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SECOND ORDER REDUCTIONS OF N-WAVE INTERACTIONS RELATED TO LOW-RANK SIMPLE LIE ALGEBRAS

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Abstract. The analysis and the classification of all reductions for the nonlinear evolution equations solvable by the inverse scattering method (ISM) is interesting and still open problem. We show how the second order reductions of the N-wave interactions related to low-rank simple Lie algebras can be embedded in the Weyl group of $\mathfrak{g}$. Some of the reduced systems find applications to nonlinear optics.

1. Introduction

It is well known that the N-wave equations [1]–[6]

$$\mathrm{i}[J, Q_t] - \mathrm{i}[I, Q_x] + [[I, Q], [J, Q]] = 0, \tag{1}$$

are solvable by the inverse scattering method (ISM) [4, 5] applied to the generalized system of Zakharov–Shabat type [4, 7, 8]:

$$L(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + [J, Q(x,t)] - \lambda J\right)\Psi(x,t,\lambda) = 0, \quad J \in \mathfrak{h}, \tag{2}$$

$$Q(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}) \in \mathfrak{g}/\mathfrak{h}, \tag{3}$$

where $\mathfrak{h}$ is the Cartan subalgebra and E_α are the root vectors of the simple Lie algebra $\mathfrak{g}$. Indeed (1) is the compatibility condition

$$[L(\lambda), M(\lambda)] = 0, \tag{4}$$

where

$$M(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + [I, Q(x,t)] - \lambda I \right) \Psi(x,t,\lambda) = 0, \qquad I \in \mathfrak{h}. \quad (5)$$

Here and below $r = \operatorname{rank}\mathfrak{g}$, Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\vec{a}, \vec{b} \in \mathbb{E}^r$ are vectors corresponding to the Cartan elements $J, I \in \mathfrak{h}$. The inverse scattering problem for (2) with real valued J [1] was reduced to a Riemann–Hilbert problem for the (matrix-valued) fundamental analytic solution of (2) [4, 7]; the action-angle variables for the N-wave equations was obtained in the preprint [1]. However often the reduction conditions require that J be complex-valued. Then the solution of the corresponding inverse scattering problem for (2) becomes more difficult [9].

The interpretation of the ISM as a generalized Fourier transform and the expansions over the "squared solutions" of (2) were derived in [8] for real J and in [10] for complex J. They were used also to prove that all N-wave type equations are Hamiltonian and possess a hierarchy of Hamiltonian structures [8, 10] $\{H^{(k)}, \Omega^{(k)}\}$, $k = 0, \pm 1, \pm 2, \ldots$. The simplest Hamiltonian formulation of (1) is given by $\{H^{(0)} = H_0 + H_{\mathrm{int}},\ \Omega^{(0)}\}$ where

$$H_0 = \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x\, \langle Q, [I, Q_x] \rangle = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x \sum_{k=1}^{r} (\vec{b}, \alpha_k)(q_{k,x} p_k - q_k p_{k,x}), \quad (6)$$

$$H_{\mathrm{int}} = \frac{1}{3} \int_{-\infty}^{\infty} \mathrm{d}x\, \langle [J, Q], [Q, [I, Q]] \rangle = \sum_{[i,j,k] \in \mathcal{M}} \omega_{j,k} H(i,j,k);$$

$$H(i,j,k) = \int_{-\infty}^{\infty} \mathrm{d}x\, (q_i p_j p_k - p_i q_j q_k), \quad \omega_{jk} = 2 \det \begin{pmatrix} (\vec{a}, \alpha_j) & (\vec{b}, \alpha_j) \\ (\vec{a}, \alpha_k) & (\vec{b}, \alpha_k) \end{pmatrix} \quad (7)$$

and the symplectic form $\Omega^{(0)}$ is equivalent to a canonical one

$$\Omega^{(0)} = \frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} dx \left\langle [J, \delta Q(x,t)] \mathop{\wedge}_{,} \delta Q(x,t) \right\rangle. \quad (8)$$

Here $\langle \cdot , \cdot \rangle$ is the Killing form of $\mathfrak{g}$ and the triple $[i,j,k]$ belongs to $\mathcal{M}$ if $\alpha_i, \alpha_j, \alpha_k \in \Delta^+$ and $\alpha_i = \alpha_j + \alpha_k$. Physically to each term $H(i,j,k)$ we relate part of a wave-decay diagram which shows how the i-th wave decays into j-th and k-th waves. In other words we assign to each root α an wave with an wave number k_α and a frequency ω_α. Each of the elementary decays preserves them, i. e.

$$k_{\alpha_i} = k_{\alpha_j} + k_{\alpha_k}, \quad \omega_{\alpha_i} = \omega_{\alpha_j} + \omega_{\alpha_k}\,.$$

Our aim is to display a number of non-trivial reductions for the N-wave equations. Thus we exhibit new examples of integrable N-wave type interactions some of which have applications to physics.

Our investigation is based on the reduction group introduced by A. V. Mikhailov [11] and further developed in [12, 13]. The examples are related to $\mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reduction groups. We point our that the reduction group can be embedded in the group of automorphisms of $\mathfrak{g}$ in several different ways which may lead to inequivalent reductions of the N-wave equations.

2. Preliminaries and General Approach

The well known reductions of N-wave systems are $\mathbb{Z}_2$-reductions realized by outer automorphisms of $\mathfrak{g}$, namely (see [4]):

$$C_1(x) = -A_1 x^\dagger A_1^{-1}, \quad \kappa_1(\lambda) = \lambda^*\,, \tag{9}$$

where A_1 belongs to the Cartan subgroup of the group $\mathfrak{G}$:

$$A_1 = \exp\left(\pi \mathrm{i} H_1\right)\,, \tag{10}$$

and $H_1 \in \mathfrak{h}$ is such that $\alpha(H_1) \in \mathbb{Z}$ for all $\alpha \in \Delta$.

Another $\mathbb{Z}_2$ reductions are related to other type of outer automorphisms:

$$C_2(x) = -A_2 x^T A_2^{-1}, \quad \kappa_1(\lambda) = -\lambda\,, \tag{11}$$

where A_2 is again of the form (10). The best known examples of NLEE obtained with the reduction (11) are the sine-Gordon and the MKdV equations which are related to $\mathfrak{g} \simeq sl(2)$. For higher rank algebras such reductions to our knowledge have not been studied. Generically reductions of type (11) lead to degeneration of the canonical Hamiltonian structure, i. e. $\Omega^{(0)} \equiv 0$; then we need to use some of the higher Hamiltonian structures (see [11, 10]) for proving their complete integrability.

In fact the reductions (9) and (11) provide us examples when the reduction is obtained with the combined use of outer and inner automorphisms.

Along with (10), (9) one may use also reductions with inner automorphisms:

$$C_3(x) = A_3 x A_3^{-1} \tag{12}$$

Since our aim is to preserve the form of the Lax pair we limit ourselves by automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. This conditions is obviously fulfilled if A_k is in the form (10). Another possibility is to choose A_1, A_2, A_3 so that they correspond to a Weyl group automorphisms.

In fact (9) is related to outer automorphisms only if $\mathfrak{g}$ is from the $\mathbf{A}_r$ series. For the $\mathbf{B}_r$, $\mathbf{C}_r$ and $\mathbf{D}_r$ series (10) is equivalent to an inner automorphism (12) with the special choose for the Weyl group element w_0 which maps all highest weight vectors into the corresponding lowest weight vectors (see Remark 1). Finally $\mathbb{Z}_2$ reductions of the form (9) in fact restrict us to the corresponding real form of the algebra $\mathfrak{g}$.

2.1. The Reduction Group

The reduction group G is a finite group which preserves the Lax representation (4), i. e. it ensures that the reduction constraints are automatically compatible with the evolution. G must have two realizations: (i) $G \subset \operatorname{Aut} \mathfrak{g}$ and (ii) $G \subset \operatorname{Conf} \mathbb{C}$, i. e. as conformal mappings of the complex λ-plane. To each $g_k \in G$ we relate a reduction condition for the Lax pair as follows [11]:

$$C_k(L(\Gamma_k(\lambda))) = L(\lambda) \qquad C_k(M(\Gamma_k(\lambda))) = M(\lambda) \tag{13}$$

where $C_k \in \operatorname{Aut} \mathfrak{g}$ and $\Gamma_k(\lambda)$ are the images of g_k. Since G is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

2.2. Finite Groups

The condition (13) is obviously compatible with the group action. Therefore it is enough to ensure that (13) is fulfilled for the generating elements of G.

In fact (see [14]) every finite group G is determined uniquely by its generating elements g_k and genetic code, e. g.:

$$g_k^{N_k} = \mathbb{1}\,, \qquad (g_j g_k)^{N_{jk}} = \mathbb{1}\,, \qquad N_k, N_{jk} \in \mathbb{Z}\,. \tag{14}$$

For example the cyclic $\mathbb{Z}_N$ and the dihedral $\mathbb{D}_N$ groups have as genetic codes

$$g^N = \mathbb{1}\,, \quad N \geq 2 \quad \text{for } \mathbb{Z}_N\,, \tag{15}$$

and

$$g_1^2 = g_2^2 = (g_1 g_2)^N = \mathbb{1}\,, \quad N \geq 2 \quad \text{for } \mathbb{D}_N\,. \tag{16}$$

2.3. Cartan–Weyl Basis and Weyl Group

Here we fix up the notations and the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g}$. We introduce $h_k \in \mathfrak{h}$, $k = 1, \ldots, r$ and E_α, $\alpha \in \Delta$ where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. Along with h_k we introduce also

$$H_\alpha = \frac{2}{(\alpha,\alpha)} \sum_{k=1}^{r} (\alpha, e_k) h_k\,, \quad \alpha \in \Delta\,, \tag{17}$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The commutation relations are given by:

$$
\begin{gathered}
[h_k, E_\alpha] = (\alpha, e_k) E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha \,, \\
[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\} \,. \end{cases}
\end{gathered} \tag{18}
$$

We will denote by $\vec{a} = \sum_{k=1}^{r} a_k e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^{r} a_k h_k$. If J is a regular real element in $\mathfrak{h}$ then without restrictions we may use it to introduce an ordering in Δ. Namely we will say that the root $\alpha \in \Delta^+$ is positive (negative) if $(\alpha, \vec{a}) > 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by:

$$
\begin{aligned}
E_{-\alpha} &= E_\alpha^T \,, & \langle E_{-\alpha}, E_\alpha \rangle &= 1 \,, \\
N_{-\alpha,-\beta} &= N_{\alpha,\beta} \,, & N_{\alpha,\beta} &= \pm(p+1) \,,
\end{aligned} \tag{19}
$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \dots, p$ and $\alpha + (p+1)\beta \notin \Delta$. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections S_α; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$
S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)} \alpha, \quad \alpha \in \Delta \,. \tag{20}
$$

All Weyl reflections S_α form a finite group $W_\mathfrak{g}$ known as the Weyl group. One may introduce in a natural way an action of the Weyl group on the Cartan–Weyl basis, namely:

$$
\begin{aligned}
S_\alpha(H_\beta) &\equiv A_\alpha (H_\beta) A_\alpha^{-1} = H_{S_\alpha \beta} \,, \\
S_\alpha(E_\beta) &\equiv A_\alpha (E_\beta) A_\alpha^{-1} = n_{\alpha,\beta} E_{S_\alpha \beta}, \quad n_{\alpha,\beta} = \pm 1 \,.
\end{aligned} \tag{21}
$$

It is also well known that the matrices A_α are given (up to a factor from the Cartan subgroup) by

$$
A_\alpha = \mathrm{e}^{E_\alpha} \, \mathrm{e}^{-E_{-\alpha}} \, \mathrm{e}^{E_\alpha} \,. \tag{22}
$$

The formula (22) and the explicit form of the Cartan–Weyl basis in the typical representation will be used in calculating the reduction condition following from (13).

2.4. Graded Lie Algebras

One of the important notions in constructing integrable equations and their reductions is the one of graded Lie algebra and Kac–Moody algebras [15]. The standard construction is based on a finite order automorphism $C \in \operatorname{Aut}\mathfrak{g}$, $C^N = \mathbb{1}$. Obviously the eigenvalues of C are ω^k, $k = 0, 1, \ldots, N-1$, where $\omega = \exp(2\pi \mathrm{i}/N)$. To each eigenvalue there corresponds a linear subspace $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ determined by

$$\mathfrak{g}^{(k)} \equiv \left\{ x;\ x \in \mathfrak{g}\,, \quad C(x) = \omega^k x \right\}. \tag{23}$$

Obviously $\mathfrak{g} = \mathop{\oplus}\limits_{k=0}^{N-1} \mathfrak{g}^{(k)}$ and the grading condition holds

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)} \right] \subset \mathfrak{g}^{(k+n)}, \tag{24}$$

where $k+n$ is taken modulo N. Thus to each pair $\{\mathfrak{g}, C\}$ one can relate an infinite-dimensional algebra of Kac-Moody type $\widehat{\mathfrak{g}}_C$ whose elements are

$$X(\lambda) = \sum_k X_k \lambda^k, \quad X_k \in \mathfrak{g}^{(k)}. \tag{25}$$

The series in (25) must contain only finite number of negative (positive) powers of λ and $\mathfrak{g}^{(k+N)} \equiv \mathfrak{g}^{(k)}$. This construction is a most natural one for Lax pairs; we see that due to the grading condition (24) we can always impose a reduction on $L(\lambda)$ and $M(\lambda)$ such that both $U(x,t,\lambda)$ and $V(x,t,\lambda) \in \widehat{\mathfrak{g}}_C$. So one of the generating elements of G will be used for introducing a grading in $\mathfrak{g}$; then the reduction condition (13) gives

$$U_0, V_0 \in \mathfrak{g}^{(0)}, \quad I, J \in \mathfrak{g}^{(1)} \cap \mathfrak{h}. \tag{26}$$

A possible second reduction condition will enforce additional constraints on U_0, V_0 and J, I.

2.5. Realizations of $G \subset \operatorname{Aut}\mathfrak{g}$

It is well known that $\operatorname{Aut}\mathfrak{g} \equiv V \otimes \operatorname{Aut}_0\mathfrak{g}$ where V is the group of outer automorphisms (the symmetry group of the Dynkin diagram) and $\operatorname{Aut}_0\mathfrak{g}$ is the group of inner automorphisms. Since we start with $I, J \in \mathfrak{h}$ it is natural to consider only those inner automorphisms that preserve the Cartan subalgebra $\mathfrak{h}$. Then $\operatorname{Aut}_0\mathfrak{g} \simeq \operatorname{Ad}_H \otimes W$ where Ad_H is the group of similarity transformations with elements from the Cartan subgroup:

$$\operatorname{Ad}_C x = C x C^{-1}, \quad C = \exp\left(\frac{2\pi \mathrm{i} H_c}{N} \right), \tag{27}$$

and W is the Weyl group of $\mathfrak{g}$. Its action on the Cartan–Weyl basis was described in (21) above. From (18) one easily finds

$$CH_\alpha C^{-1} = H_\alpha, \qquad CE_\alpha C^{-1} = \mathrm{e}^{2\pi \mathrm{i}(\alpha,\vec{c})/N} E_\alpha\,, \tag{28}$$

where $\vec{c} \in \mathbb{E}^r$ is the vector corresponding to $H_c \in \mathfrak{h}$ in (27). Then the condition $C^N = \mathbb{1}$ means that $(\alpha, \vec{c}) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Obviously H_c must be chosen so that $\vec{c} = \sum_{k=1}^{r} 2c_k\omega_k/(\alpha_k, \alpha_k)$ where ω_k are the fundamental weights of $\mathfrak{g}$ and c_k are integer. In the examples below we will use several possibilities by choosing C_k as appropriate compositions of elements from V, $\mathrm{Ad}_{\mathfrak{H}}$ and W. In fact if $\mathfrak{g}$ belongs to $\mathbf{B}_r$ or $\mathbf{C}_r$ series then $V \equiv \mathbb{1}$.

2.6. Realizations of $G \subset$ Conf $\mathbb{C}$

Generically each element $g_k \in G$ maps λ into a fraction-linear function of λ. Such action however is appropriate for a more general class of Lax operators which are fraction linear functions of λ. Since our Lax operators are linear in λ then we have the following possibilities for $\mathbb{Z}_2$:

$$\begin{aligned} &\Gamma_1(\lambda) = a_0 + \eta\lambda, \quad \eta = \pm 1, \\ &\Gamma_2(\lambda) = b_0 + \epsilon\lambda^*, \quad \epsilon = \pm 1, \quad b_0 + \epsilon b_0^* = 0. \end{aligned} \tag{29}$$

We will discuss also situations when one (or several) of the elements of G act on λ trivially, e. g. $\Gamma_k(\lambda) = \lambda$. In many cases the effect of such reductions will consist in reducing to an n-wave system for a subalgebra of $\mathfrak{g}$.

3. Inequivalent Reductions

The reduction group G may be imbedded in the Weyl group $W(\mathfrak{g})$ of the simple Lie algebra in a number of ways. Therefore it will be important to have a criterium to distinguish the nonequivalent reductions. As any other finite group, $W(\mathfrak{g})$ can be split into equivalence classes. So one may expect that reductions with elements from the same equivalence class would lead to equivalent reductions; namely the two systems of N-wave equations will be related by a change of variables.

In what follows we will describe the equivalence classes of the Weyl groups $W(\mathbf{B}_2)$, $W(\mathbf{G}_2)$ and $W(\mathbf{B}_3)$; note that $W(\mathbf{B}_l) \simeq W(\mathbf{C}_l)$. This is due to two facts: (1) the system of positive roots for $\mathbf{B}_r$ is $\Delta^+_{\mathbf{B}_r} \equiv \{e_i \pm e_j, e_i\}$, $i < j$ while the one for $\mathbf{C}_r$ series is $\Delta^+_{\mathbf{C}_r} \equiv \{e_i \pm e_j, 2e_i\}$, $i < j$; and (2) the reflection S_{e_j} with respect to the root e_j coincide with S_{2e_j} — the one with respect to the root $2e_j$. In the tables below we provide for each equivalence class: (i) the cyclic group generated by each of the automorphisms in the class; (ii) the number of elements in each class; and (iii) a representative element in it.

Remark 1. *For $\mathbf{B}_r$ and $\mathbf{C}_r$ series and for $\mathbf{G}_2$ the inner automorphism w_0 which maps the highest weight vectors into the lowest weight vectors of the algebra acts in the root space as follows:*

$$w_0(E_\alpha) = n_\alpha E_{-\alpha}, \quad w_0(H_k) = -H_k, \quad \alpha \in \Delta_+, \quad n_\alpha = \pm 1. \tag{30}$$

The Weyl group $W(\mathbf{B}_2)$ consists of 8 elements. Its genetic code is given by

$$S^2_{e_1-e_2} = S^2_{e_2} = \mathbb{1}\,, \qquad (S_{e_1-e_2}S_{e_2})^4 = \mathbb{1}\,, \tag{31}$$

i. e., it is isomorphic to the group $\mathbb{D}_4$. It has 5 equivalence classes:

$\mathbb{1}$	$-\mathbb{1}$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4$
1	1	2	2	2
$\mathbb{1}$	w_0	$S_{e_1-e_2}$	S_{e_1}	$S_{e_1-e_2}S_{e_2}$

The Weyl group $W(\mathbf{G}_2)$ has 12 elements. Its genetic code is

$$S^2_{e_1-e_2} = S^2_{e_2} = \mathbb{1}\,, \qquad (S_{e_1-e_2}S_{e_2})^6 = \mathbb{1}\,, \tag{32}$$

i. e., it is isomorphic to the group $\mathbb{D}_6$. The 6 equivalence classes are:

$\mathbb{1}$	$-\mathbb{1}$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_3$	$\mathbb{Z}_6$
1	1	3	3	2	2
$\mathbb{1}$	w_0	S_{α_1}	S_{α_2}	$(S_{\alpha_1}S_{\alpha_2})^2$	$S_{\alpha_1}S_{\alpha_2}$

The Weyl groups $W(\mathbf{B}_3)$ has 48 elements; its genetic code is

$$\begin{gathered} S^2_{e_1-e_2} = S^2_{e_2-e_3} = S^2_{e_3} = \mathbb{1}\,, \\ (S_{e_1-e_2}S_{e_2-e_3})^3 = (S_{e_2-e_3}S_{e_3})^4 = \mathbb{1}, \quad (S_{e_1-e_2}S_{e_2-e_3}S_{e_3})^6 = \mathbb{1}\,. \end{gathered} \tag{33}$$

Its 10 equivalence classes are characterized by:

$\mathbb{1}$	$-\mathbb{1}$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$
1	1	6	3	6
$\mathbb{1}$	w_0	$S_{e_1-e_2}$	S_{e_3}	$S_{e_1-e_2}S_{e_3}$
$\mathbb{Z}_2^{(4)}$	$\mathbb{Z}_3$	$\mathbb{Z}_4^{(1)}$	$\mathbb{Z}_4^{(2)}$	$\mathbb{Z}_6$
3	8	6	6	8
$S_{e_1}S_{e_2}$	$S_{e_1-e_2}S_{e_2-e_3}$	$S_{e_1}S_{e_1-e_2}$	$S_{e_1}S_{e_3}S_{e_1-e_2}$	$S_{e_1-e_2}S_{e_2-e_3}S_{e_3}$

.

We leave more detailed explanations of the general theory to other papers and turn now to the examples.

Remark 2. *In all examples below we apply the reductions to L-operators of generic form. This means that the unreduced J is a generic element of $\mathfrak{h}$ and therefore $(a, \alpha) \neq 0$. In fact we have used above the vector $\vec{a}$ for fixing up the order in the root system of $\mathfrak{g}$. The potential Q is also generic, i. e. depends on $|\Delta|$ complex-valued functions where $|\Delta|$ is the number of roots of $\mathfrak{g}$. However the reduction imposed on J may lead to qualitatively different situation in which the reduced J_r is not generic, i. e. there exist a subset of roots Δ_0 such that $(\vec{a}_r, \alpha) = 0$ for $\alpha \in \Delta_0$. Then obviously the potential $[J, Q]$ in L will depend only on $|\Delta| - |\Delta_0|$ complex-valued fields.*

In what follows whenever such situation arises we will provide the subset Δ_0 or, equivalently the list of redundant functions in Q. Obviously both the corresponding N-wave equation and its Hamiltonian structures will depend only on the fields labelled by the roots α such that $(\vec{a}_r, \alpha) \neq 0$.

Remark 3. *Several of the $\mathbb{Z}_2$-reductions below contain automorphisms which map J to $-J$. Then it is only natural that both the canonical symplectic form $\Omega^{(0)}$ and the Hamiltonian $H^{(0)}$ vanish identically. In these cases we will write down the corresponding N-wave systems of equations; their Hamiltonian formulation is discussed in Section 5 below.*

Remark 4. *In several of the examples below the action of the $\mathbb{Z}_2$ reduction group on the spectral parameter λ is trivial. Then the result is an N-wave system related to a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.*

Remark 5. *The final remark here is that under some of the reductions the corresponding Equation (1) becomes linear and trivial. This happens when the Cartan subalgebra elements invariant under the reduction form a one-dimensional subspace in $\mathfrak{h}$ and therefore $J_r \propto I_r$. For obvious reasons we have omitted these examples.*

4. Examples of $\mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ Reductions

Remark 6. *Here and below we will skip the leading zeroes in the notations of the roots, e. g. by $\{1\}$ and $\{11\}$ we mean $\{001\}$ and $\{011\}$ respectively for the $\mathbf{B}_3$ and $\mathbf{C}_3$ algebras. For $\mathbf{G}_2$ algebra by $\{1\}$ we mean $\{01\}$.*

4.1. $\mathfrak{g} \simeq \mathbf{C_2} = sp(4)$

This algebra has four positive roots $\Delta^+ = \{10, 01, 11, 21\}$ where $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$ and $jk = j\alpha_1 + k\alpha_2$. Then $Q(x, t)$ contains eight functions.

Example 1. *After the reduction of anti-hermitian type $KL(\lambda)K^{-1} = -L(\eta_1\lambda^*)^\dagger$, where $K = diag\,(s_1, s_2, 1/s_2, 1/s_1)$ and $\eta_1 = \pm 1$ we obtain*

$p_{10} = -\eta_1 s_1/s_2 q_{10}^*$, $p_1 = -\eta_1 s_2^2 q_1^*$, $p_{11} = -\eta_1 s_1 s_2 q_{11}^*$, $p_{21} = -\eta_1 s_1^2 q_{21}^*$, *and the next* 4*-wave system*

$$\begin{aligned}
&\mathrm{i}(a_1 - a_2) q_{10;t} - \mathrm{i}(b_1 - b_2) q_{10;x} - 2\kappa (s_2^2 q_{11} q_1^* - s_1 s_2 q_{21} q_{11}^*) = 0\,,\\
&\mathrm{i} a_2 q_{1;t} - \mathrm{i} b_2 q_{1;x} - 2\kappa (s_1/s_2) q_{11} q_{10}^* = 0\,,\\
&\mathrm{i} a_1 q_{21;t} - \mathrm{i} b_4 q_{21;x} + 2\kappa q_{11} q_{10} = 0\,,\\
&\mathrm{i}(a_1 + a_2) q_{11;t} - \mathrm{i}(b_1 + b_2) q_{11;x} - 2\kappa \left(q_{10} q_1 - (s_1/s_2) q_{21} q_{10}^* \right) = 0\,,
\end{aligned} \tag{34}$$

where $\kappa = a_1 b_2 - a_2 b_1$. *It is described by the following interaction Hamiltonian:*

$$H_{\text{int}} = 4\kappa \left(s_1 s_2 \left(q_{11} q_1^* q_{10}^* - \eta_1 q_{11}^* q_1 q_{10} \right) - s_1^2 \left(q_{21} q_{11}^* q_{10}^* + \eta_1 q_{21}^* q_{11} q_{10} \right) \right). \tag{35}$$

In the case $\eta_1 = 1$ *if we identify* $q_{10} = Q$, $q_{11} = E_p$, $q_{21} = E_a$ *and* $q_1 = E_s$, *where* Q *is the normalized effective polarization of the medium and* E_p, E_s *and* E_a *are the normalized pump, Stockes and anti-Stockes wave amplitudes respectively, then we obtain the system of equations studied, e. g. in [16]. This approach allowed us to derive a new Lax pair for* (34). *A particular case of* (34) *with* $s_1 = s_2 = 1$ *and* $\eta_1 = 1$ *is equivalent to the 4-wave interaction, see [4].*

Example 2. $C_2 = S_{e_1 - e_2}$. $C_2(L^*(\eta\lambda^*)) = L(\lambda)$ *and* $\eta = \pm 1$. *This reduction gives the following restrictions:*

$$\begin{gathered}
p_{10} = \eta q_{10}^*, \quad q_{11}^* = \eta q_{11}, \quad q_{21} = \eta q_1^*\,,\\
a_2 = \eta a_1^*\,, \quad b_2 = \eta b_1^*\,, \quad p_{11}^* = \eta p_{11}\,, \quad p_{21} = -\eta p_1^*\,.
\end{gathered} \tag{36}$$

Then we obtain 5-wave (2 real and 3 complex) [1] *system which is described by the Hamiltonian:*

$$H_{\text{int}} = 4\kappa \int\limits_{-\infty}^{\infty} \mathrm{d}x \left[q_{11}(q_{10}^* p_1 - q_{10} p_1^*) + \eta p_{11}(q_{10}^* q_1^* - q_{10} q_1) \right], \tag{37}$$

with $\kappa = a_1 b_1^* - a_1^* b_1$.

Example 3. $C_3 = S_{2e_2}$. $C_3(L^*(\eta\lambda^*)) = L(\lambda)$ *and* $\eta = \pm 1$. *Then we have:*

$$\begin{gathered}
a_1^* = \eta a_1, \quad a_2^* = -\eta a_2\,, \quad b_1^* = \eta b_1\,, \quad b_2^* = -\eta b_2\,,\\
q_{11} = -i\eta q_{10}^*\,, \quad p_{11} = i\eta p_{10}^*\,, \quad q_{21}^* = -\eta q_{21}\,,\\
p_{21}^* = -\eta p_{21}\,, \quad p_1 = -\eta q_1^*\,.
\end{gathered} \tag{38}$$

[1] Here and below we count as ‘real’ also the fields that are in fact purely imaginary.

which leads again to 5-wave (2 real and 3 complex) system with the Hamiltonian:

$$H_{\rm int} = -4{\rm i}\eta\kappa \int\limits_{-\infty}^{\infty} {\rm d}x\,[|q_{11}|^2(q_1 - \eta q_1^*) + |p_{11}|^2(p_{21} + q_{21})]\,, \tag{39}$$

and $\kappa = a_1b_2 - a_2b_1$.

Example 4. $C_4 = w_0$. $C_4(L^*(\eta\lambda^*)) = L(\lambda)$ *and* $\eta = \pm 1$. *Then:*

$$\begin{gathered} a_1^* = -\eta a_1\,, \quad a_2^* = -\eta a_2\,; \quad b_1^* = -\eta b_1\,, \quad b_2^* = -\eta b_2\,; \\ p_\alpha = \begin{cases} -\eta q_\alpha^* & \text{for } \alpha = \{(11)\} \\ \eta q_\alpha^* & \text{for } \alpha = \{(10),(1),(21)\}\,. \end{cases} \end{gathered} \tag{40}$$

which leads to 4-wave system with the Hamiltonian:

$$H_{\rm int} = 4\kappa \int\limits_{-\infty}^{\infty} {\rm d}x\,[q_{11}q_{10}^*q_1^* + q_{21}q_{11}^*q_{10}^* + \eta(q_{11}^*q_{10}q_1 + q_{21}^*q_{11}q_{10})]\,, \tag{41}$$

and $\kappa = a_1b_2 - a_2b_1$.

Example 5. $C_5 = w_0$. $C_5(L(-\lambda)) = L(\lambda)$. *Here we get:*

$$p_{10} = -q_{10}\,, \quad p_{11} = q_{11}\,, \quad p_1 = -q_1\,, \quad p_{21} = -q_{21}\,. \tag{42}$$

Then we obtain the following 4*-wave system (see Remark* 3*):*

$$\begin{aligned} &{\rm i}(a_1 - a_2)q_{10,t} - {\rm i}(b_1 - b_2)q_{10,x} - 2\kappa(q_{21}q_{11} + q_1q_{11}) = 0\,, \\ &{\rm i}a_2q_{1,t} - {\rm i}b_2q_{1,x} - 2\kappa q_{10}q_{11} = 0\,, \\ &{\rm i}(a_1 + a_2)q_{11,t} - {\rm i}(b_1 + b_2)q_{11,x} + 2\kappa(q_{21}q_{10} - q_1q_{11}) = 0\,, \\ &{\rm i}a_1q_{21,t} - ib_1q_{21,x} + 2\kappa q_{10}q_{11} = 0\,. \end{aligned} \tag{43}$$

with $\kappa = a_1b_2 - a_2b_1$. *Note that this reduction doesn't restrict the Cartan elements.*

4.2. $\mathfrak{g} \simeq \mathbf{G_2}$

$\mathbf{G}_2$ has six positive roots $\Delta^+ = \{10, 01, 11, 21, 31, 32\}$ where again $km = k\alpha_1 + m\alpha_2$, $\alpha_1 = 1/3e_1 - 1/3e_2 + 2/3e_3$, $\alpha_2 = e_2 - e_3$ and the interaction Hamiltonian contains the set ot triples of indices $\mathcal{M} \equiv \{[11,1,10]$, $[21,11,10]$, $[31,21,10]$, $[32,31,1]$, $[32,21,11]\}$.

Note that here if the Cartan elements are real then the N-wave equations after the reduction become trivial, see Remark 5.

4.2.1. $\mathbb{Z}_2$ reductions

Example 6. *$C_6 = S_{\alpha_1}$. $C_6(L^*(\eta\lambda^*)) = L(\lambda)$ and $\eta = \pm 1$. Then:*

$$a_2 = \begin{cases} (a_1 + a_1^*)/2 & \text{for } \eta = 1 \\ (a_1 - a_1^*)/2\mathrm{i} & \text{for } \eta = -1. \end{cases} \qquad b_2 = \begin{cases} (b_1 + b_1^*)/2 & \text{for } \eta = 1 \\ (b_1 - b_1^*)/2\mathrm{i} & \text{for } \eta = -1. \end{cases}$$
$$q_{31} = \eta q_1^*, \quad p_{10} = \eta q_{10}^*, \quad q_{21} = \eta q_{11}^*, \quad q_{32}^* = q_{32},$$
$$p_{31} = \eta p_1^*, \quad p_{21} = \eta p_{11}^*, \quad p_{32}^* = p_{32}. \tag{44}$$

so we obtain 7-wave (2 real and 5 complex) system with the Hamiltonian:

$$\begin{split} H_{\text{int}} = -6\eta\kappa[H_r(32,31,1) - H_r(32,21,11) + \eta H_r(31,21,10) \\ + 2H_r(21,11,10) + H_r(11,10,1)]\,. \end{split} \tag{45}$$

Here $H_r(i,j,k)$ is given by (7) after the present reduction and $\kappa = a_1 b_1^* - a_1^* b_1$.

Example 7. *$C_7 = S_{\alpha_2}$. $C_7(L^*(\eta\lambda^*)) = L(\lambda)$ and $\eta = \pm 1$. Then:*

$$a_1 = \begin{cases} (a_2 + a_2^*)/6 & \text{for } \eta = 1 \\ (a_2 - a_2^*)/6\mathrm{i} & \text{for } \eta = -1. \end{cases} \qquad b_1 = \begin{cases} (b_2 + b_2^*)/6 & \text{for } \eta = 1 \\ (b_2 - b_2^*)/6\mathrm{i} & \text{for } \eta = -1. \end{cases}$$
$$q_{11} = -\eta q_{10}^*, \quad p_1 = \eta q_1^*, \quad q_{21}^* = -\eta q_{21}, \quad q_{32} = \eta q_{31}^*,$$
$$p_{11} = -\eta p_{10}^*, \quad p_{21}^* = -\eta p_{21}, \quad p_{32} = \eta p_{31}^*. \tag{46}$$

so we obtain 7-wave (2 real and 5 complex) system which is described by the Hamiltonian:

$$\begin{split} H_{\text{int}} = -2\eta\kappa[H_r(32,31,1) - H_r(32,21,11) + H_r(31,21,10) \\ - 2H_r(21,11,10) + \eta H_r(11,10,1)]\,, \end{split} \tag{47}$$

with $\kappa = a_2 b_2^* - a_2^* b_2$.

Example 8. *$C_8 = w_0$. $C_8(L^*(\eta\lambda^*)) = L(\lambda)$ and $\eta = \pm 1$. This gives:*

$$a_1^* = -\eta a_1, \quad a_2^* = -\eta a_2, \quad b_1^* = -\eta b_1, \quad b_2^* = -\eta b_2;$$
$$p_\alpha = \begin{cases} -\eta q_\alpha^* & \text{for } \alpha = \{(10),(32)\} \\ q_\alpha^* & \text{for } \alpha = \{(1),(11),(21),(31)\}. \end{cases} \tag{48}$$

and a 6-wave system described by the Hamiltonian:

$$\begin{split} H_{\text{int}} = -6\eta\kappa[-\eta H_r(32,31,1) + \eta H_r(32,21,11) + H_r(31,21,10) \\ + 2H_r(21,11,10) + H_r(11,10,1)], \end{split} \tag{49}$$

with $\kappa = a_1 b_2 - a_2 b_1$.

4.2.2. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reductions

Example 9. $C_9^{(1)} = S_{\alpha_1}$, $C_9^{(1)}(L(\lambda^*)) = -L^\dagger(\lambda)$ *and* $C_9^{(2)} = S_{3\alpha_1+2\alpha_2}$, $C_9^{(2)}(L(-\lambda^*)) = -L^\dagger(\lambda)$. *The first reduction gives the following restrictions:*

$$\begin{aligned} a_2 = a_1 + a_1^* \,, \quad b_2 = b_1 + b_1^* \,, \quad q_{10}^* = q_{10} \,, \\ p_{21} = q_{11}^* \,, \quad p_1 = q_{31}^* \,, \quad p_{11} = q_{21}^* \,, \\ p_{31} = q_1^* \,, \quad p_{32} = q_{32}^* \,, \quad p_{10}^* = p_{10} \,, \end{aligned} \tag{50}$$

i. e. after the reduction there remain 7 (2 real and 5 complex) waves. Imposing the second reduction we obtain in addition the following restrictions:

$$p_{10} = q_{10} \,, \quad q_{21} = -q_{11}^* \,, \quad q_{32}^* = -q_{32} \,, \quad q_{31} = -q_1^* \,; \tag{51}$$

and this gives the next 4*-wave (1 real and 3 complex) system:*

$$\begin{aligned} &\mathrm{i}(a_1 + a_1^*)q_{10,t} - \mathrm{i}(b_1 + b_1^*)q_{10,x} - \kappa(q_{11}q_1 + q_{11}^*q_1^* - 2|q_{11}|^2) = 0, \\ &\mathrm{i}(a_1 - 2a_1^*)q_{1,t} - \mathrm{i}(b_1 - 2b_1^*)q_{1,x} - 3\kappa(q_{10}q_{11} + q_{32}q_1^*) = 0, \\ &\mathrm{i}a_1 q_{11,t} - \mathrm{i}b_1 q_{11,x} - \kappa(q_{10}q_1 + 2q_{11}^*q_{10} + q_{32}q_{11}^*) = 0, \\ &\mathrm{i}(a_1 - a_1^*)q_{32,t} - \mathrm{i}(b_1 - b_1^*)q_{32,x} - 3\kappa(|q_1|^2 - |q_{11}|^2) = 0, \end{aligned} \tag{52}$$

with $\kappa = a_1b_1 - a_1^*b_1^*$. *Since* $C_9^{(1)}\left(C_9^{(2)}(J)\right) = -J$ *then Remark* 3 *applies.*

4.3. $\mathfrak{g} \simeq \mathbf{B_3} = so(7)$

In this case there are nine positive roots $\Delta^+ = \{100,\ 010,\ 001,\ 110,\ 011,\ 111,\ 012,\ 112,\ 122\}$ where again $ijk = i\alpha_1 + j\alpha_2 + k\alpha_3$ and $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3$. Below the interaction Hamiltonian is

$$H_{\text{int}} = \sum_{[i,j,k]\in\mathcal{M}} \omega_{jk} H(i,j,k), \tag{53}$$

where the set ot triples of indices $\mathcal{M} \equiv \{[122, 112, 10]$, $[122, 111, 11]$, $[122, 12, 110]$, $[112, 111, 1]$, $[112, 12, 100]$, $[111, 110, 1]$, $[111, 11, 100]$, $[12, 11, 1]$, $[11, 1, 10]$, $[110, 10, 100]\}$.

4.3.1. $\mathbb{Z}_2$ reductions

Example 10. $C_{10} = S_{e_1-e_2}$. $C_{10}(L(\lambda)) = L(\lambda)$. *Then*

$$\begin{aligned} &p_{100} = q_{100}, \quad q_{110} = q_{10}, \quad q_{111} = q_{11}, \quad q_{112} = q_{12}, \quad q_{122} = 0, \\ &p_{110} = p_{10}, \quad p_{111} = p_{11}, \quad p_{112} = p_{12}, \quad p_{122} = 0, \\ &a_2 = a_1 \quad b_2 = b_1 \,. \end{aligned} \tag{54}$$

The interaction reduces to 8*-wave system with the Hamiltonian:*

$$H_{int}^{(1)} = 8(a_1b_3 - b_1a_3)(H(12,11,1) + H(11,1,10)). \tag{55}$$

After a proper identification of the dynamical coefficients we find that (55*) coincide with the Hamiltonian related to the subalgebra* $\mathbf{B}_2 \simeq so(5)$*. We investigated several choices for the second order automorphism* C_{10}*. Whenever* C_{10} *is a reflection with respect to a long root of* $\mathbf{B}_3$ *we again obtain a generic* $\mathbf{B}_2 \simeq so(5)$*-system. In addition* q_{100} *becomes redundant, see Remark* 2*.*

Example 11. $C_{11} = S_{e_3}$. $C_{11}(L(\lambda)) = L(\lambda)$. *Here we have*

$$\begin{aligned} &q_{112} = q_{110}, \quad q_{12} = q_{10}, \quad q_{111} = q_{11} = 0, \quad p_1 = -q_1, \\ &p_{112} = p_{110}, \quad p_{12} = p_{10}, \quad p_{111} = p_{11} = 0, \quad a_3 = b_3 = 0\,. \end{aligned} \tag{56}$$

The interaction Hamiltonian reduces to

$$H_{\rm int}^{(2)} = 4(a_1b_2 - b_1a_2)(H_r(122,110,12) + H_r(110,10,100))\,, \tag{57}$$

and contains only coefficients q_k *related to the long roots of* $\mathbf{B}_3$*, i. e. it reduces to 8-wave system related to the subalgebra* $\mathbf{D}_3 \subset \mathbf{B}_3$*.*

Reductions that act trivially on the spectral parameter naturally reduce the $\mathfrak{g}$*-wave system to a* $\mathfrak{g}_0$*-wave system where* $\mathfrak{g}_0$ *is a subalgebra of* $\mathfrak{g}$*. These reductions preserve the Hamiltonian formulation.*

Example 12. $C_{12} = S_{e_1-e_2}S_{e_3}$. $C_{12}(L(-\lambda)) = L(\lambda)$. *The nontrivial action on the spectral parameter* λ *ensures that the reduction will not be just a transition from* $\mathfrak{g}$ *to its subalgebra. Then*

$$\begin{aligned} &p_{100} = -q_{100}, \quad q_{12} = -q_{110}, \quad q_{111} = q_{11}, \quad q_{112} = -q_{10}, \quad p_1 = q_1, \\ &p_{12} = -p_{110}, \quad p_{111} = p_{11}, \quad p_{112} = -p_{10}, \quad a_2 = -a_1, \quad b_2 = -b_1. \end{aligned} \tag{58}$$

However this choice means that $C_{12}(J) = -J$ *and therefore Remark* 3 *applies. This automorphism reduces* (1) *to the following 8-wave equations:*

$$\begin{aligned} &\mathrm{i}a_1q_{100,t} - \mathrm{i}b_1q_{100,x} + \kappa(q_{10}p_{110} - q_{110}p_{10}) = 0, \\ &\mathrm{i}(a_1+a_3)q_{10,t} - \mathrm{i}(b_1+b_3)q_{10,x} + 2\kappa(q_1q_{11} - q_{100}q_{110}) = 0, \\ &\mathrm{i}a_3q_{1,t} - \mathrm{i}b_3q_{1,x} + \kappa(q_{11}p_{110} - q_{11}p_{10} + p_{11}q_{110} - p_{11}q_{10}) = 0, \\ &\mathrm{i}(a_1-a_3)q_{110,t} - \mathrm{i}(b_1-b_3)q_{110,x} + 2\kappa(q_1q_{11} + q_{100}q_{10}) = 0, \\ &\mathrm{i}a_1q_{11,t} - \mathrm{i}b_1q_{11,x} - \kappa(q_1q_{110} + q_1q_{10}) = 0, \\ &\mathrm{i}(a_1+a_3)p_{10,t} - \mathrm{i}(b_1+b_3)p_{10,x} + 2\kappa(p_{11}q_1 + q_{100}p_{110}) = 0, \\ &\mathrm{i}(a_1-a_3)p_{110,t} - \mathrm{i}(b_1-b_3)p_{110,x} + 2\kappa(p_{11}q_1 - q_{100}p_{10}) = 0, \\ &\mathrm{i}a_1p_{11,t} - \mathrm{i}b_1p_{11,x} - \kappa(q_1p_{110} + q_1p_{10}) = 0, \end{aligned} \tag{59}$$

where $\kappa = a_1b_3 - a_3b_1$ *and* q_{122}*,* p_{122} *are redundant, see Remark* 2*.*

Example 13. $C_{13} = S_{e_1} S_{e_2}$. $C_{13}(L(-\lambda)) = L(\lambda)$. *The reduction conditions give* $C_{13}(J) = -J$ *and:*

$$\begin{aligned} &p_{100} = q_{100}, \quad p_{112} = q_{110}, \quad p_{111} = -q_{111}, \quad p_{110} = q_{112}, \\ &p_1 = 0, \quad p_{122} = q_{122}, \quad p_{12} = q_{10}, \quad p_{11} = -q_{11}, \\ &p_{10} = q_{12}, \quad q_1 = 0 \quad a_3 = 0, \quad b_3 = 0\,. \end{aligned} \tag{60}$$

Again Remark 3 *applies and we obtain the next* 8*-wave system:*

$$\begin{aligned} &\mathrm{i}(a_1 - a_2) q_{100,t} - \mathrm{i}(b_1 - b_2) q_{100,x} + \kappa(q_{10} q_{112} + q_{12} q_{110} - 2 q_{11} q_{111}) = 0, \\ &\mathrm{i} a_2 q_{10,t} - \mathrm{i} b_2 q_{10,x} - \kappa(q_{100} + q_{122}) q_{110} = 0, \\ &\mathrm{i} a_1 q_{110,t} - \mathrm{i} b_1 q_{110,x} - \kappa(q_{100} - q_{122}) q_{10} = 0, \\ &\mathrm{i} a_2 q_{11,t} - \mathrm{i} b_2 q_{11,x} - \kappa(q_{100} + q_{122}) q_{111} = 0, \\ &\mathrm{i} a_1 q_{111,t} - \mathrm{i} b_1 q_{111,x} - \kappa(q_{100} - q_{122}) q_{11} = 0, \\ &\mathrm{i} a_2 q_{12,t} - \mathrm{i} b_2 q_{12,x} - \kappa(q_{100} + q_{122}) q_{112} = 0, \\ &\mathrm{i} a_1 q_{112,t} - \mathrm{i} b_1 q_{112,x} - \kappa(q_{100} - q_{122}) q_{12} = 0, \\ &\mathrm{i}(a_1 + a_2) q_{122,t} - \mathrm{i}(b_1 + b_2) q_{122,x} + \kappa(q_{10} q_{112} + q_{12} q_{110} - 2 q_{11} q_{111}) = 0, \end{aligned} \tag{61}$$

where $\kappa = a_1 b_2 - a_2 b_1$.

Example 14. $C_{14} = S_{e_1 - e_2} S_{e_3}$. $C_{14}(L^\dagger(\lambda^*)) = -L(\lambda)$. *Then:*

$$\begin{aligned} &q_{100} = -q_{100}^*, \quad p_{12} = -q_{110}^*, \quad p_{11} = q_{111}^*, \quad p_{10} = -q_{112}^*, \\ &p_{122} = q_{122}^*, \quad p_{112} = -q_{10}^*, \quad p_{111} = q_{11}^*, \quad p_{110} = -q_{12}^*, \\ &a_2 = -a_1, \quad b_2 = -b_1, \quad q_1^* = q_1, \quad p_1^* = p_1, \quad p_{100}^* = -p_{100}, \end{aligned} \tag{62}$$

and we obtain 10-wave (4 real and 6 complex) system which is described by the Hamiltonian:

$$\begin{aligned} H_{\text{int}} = 4\kappa[&H_r(112, 111, 1) - H_r(112, 12, 100) + H_r(111, 110, 1) \\ &- H_r(12, 11, 1) - H_r(11, 1, 10) - H_r(110, 10, 100)]. \end{aligned} \tag{63}$$

Here $\kappa = a_1 b_3 - a_3 b_1$ *and* q_{122}, p_{122} *are redundant, see Remark* 2.

Example 15. $C_{15} = S_{e_1} S_{e_2}$. $C_{15}(L^\dagger(\lambda^*)) = -L(\lambda)$. *As a result:*

$$\begin{aligned} &a_3 = 0, \quad b_3 = 0, \quad q_{112} = q_{110}^*, \quad q_{12} = q_{10}^*, \quad p_{112} = p_{110}^*, \quad p_{12} = p_{10}^*, \\ &q_{100}^* = q_{100}, \quad p_{100}^* = p_{100}, \quad q_{111}^* = -q_{111}, \quad p_{111}^* = -p_{111}, \\ &q_{122}^* = q_{122}, \quad p_{122}^* = p_{122}, \quad p_1 = -q_1^*, \quad q_{11}^* = q_{11}, \quad p_{11}^* = p_{11}, \end{aligned} \tag{64}$$

and we get 12-wave (8 real and 4 complex) system which is described by the Hamiltonian:

$$\begin{aligned} H_{\text{int}} = 2\kappa[&H_r(122,110,12) + H_r(122,112,10) - 2H_r(122,111,11) \\ &+ H_r(112,12,100) + H_r(110,10,100) + 2H_r(111,11,100)]. \end{aligned} \tag{65}$$

with $\kappa = a_1b_2 - a_2b_1$*; the fields* q_1 *and* p_1 *are redundant, see Remark* 2.

Example 16. $C_{16} = S_{e_1-e_2}$. $C_{16}(L^\dagger(-\lambda^*)) = -L(\lambda)$. *Therefore:*

$$\begin{aligned} &q_{100}^* = q_{100}, \quad p_{10} = q_{110}^*, \quad p_{11} = q_{111}^*, \quad p_{12} = q_{112}^*, \\ &p_{100}^* = p_{100}, \quad p_{110} = q_{10}^*, \quad p_{111} = q_{11}^*, \quad p_{112} = q_{12}^*, \\ &p_{122} = -q_{122}^*, \quad p_1 = q_1^*, \quad a_2 = a_1, \quad b_2 = b_1, \end{aligned} \tag{66}$$

which gives 8-wave system with the Hamiltonian:

$$\begin{aligned} H_{\text{int}} = 2\kappa[&H_r(122,112,10) - H_r(122,12,110) + H_r(112,111,1) \\ &+ H_r(12,11,1) + H_r(111,110,1) - H_r(11,10,1)]. \end{aligned} \tag{67}$$

Here $\kappa = a_1b_3 - a_3b_1$ *and* q_{100}*,* p_{100} *are redundant, see Remark* 2.

Example 17. $C_{17} = S_{e_3}$. $C_{17}(L^\dagger(-\lambda^*)) = -L(\lambda)$. *Then:*

$$\begin{aligned} &p_{100} = q_{100}^*, \quad p_{112} = q_{110}^*, \quad p_{111} = -q_{111}^*, \quad p_{110} = q_{112}^*, \\ &p_{122} = q_{122}^*, \quad p_{12} = q_{10}^*, \quad p_{11} = -q_{11}^*, \quad p_{10} = q_{12}^*, \\ &q_1^* = -q_1, \quad p_1^* = -p_1, \quad a_3 = 0, \quad b_3 = 0, \end{aligned} \tag{68}$$

so we obtain 8-wave system with the Hamiltonian:

$$\begin{aligned} H_{\text{int}} = 2\kappa[&H_r(122,112,10) + H_r(122,110,12) - 2H_r(122,111,11) \\ &+ H_r(112,12,100) - 2H_r(111,11,100) + H_r(110,10,100)]. \end{aligned} \tag{69}$$

Here $\kappa = a_1b_2 - a_2b_1$ *and* q_1*,* p_1 *are redundant, see Remark* 2.

4.3.2. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reductions

Example 18. $C_{18}^{(1)} = S_{e_1-e_2}$ *and* $C_{18}^{(2)} = \exp(\mathrm{i}\pi h_1))$ *with* $C_{18}^{(1)}(L(\lambda)) = L(\lambda)$ *and* $C_{18}^{(2)}(L(\lambda^*)) = -L^\dagger(\lambda)$. *The first reduction is the same as in Example* 10 *and after its action there remain* 8 *complex-valued functions. The second reduction requires in addition:*

$$p_{12} = -q_{12}^*, \quad p_{11} = q_{11}^*, \quad p_1 = q_1^*, \quad p_{10} = -q_{10}^* \tag{70}$$

Applying both reductions we obtain the 4-wave system with the Hamiltonian

$$H_{\text{int}} = 8\kappa[H_r(12,11,1) + H_r(11,1,10)], \tag{71}$$

and $\kappa = a_1b_3 - a_3b_1$. *This system is related to a* $\mathbf{B}_2$ *subalgebra, see Remark* 4.

Example 19. $C_{19}^{(1)} = S_{e_1-e_2}S_{e_3}$ *and* $C_{19}^{(2)} = \exp(\mathrm{i}\pi(h_1+h_2+h_3))$ *with* $C_{19}^{(1)}(L(-\lambda)) = L(\lambda)$ *and* $C_{19}^{(2)}(L(\lambda^*)) = -L^\dagger(\lambda)$. *The first reduction is as in Example* 12. *The second one restricts the potential also by:*

$$\begin{gathered} p_{100} = q_{100}^*, \quad p_{11} = q_{11}^*, \quad p_{10} = -q_{10}^*, \\ p_1 = -q_1^*, \quad p_{110} = -q_{110}^*. \end{gathered} \tag{72}$$

This gives the following 5-wave (2 real and 3 complex) system

$$\begin{aligned} &\mathrm{i}a_1 q_{100,t} - \mathrm{i}b_1 q_{100,x} + \kappa(q_{110}^* q_{10} - q_{10}^* q_{110}) = 0, \\ &\mathrm{i}(a_1+a_3)q_{10,t} - \mathrm{i}(b_1+b_3)q_{10,x} + \kappa(q_{110}q_{100} - q_{11}q_1) = 0, \\ &\mathrm{i}a_3 q_{1,t} - \mathrm{i}b_3 q_{1,x} + \kappa(q_{11}q_{110}^* - q_{11}q_{10}^* + q_{11}^* q_{10} - q_{11}^* q_{110}) = 0, \\ &\mathrm{i}(a_1-a_3)q_{110,t} - \mathrm{i}(b_1-b_3)q_{110,x} - 2\kappa(q_{100}q_{10} + q_1 q_{11}) = 0, \\ &\mathrm{i}a_1 q_{11,t} - \mathrm{i}b_1 q_{11,x} + \kappa(q_1 q_{110} + q_1 q_{10}) = 0, \end{aligned} \tag{73}$$

where $\kappa = a_1b_3 - a_3b_1$. *Like in Example* 12 *Remark* 3 *applies.*

4.4. $\mathfrak{g} \simeq \mathbf{C}_3 = sp(6)$

In this case there are nine positive roots $\Delta^+ = \{100, 010, 001, 110, 011, 111, 021, 121, 221\}$ where again $ijk = i\alpha_1 + j\alpha_2 + k\alpha_3$ and $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = 2e_3$ and the set ot triples of indices is $\mathcal{M} \equiv \{[110,10,100], [111,11,100], [121,21,100], [121,11,110], [21,11,10], [111,110,1], [11,1,10], [121,111,10], [221,121,100], [221,111,110]\}$.

4.4.1. $\mathbb{Z}_2$ reductions

Example 20. $C_{20} = S_{e_1-e_2}$. $C_{20}(L(\lambda)) = L(\lambda)$. *Therefore:*

$$\begin{aligned} &q_{110} = q_{10}, \quad q_{111} = q_{11}, \quad q_{221} = -q_{21}, \quad p_{100} = q_{100}, \\ &p_{110} = p_{10}, \quad p_{111} = p_{11}, \quad p_{221} = -p_{21}, \quad a_2 = a_1, \quad b_2 = b_1, \end{aligned} \tag{74}$$

and we obtain 8-wave system which is described by the Hamiltonian:

$$H_{\mathrm{int}} = 8\kappa[H_r(111,110,1) - H_r(121,111,10) - H_r(121,110,11)]\,. \tag{75}$$

Here $\kappa = a_1b_3 - a_3b_1$, *and this system is related to a* $\mathbf{C}_2$*-subalgebra, see Remark* 4. *Note that the functions* q_{121} *and* q_1 *remain unrestricted and* q_{100} *is redundant, see Remark* 2.

Example 21. $C_{21} = S_{2e_3}$. $C_{21}(L(\lambda)) = L(\lambda)$. *Then:*

$$\begin{aligned} &q_{111} = -q_{110}, \quad q_{11} = -q_{10}, \quad p_{111} = -p_{110}, \quad p_{11} = -p_{10}, \\ &p_{221} = p_{121} = p_{21} = 0, \quad q_{221} = q_{121} = q_{21} = 0, \quad p_1 = q_1^*, \\ &a_3 = 0, \quad b_3 = 0; \end{aligned} \tag{76}$$

giving 6-wave system with the Hamiltonian:

$$H_{\text{int}} = 4\kappa H_r(110, 100, 10)\,. \tag{77}$$

Here $\kappa = a_1 b_2 - a_2 b_1$. This system is related to an $\mathbf{A}_2$-subalgebra, see Remark 4.

Example 22. $C_{22} = S_{e_1-e_2} S_{2e_3}$. $C_{22}(L(-\lambda)) = L(\lambda)$. *This gives:*

$$\begin{aligned} &a_2 = -a_1, \quad b_2 = -b_1, \quad p_{100} = -q_{100}, \quad q_{110} = q_{11}, \quad q_{111} = q_{10}, \\ &q_{221} = -q_{21}, \quad p_1 = -q_1, \quad p_{110} = p_{11}, \quad p_{111} = p_{10}, \quad p_{221} = -p_{21}, \end{aligned} \tag{78}$$

and the next 8-wave system (see Remark 3):

$$\begin{aligned} &\mathrm{i}a_1 q_{100,t} - \mathrm{i}b_1 q_{100,x} + \kappa(p_{10}q_{11} - p_{11}q_{10}) = 0, \\ &\mathrm{i}(a_1+a_3)q_{10,t} - \mathrm{i}(b_1+b_3)q_{10,x} - 2\kappa(q_{21}p_{11} + q_{100}q_{11} + q_1 q_{11}) = 0, \\ &\mathrm{i}a_3 q_{1,t} - \mathrm{i}b_3 q_{1,x} - \kappa(p_{10}q_{11} - p_{11}q_{10}) = 0. \\ &\mathrm{i}(a_1-a_3)q_{11,t} - \mathrm{i}(b_1-b_3)q_{11,x} - 2\kappa(q_{21}p_{10} - q_{100}q_{10} + q_1 q_{10}) = 0, \\ &\mathrm{i}a_1 q_{21,t} - \mathrm{i}b_1 q_{21,x} - 2\kappa q_{10}q_{11} = 0, \\ &\mathrm{i}(a_1+a_3)p_{10,t} - \mathrm{i}(b_1+b_3)p_{10,x} + 2\kappa(q_1 p_{11} + q_{100}p_{11} - p_{21}q_{11}) = 0, \\ &\mathrm{i}(a_1-a_3)p_{11,t} - \mathrm{i}(b_1-b_3)p_{11,x} + 2\kappa(q_1 p_{10} - q_{100}p_{10} - p_{21}q_{10}) = 0, \\ &\mathrm{i}a_1 p_{21,t} - \mathrm{i}b_1 p_{21,x} + 2\kappa p_{10}p_{11} = 0, \end{aligned} \tag{79}$$

where $\kappa = a_1 b_3 - a_3 b_1$ and q_{121} is a redundant field, see Remark 2.

Example 23. $C_{23} = S_{e_1-e_2} S_{e_1+e_2}$. $C_{23}(L(-\lambda)) = L(\lambda)$. *Then:*

$$\begin{aligned} &a_2 = a_1, \quad b_2 = b_1, \quad p_{10} = q_{110}, \quad p_{110} = q_{10}, \quad p_{11} = -q_{111}, \\ &p_{111} = -q_{11}, \quad p_{21} = q_{221}, \quad p_{221} = q_{21}, \quad p_{121} = -q_{121}, \quad p_1 = -q_1\,. \end{aligned} \tag{80}$$

so we get the next 8-wave system (see Remark 3):

$$\begin{aligned} &\mathrm{i}(a_1-a_3)q_{10,t} - \mathrm{i}(b_1-b_3)q_{10,x} + 2\kappa(q_{121}q_{11} + q_{21}q_{111} - q_{11}q_1) = 0, \\ &\mathrm{i}a_3 q_{1,t} - \mathrm{i}b_3 q_{1,x} + 2\kappa(q_{111}q_{10} + q_{110}q_{11}) = 0, \\ &\mathrm{i}(a_1-a_3)q_{110,t} - \mathrm{i}(b_1-b_3)q_{110,x} - 2\kappa(q_{221}q_{11} + q_1 q_{111} - q_{121}q_{111}) = 0, \\ &\mathrm{i}(a_1+a_3)q_{11,t} - \mathrm{i}(b_1+b_3)q_{11,x} - 2\kappa(q_{121}q_{10} + q_1 q_{10} + q_{21}q_{110}) = 0, \\ &\mathrm{i}(a_1+a_3)q_{111,t} - \mathrm{i}(b_1+b_3)q_{111,x} - 2\kappa(q_1 q_{110} + q_{121}q_{110} - q_{221}q_{10}) = 0, \\ &\mathrm{i}a_1 q_{21,t} - \mathrm{i}b_1 q_{21,x} + 2\kappa q_{10}q_{11} = 0, \\ &\mathrm{i}a_1 q_{121,t} - \mathrm{i}b_1 q_{121,x} + \kappa(q_{111}q_{10} + q_{110}q_{11}) = 0, \\ &\mathrm{i}a_1 q_{221,t} - \mathrm{i}b_1 q_{221,x} + 2\kappa q_{110}q_{111} = 0, \end{aligned} \tag{81}$$

where $\kappa = a_1 b_3 - a_3 b_1$ and q_{100} is redundant, see Remark 2.

Example 24. $C_{24} = S_{e_1-e_2}$. $C_{24}(L^\dagger(-\lambda^*)) = -L(\lambda)$. *Therefore:*

$$\begin{aligned} &q_{100}^* = q_{100}, \quad p_{100}^* = p_{100}, \quad p_1 = q_1^*, \quad p_{121} = q_{121}^*, \\ &p_{10} = q_{110}^*, \quad p_{110} = q_{10}^*, \quad p_{11} = q_{111}^*, \quad p_{111} = q_{11}^*, \\ &p_{21} = -q_{221}^*, \quad p_{221} = -q_{21}, \quad a_2 = a_1, \quad b_2 = b_1 . \end{aligned} \tag{82}$$

and we obtain 8*-wave system which is described by Hamiltonian:*

$$\begin{aligned} H_{\text{int}} = 4\kappa[H_r(221,111,110) - H_r(121,111,10) - H_r(121,110,11) \\ - H_r(21,11,10) + H_r(111,110,1) + H_r(11,1,10)], \end{aligned} \tag{83}$$

where $\kappa = a_1 b_3 - a_3 b_1$ *and* q_{100}*,* p_{100} *are redundant fields (see Remark* 2*).*

Example 25. $C_{25} = S_{2e_3}$. $C_{25}(L^\dagger(-\lambda^*)) = -L(\lambda)$. *Then:*

$$\begin{aligned} &q_1^* = q_1, \quad p_1^* = p_1, \quad p_{100} = q_{100}^*, \quad p_{111} = -q_{110}^*, \\ &p_{110} = -q_{111}^*, \quad p_{11} = -q_{10}^*, \quad p_{10} = -q_{11}^*, \quad p_{121} = -q_{121}^*, \\ &p_{221} = -q_{221}^*, \quad p_{21} = -q_{21}^*, \quad a_3 = 0, \quad b_3 = 0 . \end{aligned} \tag{84}$$

and we get another 8-wave system with:

$$\begin{aligned} H_{\text{int}} = 2\kappa[H_r(221,121,100) + H_r(121,11,110) - 2H_r(121,21,100) \\ + H_r(121,111,10) - H_r(111,11,100) + H_r(110,10,100)], \end{aligned} \tag{85}$$

where $\kappa = a_1 b_2 - a_2 b_1$ *and* q_1*,* p_1 *are redundant fields (see Remark* 2*).*

4.4.2. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reductions

Example 26. $C_{26}^{(1)} = S_{e_1-e_2}$ *and* $C_{26}^{(2)} = \exp(\mathrm{i}\pi(h_1+h_2+h_3))$. $C_{26}^{(1)}(L(\lambda)) = L(\lambda)$ $C_{26}^{(2)}(L(\lambda)) = -L^\dagger(-\lambda^*)$*, the Cartan elements* I, J *are real. The first reduction is as in Example* 20*. This gives 8-wave system. After applying the second reduction we have in addition:*

$$p_{21} = -q_{21}^*, \quad p_{11} = q_{11}^*, \quad p_1 = q_1^*, \quad p_{10} = q_{10}^*. \tag{86}$$

Thus we obtain 4-wave system with the Hamiltonian:

$$H_{\text{int}} = 4\kappa[2H(21,11,1)) + H(11,10,1)] . \tag{87}$$

and $\kappa = a_1 b_3 - a_3 b_1$*. This system is related to a* $\mathbf{C}_2$*-subalgebra, see Remark* 4*.*

Example 27. $C_{27}^{(1)} = S_{e_1-e_2} S_{2e_3}$ *and* $C_{27}^{(2)} = \exp(\mathrm{i}\pi(h_1 + h_2 + h_3))$. $C_{27}^{(1)}(L(-\lambda)) = L(\lambda)$ $C_{27}^{(2)}(L(\lambda)) = -L^\dagger(-\lambda^*)$*, the Cartan elements* I, J *are real. The first reduction is as in Example* 22 *and the second one gives:*

$$\begin{aligned} &q_{100}^* = q_{100}, \quad q_1^* = q_1, \quad q_{110} = -q_{11}^*, \quad q_{111} = -q_{10}^* \\ &p_{221} = q_{221}, \quad p_{21} = q_{21} . \end{aligned} \tag{88}$$

The composition of both reductions leads to the following 5-wave (2 real and 3 complex) system (see Remark 3):

$$\begin{aligned}
&\mathrm{i}a_1 q_{100,t} - \mathrm{i}b_1 q_{100,x} + \kappa(q_{11}q_{10}^* - q_{11}^* q_{10}) = 0,\\
&\mathrm{i}(a_1+a_3)q_{10,t} - \mathrm{i}(b_1+b_3)q_{10,x} + 2\kappa(q_{21}q_{11}^* - q_1 q_{11} - q_{100}q_{11}) = 0,\\
&\mathrm{i}a_1 q_{1,t} - \mathrm{i}b_1 q_{1,x} + 2\kappa(q_{10}^* q_{11} - q_{10}q_{11}^*) = 0,\\
&\mathrm{i}(a_1-a_3)q_{11,t} - \mathrm{i}(b_1-b_3)q_{11,x} + 2\kappa(q_{21}q_{10}^* - q_1 q_{10} + q_{100}q_{10}) = 0,\\
&\mathrm{i}a_1 q_{21,t} - \mathrm{i}b_1 q_{21,x} + 2\kappa q_{10}q_{11} = 0\,.
\end{aligned} \tag{89}$$

with $\kappa = a_1 b_3 - a_3 b_1$.

Example 28. $C_{28}^{(1)} = S_{e_1-e_2}S_{e_1+e_2}$ *and* $C_{28}^{(2)} = \exp(\mathrm{i}\pi h_3/2)$. $C_{28}^{(1)}(L^\dagger(\lambda^*)) = L(\lambda)$ $C_{28}^{(2)}(L(\lambda)) = -L^\dagger(-\lambda^*)$, *the Cartan elements* I, J *are real. The first reduction is as in Example (23). The second one gives:*

$$\begin{aligned}
&q_{100}^* = -q_{100}, \quad q_{221}^* = -q_{221}, \quad q_{121}^* = -q_{121},\\
&q_{111} = \mathrm{i}q_{110}^*, \quad p_{111} = -\mathrm{i}q_{110}^*.
\end{aligned} \tag{90}$$

The result is a 6-wave system with the Hamiltonian:

$$\begin{aligned}
H_{\text{int}} = 4\kappa[&2H_r(121,21,100)) - H_r(121,110,11) - 2H_r(221,121,100)\\
&- H_r(110,10,100) + \mathrm{i}H_{r*}(121,110,10) + \mathrm{i}H_{r*}(110,10,100)]\,,
\end{aligned} \tag{91}$$

where $\kappa = a_1 b_2 - a_2 b_1$ *and* $H_{r*}(i,j,k) = \int_{-\infty}^{\infty} \mathrm{d}x\,(q_i q_j q_k - q_i^* q_j^* q_k^*)$ *is the term similar to* (7). *Such type of interactions are known as "blow-up instability" of waves. In this case the functions* q_{100}, q_{221} *and* q_{121} *are purely imaginary.*

5. Hamiltonian Structures of the Reduced N-wave Interactions

The generic N-wave interactions (i. e., prior to any reductions) possess a hierarchy of Hamiltonian structures. As mentioned in the Introduction the simplest one is $\{H^{(0)}, \Omega^{(0)}\}$; the symplectic form $\Omega^{(0)}$ after simple rescaling

$$q_\alpha \to w_\alpha = \frac{q_\alpha}{\sqrt{(a,\alpha)}}, \qquad p_\alpha \to y_\alpha = \frac{p'_\alpha}{\sqrt{(a,\alpha)}}, \qquad \alpha \in \Delta^+,$$

becomes canonical with w_α being canonically conjugated to y_α. However in a number of cases the reduction conditions lead to degeneracies, i.e. both $H^{(0)}$ and $\Omega^{(0)}$ vanish identically. Then it is necessary to use some of the higher Hamiltonian structures, given by:

$$
\begin{aligned}
\nabla_q H^{(k+1)} &= \Lambda \nabla_q H^{(k)}\,, \\
\Omega^{(k)} &= \frac{\mathrm{i}}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}x\,\Big\langle [J, \delta Q(x,t)] \mathop{\wedge}_{,} \Lambda^k \delta Q(x,t) \Big\rangle\,,
\end{aligned}
\tag{92}
$$

where $q(x,t) = [J, Q(x,t)]$, $\nabla_q H = (\delta H)/(\delta q^T(x,t))$. The so-called generating (or recursion) operator $\Lambda = (\Lambda_+ + \Lambda_-)/2$ is determined by:

$$
\begin{aligned}
\Lambda_\pm Z(x) &= \mathrm{ad}_J^{-1}\Bigg(\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x} + P_0\cdot([q(x), Z(x)]) \\
&\qquad + \mathrm{i}\,[q(x), I_\pm\,(\mathbb{1} - P_0)\,[q(y), Z(y)]]\Bigg), \\
P_0 S &\equiv \mathrm{ad}_J^{-1}\cdot \mathrm{ad}_J \cdot S, \qquad (I_\pm S)(x) \equiv \int\limits_{\pm\infty}^{x} \mathrm{d}y\, S(y)\,.
\end{aligned}
\tag{93}
$$

The degeneracies takes place when the reduction group contains elements transforming $J \to -J$. A $\mathbb{Z}_2$-reduction with this property degenerates $\Omega^{(2k)} \equiv 0$ and $H^{(2k)} \equiv 0$ for all $k = 0, \pm 1, \pm 2 \ldots$. Reductions of higher orders (e. g. $\mathbb{Z}_N$ with $N > 2$) degenerate all Hamiltonian structures with labels $k \neq 1(\mod N)$, see [11, 8, 10]. These results may be derived using the expressions for $\Omega^{(k)}$ and $H^{(k)}$ in terms of the scattering data of L.

6. Conclusions

We end this paper with several remarks.

1. The reductions that act trivially on λ reduce to n-wave systems for a subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$. In particular, suppose we apply $\mathbb{Z}_2$-reduction by a Weyl reflection with respect to the simple root α_k. Then the Dynkin diagram of the corresponding subalgebra $\Delta_{\mathfrak{g}_1}$ is obtained from $\Delta_{\mathfrak{g}}$ by deleting α_k.
2. The $\mathbb{Z}_2$-reductions which act on λ by $\Gamma_1(\lambda) = \lambda^*$ may be viewed as Cartan involutions and lead in fact to restricting of the system to a specific real form of the algebra $\mathfrak{g}$.
3. To all reduced systems given above we can apply the analysis in [8, 10] and derive the completeness relations for the corresponding systems of "squared" solutions. Such analysis will allow one to prove the pair-wise compatibility of the Hamiltonian structures and eventually to derive their action-angle variables, see [1, 19] for the $\mathbf{A}_n$-series.
4. These results can be extended naturally in several directions:
   - for NLEE with other dispersion laws. This would allow us to study the reductions of the multicomponent NLS-type equations, Toda type systems, etc.

- for Lax operators with more complicated λ-dependence, e. g.

$$L(\lambda)\psi = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + U_0(x,t) + \lambda U_1(x,t) + \frac{1}{\lambda}U_{-1}(x,t)\right)\psi(x,t,\lambda) = 0\,.$$

This would allow us to investigate more complicated reduction groups as e. g. $\mathbb{T}$, $\mathbb{O}$ (see [18]) and the possibilities to imbed them as subgroups of the Weyl group of $\mathfrak{g}$.

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