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ON LOWER SEMICONTINUOUS MULTIFUNCTIONS IN QUASI-UNIFORM AND VECTOR SPACES

ANDRZEJ SPAKOWSKI

Abstract. Given a cover $\mathcal{B}$ of a quasi-uniform space Y we introduce a concept of lower semicontinuity for multifunctions $F : X \to 2^Y$, called $\mathcal{B}$-lsc. In this way, we get a common description of Vietoris-lsc, Hausdorff-lsc, and bounded-Hausdorff-lsc as well. Further, we examine set-theoretical and vector operations on such multifunctions. We also point out that the convex hull of Hausdorff-lsc multifunctions need not to be Hausdorff-lsc except the case where the range space is locally convex.

1. Lower semicontinuous multifunctions

The two most known concepts of lower semicontinuity for multifunctions are the lower semicontinuity in Vietoris sense (V-lsc) and the lower semicontinuity in Hausdorff sense (H-lsc). Given a set Y we denote by 2^Y the family of all subsets of Y. Every map $F : X \to 2^Y$ will be called a *multifunction* from X to Y. Now, let X and Y be two arbitrary topological spaces. We say that a multifunction $F : X \to 2^Y$ is *V-lsc* at a point $x_0 \in X$ provided for every open $G \subset Y$ such that $F(x_0) \cap G \neq \emptyset$ there exists a neighbourhood $U(x_0)$ of x_0 such that $F(x) \cap G \neq \emptyset$ for every $x \in U(x_0)$. This is the first concept of lower semicontinuity.

Let $(Y,\mathcal{U})$ be a uniform space. Recall that every uniformity generate a topology, and a topological space is uniformizable provided it is a Tichonov space. For multifunctions from X to $(Y,\mathcal{U})$ we may formulate the second concept of lower semicontinuity. Namely, a multifunction $F : X \to 2^Y$ is called *H-lsc at* $x_0 \in X$ if for every $W \in \mathcal{U}$ there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \subset W(F(x)) \text{ for every } x \in U(x_0),$$

where

$$W(F(x)) = \{ y \in Y : (z, y) \in W \text{ for some } z \in F(x)\}.$$

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This is the second concept of lower semicontinuity. In particular, if Y is a topological vector space, with its natural uniformity generated by the neighbourhoods of 0, the condition of H-lsc can be written in the equivalent form:

$$F(x_0) \subset F(x) + V \text{ for every } x \in U(x_0),$$

where V is a neighbourhood of 0 and $C + D = \{c + d : c \in C,\, d \in D\}$ is the vector sum of sets C and D.

It is known that every topological space Y is *quasi-uniformizable* ([11, 9]). This means that there is family $\mathcal{U}$ of subsets of $Y \times Y$ such that:

(1) every $U \in \mathcal{U}$ contains the diagonal Δ of $Y \times Y$,
(2) $U, V \in \mathcal{U}$ implies that $U \cap V \in \mathcal{U}$,
(3) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$, where $V \circ V = \{(x, y) \in Y \times Y : (x, z), (z, y) \in V \text{ for some } z \in Y\}$,
(4) $U \in \mathcal{U}$ and $U \subset V$ implies $V \in \mathcal{U}$,
(5) the family $\{W(y) : W \in \mathcal{U},\ y \in Y\}$ is a neighbourhood system generating the topology of Y, where $W(y) = \{z \in Y : (y, z) \in W\}$.

Every such family is called a *quasi-uniformity of the topological space* Y.

If Y is only a set, then a family $\mathcal{U}$ of subsets of $Y \times Y$ satisfying properties (1)–(4) is called a *quasi-uniformity on* Y, and the pair $(Y, \mathcal{U})$ a *quasi-uniform space*. A quasi-uniform space $(Y, \mathcal{U})$ is a uniform space provided $\mathcal{U}$ have the following symmetric property: $W \in \mathcal{U}$ implies $W^{-1} \in \mathcal{U}$, where $W^{-1} = \{(z, y) \in Y \times Y : (y, z) \in W\}$.

In every topological space Y we have: for every $A \subset Y$ and every quasi-uniformity $\mathcal{U}$ of Y, $cl(A) = \bigcap\{W^{-1}(A) : W \in \mathcal{U}\}$. where $cl(A)$ denotes the closure of A in Y. The usage of W^{-1} is explained by the following facts:

(1) the sets $W(y)$, where $W \in \mathcal{U}$ and $y \in Y$, form a neighbourhood system for the topology of Y,
(2) for arbitrary $A \subset Y$ we have: $y \in cl(A)$, the closure of A, if and only if for every $W \in \mathcal{U}$, $W(y) \cap A \neq \emptyset$ if and only if there exists $a \in A$ such that $a \in W(y)$, or equivalently $y \in W^{-1}(a)$.

For a quasi-uniform $(Y, \mathcal{U})$ space the definition of H-lsc should be modified as follows. A multifunction $F : X \to 2^Y$ is called *H-lsc at* $x_0 \in X$ if for every $W \in \mathcal{U}$ there exists a neighbourhood V of x_0 such that

$$F(x_0) \subset W^{-1}(F(x)) \text{ for every } x \in V.$$

Note that we have the following property: F is H-lsc at x_0 if and only if $cl(F)$ is H-lsc, where $cl(F)$ is the closure multifunction of F, i.e. $cl(F)(x) = cl(F(x))$ for all $x \in X$. The basic relationships between V-lsc and H-lsc are well-known (see [4], [5]). Namely, if Y is a topological space, $\mathcal{U}$ a quasi-uniformity of Y and $F : X \to 2^Y$ a multifunction, then H-lsc of F at $x_0 \in X$ implies its V-lsc at x_0. The converse holds provided the set $F(x_0)$ is totally bounded. Recall that $A \subset Y$ is called *totally bounded* provided for every $W \in \mathcal{U}$ there exists a finite set $B \subset A$ such that $A \subset W^{-1}(B)$. In general, V-lsc need not imply H-lsc.

2. $\mathcal{B}$-LOWER SEMICONTINUITY

Penot [10] introduced a concept of bounded lower semicontinuity for multifunction from a topological space to a normed space. In [2] a similar idea is applied to convergence of sets, in particular, to convergence of epigraphs, with respect to the families of single subsets, compact subsets, weakly compact subsets, and of bounded subsets. Following this, we define an abstract concept of lower semicontinuity to unify the description of the above mentioned semicontinuities.

Let X be a topological space, $(Y,\mathcal{U})$ a quasi-uniform space, $\mathcal{B}$ a *cover* of Y, i.e., a family of nonempty subsets of Y such that $\bigcup \mathcal{B} = Y$. We say that $F : X \to 2^Y$ is $\mathcal{B}$-*lsc* at $x_0 \in X$ provided for every $W \in \mathcal{U}$ and every $B \in \mathcal{B}$ there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \cap B \subset W^{-1}(F(x)) \text{ for every } x \in U(x_0).$$

If $F(x_0) = \emptyset$, then F is trivially $\mathcal{B}$-lsc at x_0 for arbitrary cover $\mathcal{B}$. Note also the following simple observations and remarks:

(1) If $\mathcal{B} = \{Y\}$, then $\mathcal{B}$-lsc is simply the H-lsc.
(2) If $\mathcal{B}$ is a cover of Y and $\mathcal{B}_1$ the family of all finite unions of subsets of $\mathcal{B}$, then $\mathcal{B}$-lsc implies the $\mathcal{B}_1$-lsc.
(3) If $\mathcal{B} \subset \mathcal{B}_1$, then $\mathcal{B}_1$-lsc implies $\mathcal{B}$-lsc.
(4) If $\mathcal{B}$ is the family of all balls $B(r) = \{y \in Y : ||y|| < r\}$ of a normed space Y, then $\mathcal{B}$-lsc will be called also *bounded H-lsc*. This case is identical with the Penot's concept [10].
(5) If for every $B \in \mathcal{B}$ the multifunction $F_B(x) = F(x) \cap B$, $x \in X$, is H-lsc at x_0, then it is $\mathcal{B}$-lsc at this point. The converse, in general, does not hold. For instance, we can take: $F(0) = [0,1]$ and $F(x) = [0,1)$ for $x \in (0,1]$, and $\mathcal{B}$ the family of all singletons of $[0,1]$. Then F is $\mathcal{B}$-lsc but F_B, where $B = \{1\}$ is not H-lsc for $F(x) \cap \{1\} = \emptyset$ for all $x \in (0,1]$. For some positive results see [10].

Now, we show that if $\mathcal{B}$ is the family of all singletons of Y, or equivalently, the family of all finite subsets of Y, then $\mathcal{B}$-lsc is simply the V-lsc.

Theorem 2.1. *Let $\mathcal{B}$ be a cover of Y, $F : X \to 2^Y$, and consider the following three statements:*

(1) *F is H-lsc at x_0,*
(2) *F is $\mathcal{B}$-lsc at x_0,*
(3) *F is V-lsc at x_0.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*, and the converse implications does not hold.*

Proof. That (1) implies (2) is clear because

$$F(x_0) \cap B \subset F(x_0) \subset W^{-1}(F(x)).$$

Now assume that F is $\mathcal{B}$-lsc at x_0. In virtue of $\mathcal{B}$-lsc, we may assume that $\mathcal{B}$ contains all singletons of Y. We show that F is V-lsc at x_0. Given an open

$G \subset Y$ and $y_0 \in F(x_0) \cap G$ we take $W \in \mathcal{U}$ such that $W(y_0) \subset G$. By the $\mathcal{B}$-lsc there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \cap \{y_0\} \subset W^{-1}(F(x)) \text{ for all } x \in U(x_0).$$

This implies that $y_0 \in W^{-1}(y)$ for some $y \in F(x)$, or equivalently $y \in W(y_0)$ for some $y \in F(x)$. Consequently, $F(x) \cap G \neq \emptyset$ for all $x \in U(x_0)$. This proves that F is V-lsc at x_0. It remains to show that the converse implications need not to hold. Let $Y = \mathbb{R}^2$, $\mathcal{B}$ the family of all straight lines of Y through the origin, and consider the following two multifunctions: $F_1(t)$ the line $\{y = tx\}$, $F_2(t)$ the line $\{y = 1 + tx\}$, for $t \geq 1$. Observe that: F_1 is V-lsc but not $\mathcal{B}$-lsc at each point, while F_2 is $\mathcal{B}$-lsc but not H-lsc at each point. □

Theorem 2.2. *Let $F : X \to 2^Y$ be a multifunction and $\mathcal{B}$ be the family of all singletons of Y. Then F is V-lsc at x_0 if and only if it is $\mathcal{B}$-lsc at x_0.*

Proof. In virtue of the above theorem the second implication is clear. Now, assume that F is V-lsc at x_0 and $\mathcal{B}$ is the family of all singletons of Y. We show that F is $\mathcal{B}$-lsc at x_0. Let $B = \{y_0\}$. Then $F(x_0) \cap B$ is empty or equal B. Let $W \in \mathcal{U}$ be arbitrary. Then $W(y_0)$ is open and $W(y_0) \cap F(x_0) \neq \emptyset$. By the V-lsc of F at x_0 there exists a neighbourhood $U(x_0)$ of x_0 such that $W(y_0) \cap F(x) \neq \emptyset$ for all $x \in U(x_0)$. Thus for every $x \in U(x_0)$ there exists $y \in W(y_0) \cap F(x)$. From this we infer that $(y_0, y) \in W$, or equivalently $(y, y_0) \in W^{-1}$. Consequently, $y_0 \in W^{-1}(y) \subset W^{-1}(F(x))$. This proves that F is $\mathcal{B}$-lsc at x_0. □

A general, similar theorem exists for $\mathcal{B}$-lsc and V-lsc. First, let us introduce a generalized concept of a totally bounded set. Let $\mathcal{B}$ be a cover of Y, and $(Y,\mathcal{U})$ a quasi-uniform space. A set $A \subset Y$ will be called *$\mathcal{B}$-totally bounded* if for every $B \in \mathcal{B}$, the set $A \cap B$ is totally bounded. Note that, if $\mathcal{B} = \{Y\}$, then the $\mathcal{B}$-total boundedness is simply the usual total boundedness, and if $\mathcal{B}$ is the family of all singletons, then the $\mathcal{B}$-total boundedness is trivial: each subset of Y is $\mathcal{B}$-totally bounded.

Theorem 2.3. *Let $F : X \to 2^Y$, $x_0 \in X$, and $F(x_0)$ be $\mathcal{B}$-totally bounded, where $\mathcal{B}$ is a cover of Y. Then F is V-lsc at x_0 if and only if it is $\mathcal{B}$-lsc at x_0.*

Proof. It is clear that $\mathcal{B}$-lsc implies V-lsc. Now, let us assume that F is V-lsc at x_0 and $F(x_0)$ is $\mathcal{B}$-totally bounded. Let $B \in \mathcal{B}$ and $W \in \mathcal{U}$ be arbitrary. Let $V \in \mathcal{U}$ be such that $V \circ V \subset W$. Hence $V^{-1} \circ V^{-1} \subset W^{-1}$, and there exist $y_1, \dots, y_n \in F(x_0)$ such that

$$F(x_0) \cap B \subset \bigcup_{i=1}^{n} V^{-1}(y_i) = \bigcup_{i=1}^{n} V^{-1}(F(x_0) \cap \{y_i\}).$$

In virtue of the V-lsc at x_0 there exists a neighbourhood $U(x_0)$ of x_0 such that $F(x_0) \cap \{y_i\} \subset V^{-1}(F(x))$, for $i = 1, \dots, n$, and $x \in U(x_0)$. This implies that $F(x_0) \cap B \subset V^{-1}(V^{-1}(F(x)) \subset W^{-1}(F(x))$, which ends the proof. □

As a corollary, for $\mathcal{B} = \{Y\}$, we get the well-known equivalence between V-lsc and H-lsc whenever we consider totally-bounded valued multifunctions.

Remarks.

One can examine a concept of $\mathcal{B}$-usc described as follows: for every $B \in \mathcal{B}$ and every $W \in \mathcal{U}$ there exists a neighbourhood $U(x_0)$ of x_0 such that $F(x) \cap B \subset W(F(x_0))$ for every $x \in U(x_0)$. It is clear that V-usc always implies $\mathcal{B}$-usc. Unfortunately, if $\mathcal{B}$ is the family of all singletons, $\mathcal{B}$-usc multifunctions need not to be V-usc. For example, let $Y = \mathbb{R}^2$, $\mathcal{B}$ the family of all singletons of Y, and we take the following multifunction: $F(0)$ is the line $\{y = 0\}$, and $F(t)$ is the line $\{y = tx\}$, for $t > 0$. F is $\mathcal{B}$-usc at 0 but not V-usc at this point.

See [3] for the bounded H-usc, i.e. for the $\mathcal{B}$-usc with $\mathcal{B}$ being the family of all bounded subsets of Y.

It is clear that $\mathcal{B}$-lsc is topologizable provided $\mathcal{B}$ is the family of all singletons or $\mathcal{B} = \{Y\}$. In general, $\mathcal{B}$-lsc is not topologizable. To show this we use the following Diagonalization Criterion (see e.g. [4]):

Theorem 2.4. *Let T be a directed set and for each $t \in T$ there is another directed set $E(t)$. Then we define a new directed set $D = T \times \Pi_{t\in T} E(t)$ ordered as follows: $(t, (\alpha(t))) \leq (s, (\beta(t)))$ if and only if $t \leq s$ and $\alpha(t) \leq \beta(t)$ for each $t \in T$. Suppose that $z(t,\gamma)$, $t \in T$, $\gamma \in E(t)$, are elements of a topological space Z. Consider the following net: $z(t,\alpha) = z(t,\alpha(t))$, $(t,\alpha) \in D$, where $\alpha(t)$ is the t-coordinate of α. If $\lim_{t\in T} \lim_{\gamma\in E(t)} z(t,\gamma) = z$ then $\lim_{(t,\alpha)\in D} z(t,\alpha(t)) = z$.*

Now, we can construct an example of a non-topologizable $\mathcal{B}$-lsc. Observe first that $\mathcal{B}$-lsc is simply continuity with respect to the following $\mathcal{B}^-$-convergence: $A_\lambda \to A_0$ whenever for every $W \in \mathcal{U}$ and every $B \in \mathcal{B}$ there exists λ_0 such that

$$A_0 \cap B \subset W^{-1}(A_\lambda) \text{ for every } \lambda > \lambda_0.$$

Let $T = \mathbb{N}$ and for each $n \in T$ we take $E(n) = \mathbb{N}$. Let $Y = \mathbb{R}^2$ and $\mathcal{B}$ be the family of all straight line through the $(0,0)$. For every $n, k \in \mathbb{N}$ we denote: $A(n,k) =$ the line $\{y = (1/k)x + 1/n\}$, $A(n) =$ the line $\{y = 1/n\}$, $A(0) = B(0) =$ the line $\{y = 0\}$. It is easy to check $\mathcal{B}^-$-convergence: $\lim_n \lim_k A(n,k) = \lim_n A(n) = A(0)$.

On the other hand the convergence $\lim_{(n,\alpha)} A(n,\alpha(n)) = A_0$ does not holds for $A(0) \cap B(0) =$ the line $\{y = 0\}$ is not contained in any $A(n,\alpha(n)) + V$, where V is a neighbourhood of $(0,0)$. Thus the $\mathcal{B}^-$-convergence is not topologizable.

3. Unions and cartesian products

In this paragraph we deal with some set-theoretical operations on multifunctions, namely with unions and cartesian products (see [4]). Operation of intersection of multifunctions will be examined separately in the next paragraph.

Unions. Let X and Y be spaces and $F_i : X \to 2^Y$, $i \in I$, a family of multifunctions. The *union* $F = \bigcup_{i\in I} F_i$ *of multifunctions* F_i is defined by $F(x) = \bigcup_{i\in I} F_i(x)$, $x \in X$. It is known and easy to prove that the union of an arbitrary family of V-lsc multifunctions is V-lsc. However, the union of an infinite family of H-lsc multifunctions need not to be H-lsc. For instance, define $F_n(0) = [0, n]$ and $F_n(x) = [0, 1/x]$ for $x \in (0, 1]$, $n = 1, 2, \dots$. Then the multifunctions F_n are H-lsc at 0. But, their union $F = \bigcup_n F_n$ is not H-lsc at 0 for $F(0) = [0, +\infty)$.

Theorem 3.1. *Let X be a topological space, $(Y, \mathcal{U})$ a quasi-uniform space and $F_1, F_2 : X \to 2^Y$ multifunctions $\mathcal{B}$-lsc at $x_0 \in X$. Then the union multifunction $F = F_1 \cup F_2$ is $\mathcal{B}$-lsc at x_0.*

Proof. The proof is a consequence of the formula:

$$W^{-1}(F_1(x)) \cup W^{-1}(F_2(x)) = W^{-1}(F_1(x) \cup F_2(x)),$$

where $W \in \mathcal{U}$ and $x \in X$. □

It is clear that the above theorem holds for finitely many multifunctions, and need not hold when we consider an infinite family of multifunctions.

Products. Now, let us describe the cartesian product of multifunctions. Let X and Y_1, Y_2 be spaces and $F_i : X \to 2^{Y_i}$, $i = 1, 2$, multifunctions. The *product* of two multifunctions F_1 and F_2 is defined as the multifunction $F = F_1 \times F_2 : X \to 2^{Y_1 \times Y_2}$ such that $F(x) = F_1(x) \times F_2(x)$, $x \in X$. In particular, if $F_2(x) = Y_2$ for all x, or $F_1(x) = Y_1$ for all x, we will write simply, $F_1 \times Y_2$, or $Y_1 \times F_1$, respectively. Analogously, we define the product $\Pi_{i\in I} F_i$ of an arbitrary family of multifunctions F_i, $i \in I$. It is known that the product of an arbitrary family of V-lsc (H-lsc) multifunctions is also V-lsc (H-lsc). Remark that the product of H-lsc multifunctions has more complicated nature than the product of V-lsc ones. To formulate a general theorem for $\mathcal{B}$-lsc we need to consider the product of quasi-uniform spaces. First, we describe the product of two quasi-uniform spaces. Let $(Y_i, \mathcal{U}_i)$, $i = 1, 2$, be quasi-uniform spaces, and $P_i : Y_1 \times Y_2 \to Y_i$, $i = 1, 2$, be the projections, i.e. $P_i(y_1, y_2) = y_i$, $i = 1, 2$. By the *product quasi-uniformity* $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ *in* $Y = Y_1 \times Y_2$ we mean a quasi-uniformity generated by the base consisting of sets of the form

$$[W_1, W_2] = \{(z_1, z_2) \in Y \times Y : (P_i(z_1), P_i(z_2)) \in W_i,\ i = 1, 2\},$$

where $W_i \in \mathcal{U}_i$, $i = 1, 2$. In other words, the set $[W_1, W_2]$ has a form:

$$\{(s_1, y_1, s_2, y_2) \in Y_1 \times Y_2 \times Y_1 \times Y_2 : (s_1, s_2) \in W_1,\ (y_1, y_2) \in W_2\}.$$

Remark that $[W_1, W_2] = \tilde{W}_1 \cap \tilde{W}_2$, where

$$\tilde{W}_i = \{(z_1, z_2) \in Y \times Y : (P_i(z_1), P_i(z_2)) \in W_i\},\ i = 1, 2.$$

The sets $\tilde{W}_i$, $W_i \in \mathcal{U}_i$, $i = 1, 2$, form a subbase of the product quasi-uniformity $\mathcal{U}_1 \times \mathcal{U}_2$. In case of an arbitrary family of quasi-uniform spaces we proceed similarly as above and as in the construction of product topological structures. Let $(Y_i, \mathcal{U}_i)$, $i \in I$, be a family od quasi-uniform spaces. Denote: $Y = \Pi_{i\in I} Y_i$, P_i the projection on the i-th axis, i.e. $P_i(y) = y_i$, where y_i is the i-th coordinate of $y \in Y$, $i \in I$. By the *product quasi-uniformity* $\mathcal{U} = \Pi_{i\in I}\mathcal{U}_i$ *in* Y we mean a quasi-uniformity generated by the subbase consisting of sets of the form

$$\tilde{W}_i = \{(z_1, z_2) \in Y \times Y : (P_i(z_1), P_i(z_2)) \in W_i\},$$

where $W_i \in \mathcal{U}_i$, $i \in I$. Observe that if $\mathcal{B}_i$ is a cover of Y_i, $i \in I$, then $\Pi_{i\in I}\mathcal{B}_i$, i.e., the family of all sets of the form $\Pi_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ and $B_i = Y_i$ for all but a finite number of $i \in I$, is a cover of $\Pi_{i\in I} Y_i$.

Theorem 3.2. *Let* $(Y_i, \mathcal{U}_i)$, $i = 1, 2$, *be quasi-uniform spaces and* $A_i \subset Y_i$, $i = 1, 2$, *be arbitrary subsets. Then*

(1) $[W_1, W_2](A_1 \times A_2) = W_1(A_1) \times W_2(A_2)$,
(2) $[W_1, W_2]^{-1}(A_1 \times A_2) = W_1^{-1}(A_1) \times W_2^{-1}(A_2)$.

Proof. We have

$$\begin{aligned}
&[W_1, W_2](A_1 \times A_2)\\
&\quad = \{(s_1, s_2) \in Y_1 \times Y_2 : \exists y_1 \in A_1\ \exists y_2 \in A_2\ (y_1, y_2, s_1, s_2) \in [W_1, W_2]\}\\
&\quad = \{(s_1, s_2) \in Y_1 \times Y_2 : \exists y_1 \in A_1\ \exists y_2 \in A_2\ (y_1, s_1) \in W_1,\ (y_2, s_2) \in W_2\}\\
&\quad = \{(s_1, s_2) \in Y_1 \times Y_2 : s_1 \in W_1(A_1),\ s_2 \in W_2(A_2)\}\\
&\quad = W_1(A_1) \times W_2(A_2),
\end{aligned}$$

which proves (1). The proof of (2) is similar. □

Theorem 3.3. *Let* X *be a topological space,* $(Y_i, \mathcal{U}_i)$, $i \in I$, *a family of quasi-uniform spaces and* $F_i : X \to 2^{Y_i}$ *a multifunction* $\mathcal{B}_i$*-lsc at* $x_0 \in X$, *where* $\mathcal{B}_i$ *is a cover of* Y_i, $i \in I$. *Then the product multifunction* $\Pi_{i\in I} F_i$ *is* $\Pi_{i\in I}\mathcal{B}_i$*-lsc at* x_0.

Proof. By the construction of the product of quasi-uniform spaces $(Y_i, \mathcal{U}_i)$, $i \in I$, it is sufficient to consider only the case $I = \{1, 2\}$. In general case, the proof is similar. Let multifunctions $F_i : X \to 2^{Y_i}$, $i = 1, 2$, be $\mathcal{B}_i$-lsc at $x_0 \in X$ and $W \in \mathcal{U}_1 \times \mathcal{U}_2$ be arbitrary. There exist $W_i \in \mathcal{U}_i$, $i = 1, 2$, such that $[W_1, W_2] \subset W$. Now, let $B_i \in \mathcal{B}_1$. There exists a neighbourhood $U(x_0)$ of x_0 such that $F_i(x_0) \cap B_i \subset W_i^{-1}(F_i(x))$, $i = 1, 2$, $x \in U(x_0)$, and, by the Lemma 3.2, we get

$$(F_1(x_0) \cap B_1) \times (F_2(x_0) \cap B_2) \subset [W_1, W_2]^{-1}(F_1(x) \times F_2(x)),$$

for every $x \in U(x_0)$. This shows the $\mathcal{B}_1 \times \mathcal{B}_2$-lsc of $F_1 \times F_2$ because

$$(F_1(x_0) \cap B_1) \times (F_2(x_0) \cap B_2) = (F_1(x_0) \times F_2(x_0)) \cap (B_1 \times B_2).$$

□

Remark. The converse theorem also holds. Namely, if a product multifunction $\Pi_{i\in I}F_i$ is $\Pi_{i\in I}\mathcal{B}_i$-lsc at x_0, then for every $i \in I$ the multifunction F_i is $\mathcal{B}_i$-lsc at x_0.

4. Intersections

In optimization theory the lower semicontinuity properties of intersections of multifunctions play an important role [10]. The most wanted theorems are ones with no boundedness conditions on the values of intersecting multifunctions. Here we formulate a theorem of this kind. Let Y be a normed space. If we assume that the considered multifunctions are *boundedly H-lsc*, i.e., $\mathcal{B}$-lsc with $\mathcal{B}$ being the family of all balls $B(r)$, $r > 0$, then we may formulate a theorem on intersection, without boundedness conditions on $F(x_0)$. For this we need a lemma from [6] on interiority properties of convex, bounded, and with the nonempty interior subsets of a normed space.

Theorem 4.1. *Let Y be a normed space and $A \subset Y$ be convex, bounded, and with the nonempty interior. Then for every $\varepsilon > 0$ there exist a set $C \subset int(A)$ and $\delta > 0$ such that $C + B(\delta) \subset A \subset C + B(\varepsilon)$.*

We need also the following well-known (see [12, 14]) and very useful *law of cancellation*:

Theorem 4.2. *Let A, B and C be subsets of a topological vector space Y. Assume that B is bounded, and C is nonempty, closed and convex. Then $A + B \subset cl(C + B)$ implies $A \subset C$. In particular, $A + B \subset C + B$ implies $A \subset C$, and $cl(A + B) \subset cl(C + B)$ implies $A \subset C$.*

Let X be a topological space and Y a topological vector space. A multifunction F from X to Y is called *locally convex-valued* (*locally closed-valued*) *at* $x_0 \in X$ if there is a neighbourhood U of x_0 such that $F(x)$ is convex (closed) for every $x \in U$.

Theorem 4.3. *Let X be a topological space, Y a normed space, $\mathcal{B}$ the family of balls $B(r) \subset Y$, $0 < r$, F_1 and F_2 two multifunctions from X to Y and $F = F_1 \cap F_2$. If F_1 and F_2 are $\mathcal{B}$-lsc at $x_0 \in X$, locally convex- and locally closed-valued at x_0, and $int F(x_0) \neq \emptyset$, then F is $\mathcal{B}$-lsc at x_0, and hence, V-lsc at x_0.*

Proof. By the assumption on the interior of $F(x_0)$ in Y there exists $r > 0$ such that

$$int(F(x_0) \cap B(r)) \neq \emptyset. \tag{1}$$

Let $\varepsilon > 0$ be arbitrary. By the Lemma 4.1 there exist a subset $C \subset F(x_0) \cap B(r)$ and $\delta > 0$ such that $C + B(\delta) \subset F(x_0) \cap B(r) \subset C + B(\varepsilon)$. In virtue of the $\mathcal{B}$-lsc at x_0 there exists a neighbourhood $U(x_0)$ of x_0 such that

$$C + B(\delta) \subset F_i(x_0) \cap B(r) \subset F_i(x) + B(\delta) \text{ for } x \in U(x_0) \text{ and } i = 1, 2.$$

We can assume that the multifunctions are closed- and convex-valued on $U(x_0)$. Applying the law of cancellation, we infer that $C \subset F_1(x) \cap F_2(x)$ for every $x \in U(x_0)$. But this implies that

$$\begin{aligned} F_1(x_0) \cap F_2(x_0) \cap B(r) &\subset C + B(\varepsilon) \\ &\subset F_1(x) \cap F_2(x) + B(\varepsilon) \end{aligned}$$

for all $x \in U(x_0)$. This shows that the intersection $F = F_1 \cap F_2$ is $\mathcal{B}$-lsc at x_0 and ends the proof. □

Remark. If Y is finite dimensional we can omit in the above theorem the assumption that the multifunctions are locally closed-valued, and then proceed in a manner as in [6] using the below theorem on local interior property of $\mathcal{B}$-lower semicontinuous multifunctions.

5. Vector operations

Here we consider vector sum and convex hull operations on lower semicontinuous multifunctions with values in a topological vector space (see e.g. [7]). Let X be a topological space, Y a topological vector space and $F, G : X \to 2^Y$. We define two multifunctions:

$$\begin{aligned} (F+G)(x) &= F(x) + G(x) \\ &= \{a + b : a \in F(x),\ b \in G(x)\},\ x \in X, \end{aligned}$$

called the *vector sum* of F and G, and $conv(F)(x) = conv(F(x))$ the convex hull of $F(x)$, $x \in X$, called the *convex hull* of F.

Vector sum. It is known and easy to prove that the vector sum of two H-lsc multifunctions is H-lsc. We state some further results and show that, in general, the vector sum of two $\mathcal{B}$-lsc multifunctions need not to be $\mathcal{B}$-lsc.

Theorem 5.1. *Let $F, G : X \to 2^Y$ be V-lsc at $x_0 \in X$. Then the vector sum $F + G$ is V-lsc at x_0.*

Proof. Let V be an arbitrary neighbourhood of 0 in Y, F and G be V-lsc at x_0, and recall that V-lsc is equivalent to $\mathcal{B}$-lsc with $\mathcal{B}$ equals the family of all singletons of Y. Let $b \in Y$ be such that $(F(x_0) + G(x_0)) \cap \{b\} \neq \emptyset$. Then $b = b_1 + b_2$ with $b_1 \in F(x_0)$ and $b_2 \in G(x_0)$. By the $\mathcal{B}$-lsc of F and G there exists a neighbourhood $U(x_0)$ such that

$$\begin{aligned} &F(x_0) \cap \{b_1\} \subset F(x) + V \text{ and} \\ &G(x_0) \cap \{b_2\} \subset G(x) + V \end{aligned}$$

for all $x \in U(x_0)$. This implies that

$$b_1 + b_2 \in F(x) + G(x) + V + V$$

for $x \in U(x_0)$, or equivalently,

$$(F(x_0) + G(x_0)) \cap \{b\} \subset F(x) + G(x) + V + V$$

for $x \in U(x_0)$, which shows $\mathcal{B}$-lsc of $F + G$ at x_0 and ends the proof. □

We say that a cover $\mathcal{B}$ of Y is *translation invariant* if $B + c \in \mathcal{B}$ for every $B \in \mathcal{B}$ and every vector $c \in Y$.

Theorem 5.2. *Let $\mathcal{B}$ be a translation invariant cover of Y, $F : X \to 2^Y$ a multifunction $\mathcal{B}$-lsc at $x_0 \in X$ and $g : X \to Y$ a function continuous at x_0. Then the vector sum $(F + g)(x) = F(x) + g(x)$, $x \in X$, is a multifunction $\mathcal{B}$-lsc at x_0.*

Proof. Let V be a neighbourhood of 0 in Y and $B \in \mathcal{B}$. Note that

$$(F(x_0) + g(x_0)) \cap B = F(x_0) \cap (B - g(x_0)) + g(x_0)$$

By the assumptions for all x in a neighbourhood of x_0 we have

$$\begin{aligned} (F(x_0) + g(x_0)) \cap B &\subset (F(x) + g(x_0) + V) \cap B \\ &\subset (F(x) + g(x) + V + V). \end{aligned}$$

This shows $\mathcal{B}$-lsc of $F + g$ at x_0 and ends the proof. □

The following example shows that translation invariantness of $\mathcal{B}$ is not sufficient to get $\mathcal{B}$-lsc of the vector sum of a two $\mathcal{B}$-lsc multifunctions.

Example. Let $Y = \mathbb{R}^3$, $\mathcal{B}$ consists only of the plane $\{y = x\}$ and all of its translations. Define two multifunctions: $F(t)$ the line $\{z = ty,\ x = 0\}$ and $G(t)$ the line $\{y = 0,\ z = 0\}$, $t \geq 0$. Observe that F and G are $\mathcal{B}$-lsc but their vector sum $F + G$ is not $\mathcal{B}$-lsc at each point.

Convex hull. It is known [8] and easy to proof that for every V-lsc multifunction F, the convex hull of F is also V-lsc. We use the concept of $\mathcal{B}$-lsc to get a general result provided the space Y is locally convex. In particular, we get a result for H-lsc.

Theorem 5.3. *Let Y be a locally convex space and $F : X \to 2^Y$ be $\mathcal{B}$-lsc at $x_0 \in X$. Then the convex hull of F is $\mathcal{B}$-lsc at x_0.*

Proof. Let $V \subset Y$ be a convex neighbourhood of 0 in Y and $B \in \mathcal{B}$. By the $\mathcal{B}$-lsc of F at x_0 there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0) \cap B \subset F(x) + V \text{ for all } x \in U(x_0). \tag{2}$$

We claim that

$$conv(F(x_0)) \cap B \subset conv(F(x)) + V \text{ for all } x \in U(x_0).$$

Indeed, let $y \in conv(F(x_0)) \cap B$ and $x \in U(x_0)$ be arbitrary. In virtue of (2) there exist $n \in \mathbb{N}$, $y_1, \ldots, y_n \in F(x_0)$, $z_1, \ldots, z_n \in F(x)$, $v_1, \ldots, v_n \in V$, and positive numbers $t_1, \ldots, t_n$ such that $y = t_1 y_1 + \cdots + t_n y_n$, $t_1 + \cdots + t_n = 1$, and $y_i = z_i + v_i$ for every $1 \leq i \leq n$. This implies that

$$y = t_1 z_1 + \cdots + t_n z_n + t_1 v_1 + \cdots + t_n v_n \in conv(F(x)) + V$$

for V is convex, which ends the proof. □

Remark. If the topological vector space Y is not locally convex then the convex hull operation does not preserve H-lsc. Indeed, if Y is metrizable and not locally convex then there exists a sequence $y_n \in Y$ which converges to 0 such that the convex hull of the set $\{y_n : n = 1, 2, \ldots\}$ is not bounded [1]. Now, observe that the multifunction F defined by: $F(0) = \{0, y_1, y_2, \ldots\}$, $F(1/n) = \{0, y_1, \ldots, y_n\}$, $n = 1, 2, \ldots$, is H-lsc at 0 but the convex hull of F is not. For a simple example of such sequence $y_n \in l^p$ $(0 < p < 1)$ see [13] or [1].

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Instytut Matematyki i Informatyki, Uniwersytet Opolski,, Oleska 48, 45-052 Opole, Poland

E-mail address: aspakow@uni.opole.pl