Nonabsolutely convergent Poisson integrals

Erik Talvila

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton AB Canada T6G 2E2
etalvila@math.ualberta.ca

Abstract. If a function f has finite Henstock integral on the boundary of the unit disk of $\mathbb{R}^2$ then its Poisson integral exists for $|z| < 1$ and is $o((1-|z|)^{-1})$ as $|z| \to 1^-$. It is shown that this is the best possible uniform pointwise estimate. For an L^1 measure the best estimate is $O((1-|z|)^{-1})$.
AMS (MOS) subject classification: 26A39, 31A20

In this paper we consider estimates of Poisson integrals on the unit circle with respect to Alexiewicz and L^p norms. Define the open disk in $\mathbb{R}^2$ as $D := \{re^{i\theta} \,|\, 0 \le r < 1,\, -\pi < \theta \le \pi\}$ and let the unit circle T be its boundary. Let $f\colon T \to \mathbb{R}$. The Poisson integral of f with respect to Lebesgue measure is

$$P[f](re^{i\theta}) = \frac{(1-r^2)}{2\pi} \int\limits_{\phi=-\pi}^{\pi} \frac{f(\phi)\,d\phi}{1-2r\cos(\phi-\theta)+r^2}.$$

For the Poisson integral of f with respect to measure μ we write $P[f,\mu]$. Since T has no end points, an appropriate form of the Alexiewicz norm of f is $\|f\| := \sup_{I\subseteq T} \left|\int_I f\right|$ where I is an interval in T. Hence, we can have $I = [\alpha,\beta]$ where $\alpha,\beta \in \mathbb{R}$ and $0 \le \beta-\alpha \le 2\pi$. The Alexiewicz norm was introduced in [1]. The variation of f on T is $\sup \sum_{i=1}^N |f(x_{i-1}) - f(x_i)|$ where the supremum is taken over all finite sets of disjoint intervals $\{(x_{i-1},x_i)\}_{i=1}^N$ in $[-\pi,\pi]$. We denote the variation of f over $I \subseteq T$ as $V_I f$.

The following results are well known (see [4]). Suppose that $1 \le p \le \infty$ and $f \in L^p(T)$. If $e^{i\theta_0} \in T$ and $z \in D$, we say that $z \to e^{i\theta_0}$ *nontangentially* if there is $0 \le \alpha < \pi/2$ such that $z \to e^{i\theta_0}$ with z remaining in the cone

$K_\alpha(e^{i\theta_0}) := \{\zeta \in D : |\arg(\zeta - e^{i\theta_0}) - \theta_0| < \alpha\}$. Write $u_r(\theta) = P[f](re^{i\theta})$. Then

1. u_r is harmonic in D
2. $\|u_r\|_p \leq \|f\|_p$ for all $0 \leq r < 1$
3. If $1 \leq p < \infty$ then $\|u_r - f\|_p \to 0$ as $r \to 1^-$
4. $u_r \to f$ almost everywhere on T as $r \to 1$ nontangentially in D.

We examine analogues of these results when f is Henstock integrable. All the results also hold when we use the wide Denjoy integral.

Necessary and sufficient for the existence of $P[f]$ as a Henstock integral on D is that f be integrable, i.e., the Henstock integral $\int_{-\pi}^{\pi} f$ is finite. This is because the kernel $(1-r^2)/[1-2r\cos(\phi-\theta)+r^2]$ is bounded away from 0 and is of bounded variation in ϕ for each $re^{i\theta} \in D$. In [2], integration by parts was used to show that we can differentiate under the integral sign. This in turn shows that $P[f]$ is harmonic in D and that $P[f] \to f$ nontangentially, almost everywhere in T (4. above). In [3], Theorem 4, p. 238, necessary and sufficient conditions were given for determining when a function that is harmonic on D is the Poisson integral of a Henstock integrable function. Corresponding results when $\|u_r\|_p$ are uniformly bounded have been known for some time ([4], Theorem 11.30).

Our first result is to show that $P[f](re^{i\theta}) = o(1/(1-r))$ as $r \to 1^-$. That is, $\sup_{\theta\in[-\pi,\pi]}(1-r)|P[f](re^{i\theta})| \to 0$ as $r \to 1^-$. Thus, the manner of approach to the boundary is unrestricted. This same estimate was obtained for Lebesgue integrable functions in [6]. We show it is the best possible pointwise estimate under our minimal existence hypothesis.

Theorem 1 *i) Let $f : T \to \mathbb{R}$. If f is integrable then $P[f](re^{i\theta}) = o(1/(1-r))$ as $r \to 1^-$. This estimate is sharp in the sense that if $\psi : D \to \mathbb{R}$ and $\psi(re^{i\theta}) = o(1/(1-r))$ then there is an integrable function f such that $P[f] \neq o(\psi)$.*
ii) Let μ be a positive measure on T. If f is in $L^1(\mu)$ then $P[f, d\mu](re^{i\theta}) = O(1/(1-r))$. This estimate is sharp in the same sense as in i).

Proof: Let $\Phi_r(\phi) := (1-r)^2/(1-2r\cos\phi + r^2)$ with $\Phi_1(0) := 1$ and $f_\theta(\phi) :=$

$f((\phi+\theta) \mod 2\pi)$. Then

$$(1-r)P[f](re^{i\theta}) = \frac{(1+r)}{2\pi}\int_{\phi=-\pi}^{\pi} f_\theta(\phi)\,\Phi_r(\phi)\,d\phi.$$

Write

$$\frac{2\pi(1-r)P[f](re^{i\theta})}{1+r} = \int_{|\phi|<\delta} f_\theta(\phi)\Phi_r(\phi)\,d\phi \quad + \int_{\delta<|\phi|<\pi} f_\theta(\phi)\Phi_r(\phi)\,d\phi.$$

Let $F_\theta(\phi)=\int_{-\delta}^{\phi} f_\theta$ and integrate by parts. Then

$$\begin{aligned}\left|\int_{|\phi|<\delta} f_\theta(\phi)\Phi_r(\phi)\,d\phi\right| &= \left|F_\theta(\delta)\Phi_r(\delta) - \int_{\phi=-\delta}^{\delta} F_\theta\,d\Phi_r(\phi)\right|\\ &\leq \left|\int_{\theta-\delta}^{\theta+\delta} f\right|(1+V_T\Phi_r)\,. \qquad (1)\end{aligned}$$

But $V_T\Phi_r = 8r/(1+r)^2 \leq 2$. And, since the integral is continuous with respect to its limits of integration, by taking $\delta>0$ small enough we can make the right side of (1) as small as we please.

Letting $G_\theta(\phi) := \int_\delta^\phi f_\theta$, we have

$$\begin{aligned}\left|\int_{\phi=\delta}^{\pi} f_\theta(\phi)\Phi_r(\phi)\,d\phi\right| &= \left|G_\theta(\pi)\Phi_r(\pi) - \int_{\phi=\delta}^{\pi} G_\theta\,d\Phi_r(\phi)\right|\\ &\leq \|f\|\left|\left(\frac{1-r}{1+r}\right)^2 + \frac{(1-r)^2}{1-2r\cos\delta+r^2}\right|\\ &\to 0 \quad \text{as } r\to 1.\end{aligned}$$

Similarly, $\int_{-\pi}^{-\delta} f_\theta(\phi)\Phi_r(\phi)\,d\phi \to 0$ as $r\to 1$.

To prove sharpness, suppose $\psi\colon D\to\mathbb{R}$ is given. It suffices to show that $P[f](r_ne^{i\theta_n}) \neq o(\psi(r_ne^{i\theta_n}))$ for some sequence $\{r_ne^{i\theta_n}\}\in D$ with $r_n\to 1^-$. Take $r_ne^{i\theta_n}\to 1$ and $\theta_n \downarrow 0$. Let $a_n = |\psi(r_ne^{i\theta_n})|$ and let $\{\alpha_n\}$ and $\{f_n\}$ be sequences of positive numbers. Define

$$f(\phi) = \begin{cases} f_n, & |\phi-\theta_n|\leq\alpha_n \quad \text{for some } n\\ 0, & \text{otherwise.}\end{cases}$$

For $n = 1, 2, 3, \cdots$ take $0 < \alpha_n \leq \pi - \theta_n$ and small enough so that the intervals $(\theta_n - \alpha_n, \theta_n + \alpha_n)$ are disjoint. This will be so if $\alpha_n \leq \frac{1}{2}\min(\theta_{n-1} - \theta_n, \theta_n - \theta_{n+1})$ $(\theta_0 := \pi)$. Now,

$$\begin{aligned}
\pi P[f](r_n e^{i\theta_n}) &= (1 + r_n)(1 - r_n)\sum_{k=1}^{\infty} f_k \int_{\phi=\theta_k-\alpha_k}^{\theta_k+\alpha_k} \frac{d\phi}{r_n^2 - 2r_n\cos(\theta_n - \phi) + 1} \\
&\geq \frac{2(1 + r_n)(1 - r_n) f_n\, \alpha_n}{r_n^2 - 2r_n\cos(\alpha_n) + 1} \\
&\geq \frac{2(1 + r_n)(1 - r_n) f_n\, \alpha_n}{(1 - r_n)^2 + r_n\alpha_n^2}.
\end{aligned}$$

Hence, taking $\alpha_n = \min(\frac{1}{2}(\theta_{n-1} - \theta_n), \frac{1}{2}(\theta_n - \theta_{n+1}), 1 - r_n)$ and $f_n = \pi(1 - r_n)a_n/\alpha_n$ gives $P[f](r_n e^{i\theta_n}) \geq a_n$. And, $f \in L^1$ if $\sum f_k\, \alpha_k = \pi \sum (1 - r_k)a_k < \infty$. Since $(1 - r_k)a_k \to 0$ there is a subsequence $\{(1 - r_n)a_n\}_{n\in I}$ defined by an unbounded index set $I \subset \mathbb{N}$ such that $\sum_{k\in I} a_k < \infty$. Then, $f \in L^1$ and $P[f](r_n e^{i\theta_n}) \geq |\psi(r_n e^{i\theta_n})|$ for all $n \in I$.

For ii), let $f \in L^1(d\mu)$. Then

$$|P[f, \mu](r, \theta)| \leq \frac{1 - r^2}{2\pi(1 - r)^2} \int_{-\pi}^{\pi} |\ f\ |\ d\mu = O\left(\frac{1}{1 - r}\right).$$

The estimate is realised with the Dirac measure, i.e., if $\phi_0 \in [-\pi, \pi]$ then $(1 - r)P[1, \delta_{\phi_0}](r, \phi_0) = (1 + r)/(2\pi) \to 1/\pi$ as $r \to 1^-$. ■

In part i), the sharpness is in fact realised with data that is positive (and hence L^1). The electrostatic interpretation of ii) is a unit charge at $z = 1$.

The analogues of properties 2. and 3. are now considered for the Alexiewicz norm.

Theorem 2 *Let $f : T \to \mathbb{R}$ be integrable. For $re^{i\theta} \in D$ define $u_r(\theta) := P[f](re^{i\theta})$. Then*

i) $\|u_r\| \leq \|f\|$ *for all* $0 \leq r < 1$

ii) $\|u_r - f\| \to 0$ *as* $r \to 1^-$

iii) In ii), the decay of $\|u_r - f\|$ *can be arbitrarily slow.*

Proof: i) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. Then

$$\int_{\theta=\alpha}^{\beta} u_r(\theta)\,d\theta = \int_{\theta=\alpha}^{\beta} \frac{(1-r^2)}{2\pi} \int_{\phi=-\pi}^{\pi} \frac{f(\phi)\,d\phi\,d\theta}{1-2r\cos(\phi-\theta)+r^2}. \tag{2}$$

If $r = 0$ it is clear that $\|u_0\| \leq \|f\|$ so assume $0 < r < 1$. By Theorem 57 (p. 58) or Theorem 58 (p. 60) in [3] or by [5] we can interchange the orders of integration in (2). Let $v_r(\theta) = P[\chi_{[\alpha,\beta]}](re^{i\theta})$. Then

$$\int_{\theta=\alpha}^{\beta} u_r(\theta)\,d\theta = \int_{\phi=-\pi}^{\pi} f(\phi)v_r(\phi)\,d\phi.$$

If $\beta - \alpha = 2\pi$ then $v_r = 1$ on T and the result is immediate. Now assume $0 < \beta - \alpha < 2\pi$. For fixed r the function v_r has one maximum, at $\phi_1 := (\alpha+\beta)/2$, and one minimum, at $\phi_2 := \phi_1 + \pi$. Now use the Bonnet form of the Second Mean Value Theorem for integrals ([3], p. 34) to write

$$\begin{aligned}
\int_{\theta=\alpha}^{\beta} u_r(\theta)\,d\theta &= \int_{\phi=\phi_1}^{\phi_2} f(\phi)v_r(\phi)\,d\phi + \int_{\phi=\phi_2}^{\phi_2+\pi} f(\phi)v_r(\phi)\,d\phi \\
&= v_r(\phi_1)\int_{\phi_1}^{\xi_1} f + v_r(\phi_1)\int_{\xi_2}^{\phi_2+\pi} f \\
&= v_r(\phi_1)\int_{\xi_2}^{\xi_1} f
\end{aligned}$$

where $\phi_1 < \xi_1 < \phi_2$ and $\phi_2 < \xi_2 < \phi_2 + \pi$. Now,

$$\begin{aligned}
\left|\int_{\alpha}^{\beta} u_r\right| &\leq \max_{\theta\in[-\pi,\pi]} v_r(\theta)\left|\int_{\xi_2}^{\xi_1} f\right| \\
&\leq \|f\|.
\end{aligned}$$

It now follows that $\|u_r\| \leq \|f\|$.

ii) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. Suppose $\epsilon > 0$ is given. There are functions $a\colon (0,1) \to [-\pi,\pi]$ and $b\colon (0,1) \to [-\pi,\pi]$ such that for each $0 < r < 1$ we have

$$\|u_r - f\| \leq \left|\int_{\theta=a(r)}^{b(r)} [u_r(\theta) - f(\theta)]\,d\theta\right| + \epsilon.$$

Let $r_n \uparrow 1$. There is then a subsequence $\{r_{n_m}\}$ on which a converges, say $a(r_{n_m}) \to \alpha$ as $r_{n_m} \to 1$. By taking a piecewise linear function that agrees with a at the points r_{n_m}, we can assume a is continuous and has limit α. Similarly, we can assume b is continuous and has limit β where $0 < \beta - \alpha < 2\pi$. Note that

$$\int_{\theta=a(r)}^{b(r)} u_r(\theta)\, d\theta - \int_{\theta=\alpha}^{\beta} u_r(\theta)\, d\theta = \int_{\theta=-\pi}^{\pi} f(\theta) P[\chi_{[a(r),b(r)]} - \chi_{[\alpha,\beta]}](re^{i\theta})\, d\theta. \quad (3)$$

The variation of $P[\chi_{[a(r),b(r)]} - \chi_{[\alpha,\beta]}](re^{i\theta})$ over $\theta \in T$ is at most 4. And, $P[\chi_{[a(r),b(r)]} - \chi_{[\alpha,\beta]}](re^{i\theta}) \to 0$ as $r \to 1$ for $\theta \neq \alpha, \beta$. We can bring this limit under the integral sign and it follows that both sides of (3) tend to 0 as $r \to 1$. Since integrals are continuous with respect to their limits of integration we will have

$$\|u_r - f\| \leq \left| \int_{\theta=\alpha}^{\beta} [u_r(\theta) - f(\theta)]\, d\theta \right| + 2\epsilon,$$

for r close enough to 1. And,

$$\begin{aligned} \int_{\theta=\alpha}^{\beta} [u_r(\theta) - f(\theta)]\ d\theta &= \int_{\phi=-\pi}^{\pi} f(\phi) v_r(\phi)\, d\phi - \int_{\phi=\alpha}^{\beta} f(\phi)\, d\phi \\ &= \int_{\phi=-\pi}^{\pi} f(\phi) \psi_r(\phi)\, d\phi \qquad (4) \end{aligned}$$

where $\psi_r := v_r - \chi_{[\alpha,\beta]}$.

Now, ψ_r has variation at most 2. Hence, it is of bounded variation, uniformly with respect to $0 \leq r \leq 1$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $0 \leq \beta - \alpha \leq 2\pi$. And,

$$\psi_r(\phi) \to \begin{cases} 0, & \phi \neq \alpha, \beta \\ -1/2, & \phi = \alpha, \beta. \end{cases}$$

Taking the limit $r \to 1^-$ inside the integral (4) now gives $\|u_r - f\| \to 0$ as $r \to 1^-$.

iii) Let f be positive on $(0,1)$ and vanish elsewhere. Then u_r is positive for $0 \leq r < 1$. We then have

$$\begin{aligned}
\|u_r - f\| &\geq \int_{\phi=-\pi}^{0} u_r(\phi)\,d\phi \\
&= \int_{\phi=-\pi}^{0} \frac{1-r^2}{2\pi} \int_{\theta=0}^{1} \frac{f(\theta)\,d\theta\,d\phi}{1-2r\cos(\theta-\phi)+r^2} \\
&= \int_{\theta=0}^{1} f(\theta) P[\chi_{[-\pi,0]}](re^{i\theta})\,d\theta.
\end{aligned}$$

Now, as $r \to 1$

$$P[\chi_{[-\pi,0]}](re^{i\theta}) \to \begin{cases} 0, & 0 < \theta < \pi \\ 1/2, & \theta = -\pi, 0 \\ 1, & -\pi < \theta < 0. \end{cases}$$

But, the convergence is not uniform. Let a decay rate be given by $A\colon(0,1) \to (0,1/2)$, where $A(r)$ decreases to 0 as r increases to 1. By keeping θ close enough to 0 we can keep $P[\chi_{[-\pi,0]}](re^{i\theta})$ bounded away from 0 for all r. To see this, write $\rho := (1+r)/(1-r)$. Then

$$\begin{aligned}
\|u_r - f\| &\geq \int_{\theta=0}^{1-r} f(\theta) P[\chi_{[-\pi,0]}](re^{i\theta})\,d\theta \\
&= \frac{1}{\pi} \int_{\theta=0}^{1-r} f(\theta) \left\{ \frac{\pi}{2} - \arctan\left[\rho \tan\left(\frac{\theta}{2}\right)\right] + \arctan\left[\frac{1}{\rho} \tan\left(\frac{\theta}{2}\right)\right] \right\} d\theta \\
&\geq \int_{\theta=0}^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{1}{\pi} \arctan\left[\rho \tan\left(\frac{\theta}{2}\right)\right] \right\} d\theta \\
&\geq \int_{\theta=0}^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{\rho\theta}{2\pi} \right\} d\theta \\
&\geq \left(\frac{1}{2} - \frac{1}{\pi}\right) \int_{\theta=0}^{1-r} f(\theta) d\theta.
\end{aligned}$$

We can now let

$$f(\theta) := \begin{cases} -\left(\frac{1}{2}-\frac{1}{\pi}\right)^{-1} A'(1-\theta), & 0<\theta<1 \\ 0, & \text{otherwise.} \end{cases}$$

And,

$$\|u_r - f\| \geq -\int_{\theta=0}^{1-r} A'(1-\theta)\,d\theta = A(r). \quad \blacksquare$$

Remarks.
1. We have equality in i) when f is of one sign.
2. The triangle inequality and ii) show that $\|u_r\| \to \|f\|$ as $r \to 1$.
3. In iii), the decay of $\|u_r - f\|$ can be arbitrarily rapid. Take f to be constant!
4. The same proof shows that we can choose $f \in L^1$ to make $\|u_r - f\|_1$ tend to 0 arbitrarily slowly. Jensen's inequality then shows the same holds true for $\|u_r - f\|_p$ for some $f \in L^p$, for each $1 \leq p < \infty$.

We now look at the interplay between the Alexiewicz and L^p norms. In Theorem 1 we saw that $P[f](re^{i\theta})$ has the same best pointwise estimate $o(1/(1-r)$ when f is Henstock integrable or in L^1. The L^∞ norm is thus too coarse for it to show a size difference. However, for $1 \leq p < \infty$ the L^p norms of $P[f]$ are substantially larger when $P[f]$ can converge conditionally.

Theorem 3 *Let $f : T \to \mathbb{R}$ be integrable. For $re^{i\theta} \in D$ define $u_r(\theta) := P[f](re^{i\theta})$. Then $\|u_r\|_p = o(1/(1-r))$ for $1 \leq p < \infty$.*

Proof: From Theorem 1 we can write $u_r(\theta) = w_r(\theta)/(1-r)$ where $\sup_{\theta\in[-\pi,\pi]} |w_r(\theta)| \to 0$ as $r \to 1$. And, w_r is periodic and real analytic on $[-\pi, \pi]$ for each $0 \leq r < 1$. Let $1 \leq p < \infty$. Then

$$\begin{aligned} \|u_r\|_p &= \frac{1}{1-r}\left[\int_{\theta=-\pi}^{\pi} |w_r(\theta)|^p\,d\theta\right]^{1/p} \\ &\leq \frac{(2\pi)^{1/p}}{1-r} \sup_{\theta\in[-\pi,\pi]} |w_r(\theta)|. \end{aligned}$$

Hence, $\|u_r\|_p = o(1/(1-r))$ as $r \to 1$. $\blacksquare$

It is not known at this time whether or not this estimate is sharp. However, an example shows that for each $0 < \alpha < 1$ and $1 \leq p < \infty$ there is an integrable function f so that $\limsup ||u_r||_p(1-r)^\alpha = \infty$ as $r \to 1$.

References

[1] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, Colloq. Math. **1** (1948) 289-293.

[2] M. Benedicks and W.F. Pfeffer, *The Dirichlet problem with Denjoy–Perron integrable boundary condition*, Canad. Math. Bull. **28** (1985) 113–119.

[3] V.G. Čelidze and A.G. Džvaršeĭšvili, *The theory of the Denjoy integral and some applications* (trans. P.S. Bullen), Singapore, World Scientific, 1989.

[4] W. Rudin, *Real and complex analysis*, New York, McGraw-Hill, 1987.

[5] E. Talvila, *Limits and Henstock integrals of products*, Real. Anal. Exchange **25** (1999/00) 907-918.

[6] F. Wolf, *The Poisson integral. A study in the uniqueness of harmonic functions*, Acta. Math. **74** (1941) 65-100.