) < r \wedge y \le z \wedge y \ne z\} \in \sum_1 (Y, d) \cap \prod_1 (Y, d).$$

Note that this topology $\mathcal{T}(Y)$ may be viewed as a natural generalization of the known Sorgenfrey topology on the real line.

Corollary 1. *Let f be a real function defined on the product of perfectly normal toplogical space X and the real line $\mathbb{R}$. Let us suppose that all x-sections of f are right-continuous and all y-sections of f are of upper Young class α. Then f is of lower class $\alpha + 1$ on $X \times \mathbb{R}$, i.e. it may be represented as a point-limit of an increasing sequence of functions of upper Young class α.*

Proof. Let us note that a multivalued function $F : X \times \mathbb{R} \to \mathbb{R}$ defined by formula

$$F(x, y) = [2 - \arctan f(x, y), 2 + \arctan f(x, y)]$$

is of lower class $\alpha + 1$, by Theorem 1. Moreover for $a < b$ we have

$$F^-((a, b)) = \{(x, y) : 2 - \arctan f(x, y) < b\} \cap \{(x, y) : 2 + \arctan f(x, y) > a\}.$$

By (3) the function $g(x, y) = 2 - \arctan f(x, y)$ is of upper class $\alpha + 1$ and the function $h(x, y) = 2 + \arctan f(x, y)$ is of lower class $\alpha + 1$ in the Young classification, which finishes the proof of Corollary 1. □

The next theorem is a dualization of Theorem 1.

Theorem 2. *Let (Y, d) be a metric space and $(X, \mathcal{T}(X)), (Z, \mathcal{T}(Z))$ two perfectly normal topological spaces. Let $\mathcal{T}(Y)$ be a topology on Y which is finer than the metric one and such that $(Y, \mathcal{T}(Y))$ is seperable. Let S be a countable $\mathcal{T}(Y)$-dense subset of Y. Suppose that to every point $v \in Y$ there corresponds a subset $U(v) \in \mathcal{T}(Y)$ such that*

$$\forall y \in S \ B(y) = \{v : y \in U(v)\} \in \sum_\alpha (Y, d)$$

and

$$\forall v \in Y \ \mathcal{N}(v) = \{U(v) \cap B(v, 2^{-n}) : n = 1, 2, \ldots\},$$

forms a filterbase of $\mathcal{T}(Y)$-neighbourghoods of the point v. Let $F : X \times Y \to Z$ be a compact-valued multivalued function whose all y-sections are of lower class α and all x-sections are $\mathcal{T}(Y)$-continuous. Then F is of upper class $\alpha + 1$ on the product $(X, \mathcal{T}(X)) \otimes (Y, d)$.

Proof. Let D be an arbitrary $\mathcal{T}(Z)$-closed subset of Z and let $S = \{y_k : k \in \mathbb{N}$. We will first prove that

$$F^-(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \cap G_n \neq \emptyset\} \times V_n(y_k)), \tag{16}$$

where G_n are open subsets of Z fulfilling (4) and (5), while $V_n(y_k)$ is defined by the formula (7).

If

$$(u, v) \in F^-(D) = \{(x, y) : F(x, y) \cap D \neq \emptyset\},$$

then by (4) $F(u, v)$ has nonempty intersection with G_n for each $n \in \mathbb{N}$. Let n be fixed and arbitrary. By $\mathcal{T}(Y)$-lower semicontinuity of u-section of F at the point v there exists a $\mathcal{T}(Y)$-open neighbourhood $U(v) \in \mathcal{N}(v)$ of v such that $F(u, y) \cap G_n \neq \emptyset$ for all $y \in U(v)$. Taking k such that $v \in V_n(y_k)$ we have

$$(u, v) \in [F^{y_k}]^-(G_n) \times V_n(y_k) = \{x : F(x, y_k) \cap G_n \neq \emptyset\} \times V_n(y_k),$$

which gives

$$F^-(D) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \cap G_n \neq \emptyset\} \times V_n(y_k)).$$

Now let us suppose that

$$(u, v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \cap G_n \neq \emptyset\} \times V_n(y_k)).$$

Then to each n there corresponds an index $k = k(n)$ such that for $y_{k(n)} \in S$ we have $F(u, y_{k(n)}) \cap G_n \neq \emptyset$, and then by (5)

$$F(u, y_{k(n+j)}) \cap G_n \neq \emptyset \text{ for any } j \in \mathbb{N}. \tag{17}$$

If (u, v) were not in $F^-(D)$, by (4) we would have

$$F(u, v) \subset Z \setminus D = \bigcup_{n\in\mathbb{N}} (Z \setminus \mathrm{Cl}(G_n)).$$

The value $F(u, v)$ is a compact subset of Z and the sets $Z \setminus \mathrm{Cl}(G_n)$, $n \in \mathbb{N}$, create a decreasing sequence of open sets, i.e.

$$Z \setminus \mathrm{Cl}(G_n) \subset Z \setminus \mathrm{Cl}(G_{n+1}).$$

Therefore for some $m \in \mathbb{N}$ we have $F(u, v) \subset Z \setminus \mathrm{Cl}(G_m)$. Then by the $\mathcal{T}(Y)$-upper semicontinuity of u-section of F at the point $v \in Y$ we have $F(u, y) \subset Z \setminus \mathrm{Cl}(G_m)$ for $y \in W(v)$, where $W(v)$ is a certain neighbourhood of the point v, chosen from the postulated filterbase $\mathcal{N}(v)$. Since $y_{k(n)}$ tends in (Y,d) to v as n tends to infinity, by the above there exists an index n_0 such that $y_{k(n)} \in W(v)$ for $n > n_0$. Therefore

$$F(u, y_{k(n)}) \subset Z \setminus \mathrm{Cl}(G_m) \text{ for any } n > n_0. \tag{18}$$

Taking $n = m$ in (17) we have $F(u, y_{k(m+j)}) \cap G_m \neq \emptyset$ for any $j \in \mathbb{N}$, which contradicts (18). Thus the equality (16) is proved.

Since the y_k-section of F is of lower class α, we have

$$\{x : F(x, y_k) \cap G_n \neq \emptyset\} \in \sum_\alpha (X).$$

Moreover under the assumption of our theorem we have $V_n(y_k) \in \sum_\alpha (Y,d)$. Thus we conclude from (16) that

$$F^-(D) \in \sum_\alpha (X) \otimes \sum_\alpha (Y,d) \subset \sum_\alpha (X \otimes Y) \subset \prod_{\alpha+1} (X \otimes Y),$$

where $X \otimes Y$ is the product of topological spaces $(X, \mathcal{T}(X))$ and (Y,d), as required. The proof of Theorem 2 is finished. □

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