Tenth International Conference on
Geometry, Integrability and Quantization
June 6–11, 2008, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2009, pp 158–163

Geometry, Integrability and Quantization®

PSEUDO-FERMIONIC COHERENT STATES*

OMAR CHERBAL and MAHREZ DRIR

Theoretical Physics Laboratory, Faculty of Physics USTHB
B.P 32 El-Alia, Bab Ezzouar, 16111 Algiers, Algeria

Abstract. We have generalized the fermionic coherent states to pseudo-fermion oscillator system. The system of coherent states constructed consist of two subsets, which are bi-normalized and bi-overcomplete. The two subsets are built up as eigenstates of two annihilation operators b and $\tilde{b} = \eta b \eta^{-1}$ of respectively H and H^+ where η is the Hermitian and invertible operator that ensures the pseudo-Hermiticity of the Hamiltonian $H = \eta^{-1} H^+ \eta$.

1. Introduction

The coherent states which provide a quantum description of the evolution of a classical system [4] has been generalized to several quantum systems [9, 12]. In last years the concept of coherent states was also introduced to non-Hermitian quantum mechanics [1, 10]. In this perspective, we have constructed in a recent paper [3] pseudo-fermionic coherent states for pseudo-Hermitian two-level Hamiltonians with real spectrum.

Our aim is to develops the ideas of [3] in the case of the single pseudo-fermion or called "phermion" oscillator described by the Hamiltonian $H = \omega\left(b^{\#} b - \frac{1}{2}\right)$. First we start with a review in Section 2 of some main results on the pseudo-Hermiticity. In Section 3 we construct pseudo-fermionic or "phermionic" coherent states for the single phermion oscillator. In Section 4 we study the time evolution of coherent states constructed. The paper ends with concluding remarks.

*Reprinted from JGSP **14** (2009) 13–19.

2. Some Main Results on Pseudo-Hermiticity

By definition [5], an Hamiltonian H is called **pseudo-Hermitian** if it satisfies the relation

$$H^{+} = \eta H \eta^{-1} \tag{1}$$

where η is a linear, Hermitian, and invertible operator. One can also express the definition (1) in the form

$$H^{\#} = H \tag{2}$$

where

$$H^{\#} = \eta^{-1} H^{+} \eta \tag{3}$$

is the η-pseudo adjoint of H [5]. The condition (1) reduces to hermicity when the operator η is equal to the identity. The pseudo-Hermitian conjugation $^{\#}$ has the same properties as the Hermitian conjugation $^{+}$, namely

a) $(A^{\#})^{\#} = A$

b) $(AB)^{\#} = B^{\#} A^{\#}$

c) $(\alpha A + \beta B)^{\#} = \alpha^{*} A^{\#} + \beta^{*} B^{\#}$, where A and B are linear operators, and α and β are complexes numbers.

3. Pseudo-Fermionic or "Phermionic" Coherent States

We consider the single pseudo-fermion "phermion" oscillator described by the following Hamiltonian

$$H = \omega \left(b^{\#} b - \frac{1}{2} \right) \tag{4}$$

where ω is constant, $b^{\#}$ and b are respectively the creation and annihilation operators of the single-degree of freedom of what is called the pseudo-Hermitian fermion or a **phermion** [6], which satisfies the standard anticommutation relations

$$\left[b, b^{\#} \right]_{+} = b b^{\#} + b^{\#} b = 1, \qquad (b)^2 = (b^{\#})^2 = 0 \tag{5}$$

$b^{\#} = \eta^{-1} b^{+} \eta$ [5], where η is a linear, Hermitian and invertible operator. The phermion number operators is $N = b^{\#} b$ satisfy

$$[b, N] = b, \qquad \left[b^{\#}, N \right] = -b^{\#}, \qquad \left[b, b^{\#} \right] = 1 - 2N. \tag{6}$$

H^{+} satisfies the pseudo-hermiticity relation [5] $H^{+} = \eta H \eta^{-1}$. We note that if $\eta = 1$, thus $b^{\#} = b^{+}$, the pseudo-Hermitian fermion (phermion) algebra (5) reduces to the usual fermion algebra [6]. By analogy with the Fock space representation of the fermion algebra, the Fock space representation of the phermion algebra is spanned by the two-dimensional simultaneous eigenbasis $\{ |\psi_1\rangle , |\psi_2\rangle \}$ of the corresponding

number operator $b^{\#}b$. The operators b and $b^{\#}$ allow transitions between the states as

$$b\,|\psi_1\rangle = 0, \qquad b\,|\psi_2\rangle = |\psi_1\rangle \tag{7}$$

$$b^{\#}\,|\psi_2\rangle = 0, \qquad b^{\#}|\psi_1\rangle = |\psi_2\rangle. \tag{8}$$

The operator b annihilates the lowest eigenstates $|\psi_1\rangle$, and $b^{\#}$ brings this state onto the upper eigenstates $|\psi_2\rangle$.

We define the **phermionic coherent states** $|\xi\rangle$ in an analogue scheme as the fermionic coherent states [2, 8] as follow

$$|\xi\rangle = \mathrm{e}^{\left(b^{\#}\xi - \xi^{*}b\right)}\,|\psi_1\rangle = \mathrm{e}^{-\frac{1}{2}\xi^{*}\xi}\left(|\psi_1\rangle - \xi\,|\psi_2\rangle\right)$$

where ξ and ξ^{*} are Grassmannian variables which satisfy the anticommutation relations

$$\{\xi, \xi^{*}\} = \xi\xi^{*} + \xi^{*}\xi = 0, \qquad \{\xi, \xi\} = 0, \qquad \{\xi^{*}, \xi^{*}\} = 0. \tag{9}$$

The ξ and ξ^{*} anticommute with b and $b^{\#}$

$$\begin{gathered} \xi b = -b\xi, \qquad \xi^{*}b = -b\xi^{*} \\ \xi b^{\#} = -b^{\#}\xi, \quad \xi^{*}b^{\#} = -b^{\#}\xi^{*} \end{gathered} \tag{10}$$

and have the following properties

$$\xi|\psi_1\rangle = |\psi_1\rangle\xi, \qquad \xi|\psi_2\rangle = -|\psi_2\rangle \tag{11}$$

$$\xi|\phi_1\rangle = |\phi_1\rangle\xi, \qquad \xi|\phi_2\rangle = -|\phi_2\rangle\xi. \tag{12}$$

The pseudo-Hermitian conjugation reverses the order of all fermionic quantities, both the operators and the Grassmann variables

$$(b^{\#}\xi + \xi^{*}b)^{\#} = \xi^{*}b + b^{\#}\xi. \tag{13}$$

The Grassmann integration and differentiation over the complex Grassmann variables are given by

$$\int \mathrm{d}\xi\; 1 = 0, \qquad \int \mathrm{d}\xi\xi = 1, \qquad \int \mathrm{d}\xi^{*}\; 1 = 0, \qquad \int \mathrm{d}\xi^{*}\xi^{*} = 1 \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}1 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\xi = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi^{*}}1 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi^{*}}\xi^{*} = 1. \tag{15}$$

The Grassmann integration of any function is equivalent to the left differentiation

$$\int \mathrm{d}\xi\; f(\xi) = \frac{\partial}{\partial\xi}\, f(\xi). \tag{16}$$

The Hermitian adjoint of the coherent state is

$$\langle\xi| = \mathrm{e}^{-\frac{1}{2}\xi^{*}\xi}\left(\langle\psi_1| + \xi^{*}\langle\psi_2|\right). \tag{17}$$

In the same way, we introduce another family of coherent states associated to H^+ as follows

$$|\widetilde{\xi}\rangle = \mathrm{e}^{-\frac{1}{2}\xi^*\xi}\left(|\phi_1\rangle - \xi|\phi_2\rangle\right) \tag{18}$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the eigenstates of H^+. The Hermitian adjoint of $|\widetilde{\xi}\rangle$ is

$$\langle\widetilde{\xi}| = \mathrm{e}^{-\frac{1}{2}\xi^*\xi}\left(\langle\phi_1| + \xi^*\langle\phi_2|\right). \tag{19}$$

The scalar product between $\langle\widetilde{\xi}|\widetilde{\xi}\rangle$ takes the form

$$\langle\widetilde{\xi}|\widetilde{\xi}\rangle = \langle\phi_1|\phi_1\rangle + (\langle\phi_2|\phi_2\rangle - \langle\phi_1|\phi_1\rangle)\xi^*\xi - 2\mathrm{i}\,\mathrm{Im}(\xi\langle\phi_1|\phi_2\rangle) \neq 1 \tag{20}$$

while

$$\langle\widetilde{\xi}|\xi\rangle = \langle\phi_1|\psi_1\rangle + (\langle\phi_2|\psi_2\rangle - \langle\phi_1|\psi_1\rangle)\xi^*\xi - 2\mathrm{i}\,\mathrm{Im}(\xi\langle\phi_1|\psi_2\rangle) = 1 \tag{21}$$

and

$$\langle\xi|\widetilde{\xi}\rangle = \langle\psi_1|\phi_1\rangle + (\langle\psi_2|\phi_2\rangle - \langle\psi_1|\phi_1\rangle)\xi^*\xi - 2\mathrm{i}\,\mathrm{Im}(\xi\langle\psi_1|\phi_2\rangle) = 1 \tag{22}$$

or more generally

$$\langle\xi_1|\widetilde{\xi_2}\rangle = \langle\psi_1|D^+(\xi_1)\tilde{D}(\xi_2)|\phi_1\rangle = \xi_1^*\xi_2 + \frac{1}{4}(2 - \xi_1^*\xi_1)(2 - \xi_2^*\xi_2). \tag{23}$$

By means of the two type of states $|\xi\rangle$ and $|\widetilde{\xi}\rangle$ the resolution of the identity is realized in the following way

$$1 = \int \mathrm{d}\xi^*\mathrm{d}\xi\,|\xi\rangle\langle\widetilde{\xi}| = \int \mathrm{d}\xi^*\mathrm{d}\xi\,|\widetilde{\xi}\rangle\langle\xi|. \tag{24}$$

This leads to the statement: The system of phermionic coherent states $\{|\xi\rangle, |\widetilde{\xi}\rangle\}$ consists of two subsets $\{|\xi\rangle\}$ and $\{|\widetilde{\xi}\rangle\}$, which are bi-normalized and bi-overcomplete.

In the next Section we show that these phermionic coherent states satisfy also the temporal stability property.

4. Time Evolution of Phermionic Coherent States

We study the time evolution of the phermionic coherent states constructed above. We said that the evolution of a given coherent state is time-stable if the time evolution of any initial state from the set, governed by the Hamiltonian, leaves the state in the set for any t [7, 11]. In the case of our phermionic coherent states $\{|\xi\rangle, |\widetilde{\xi}\rangle\}$ the set parameter is the complex Grassmann variable ξ, the eigenvalue of the lowering operators b or $\tilde{b}$. The time evolution is stable if the evolved states $|\xi; t\rangle$ and

$\widetilde{|\xi;t\rangle}$ remain eigenstates of the operators b and $\tilde{b}$ respectively

$$b|\xi;t\rangle = \xi(t)|\xi;t\rangle \tag{25}$$

$$\tilde{b}\widetilde{|\xi;t\rangle} = \xi(t)\widetilde{|\xi;t\rangle}. \tag{26}$$

This implies that the time evolved coherent states $|\xi;t\rangle$ and $\widetilde{|\xi;t\rangle}$ should form bi-normal and bi-overcomlete system. Let us first consider the time evolution of an initial coherent states $|\xi\rangle$. Clearly we have

$$|\xi;t\rangle = \mathrm{e}^{(-\mathrm{i}Ht)}|\xi\rangle, \qquad |\xi;0\rangle \equiv |\xi\rangle. \tag{27}$$

Using the form (9) of $|\xi\rangle$ and the facts that $|\psi_{1,2}\rangle$ are eigenstates of H (with eigenvalues $E_{1,2}$) we get

$$|\xi;t\rangle = \mathrm{e}^{-\mathrm{i}E_1 t}\left(1 - \frac{1}{2}\xi^*\xi\right)|\psi_1\rangle - \mathrm{e}^{-\mathrm{i}E_2 t}\xi|\psi_2\rangle. \tag{28}$$

Taking into account that $E_1 = -E$ and $E_2 = E$ we put $\xi(t) = \mathrm{e}^{-\mathrm{i}2Et}\xi$ and rewrite the last equation in the form

$$|\xi;t\rangle = \mathrm{e}^{\mathrm{i}Et}\left(\left(1 - \tfrac{1}{2}\xi(t)^*\xi(t)\right)|\psi_1\rangle - \xi(t)|\psi_2\rangle\right) = \mathrm{e}^{\mathrm{i}Et}|\xi(t)\rangle \tag{29}$$

which manifests the stability of the time evolution of coherent states $|\xi\rangle$. In a similar manner we establish, that the time evolution $\widetilde{|\xi;t\rangle}$ of an initial $\widetilde{|\xi\rangle}$, is stable (remains eigenstate of $\tilde{b}$)

$$\widetilde{|\xi;t\rangle} = \mathrm{e}^{(\mathrm{i}Et)}\left(\left(1 - \tfrac{1}{2}\xi(t)^*\xi(t)\right)|\phi_1\rangle - \xi(t)|\phi_2\rangle\right) = \mathrm{e}^{(\mathrm{i}Et)}\widetilde{|\xi(t)\rangle}. \tag{30}$$

The results (29) and (30) reveal the bi-normality and bi-overcompleteness of the family of time evolved states $\{|\xi;t\rangle,\ \widetilde{|\xi;t\rangle}\}$ of the phermionic oscillator system (4) – one has $\langle t;\xi|\widetilde{\xi;t}\rangle = 1$, and

$$1 = \int \mathrm{d}\xi^*\mathrm{d}\xi|\xi;t\rangle\widetilde{\langle t;\xi|} = \int \mathrm{d}\xi^*\mathrm{d}\xi\widetilde{|\xi;t\rangle}\langle t;\xi|. \tag{31}$$

We observe that here the time evolved states $|\xi;t\rangle$ and $\widetilde{|\xi;t\rangle}$ differ from coherent states $|\xi(t)\rangle$ and $\widetilde{|\xi(t)\rangle}$ only in phase factors.

5. Concluding Remarks

In this paper, we have constructed phermionic coherent states for the single phermionic oscillator. We have shown that these coherent states satisfy the usual properties of the coherent states: a) continuity of labelling, b) the resolution of identity, c) the temporal stability.

References

[1] Bagchi B. and Quesne C., *Creation and Annihilation Operators and Coherent States for the PT-Symmetric Oscillator*, Mod. Phys. Lett. A **16** (2001) 2449–2455.

[2] Cahill K. and Glauber R., *Density Operators for Fermions*, Phys. Rev. A **59** (1999) 1538–1555.

[3] Cherbal O., Drir M., Maamache M. and Trifonov D., *Fermionic Coherent States for Pseudo-Hermitian Two-Level Systems*, J. Phys. A: Math. Theor. **40** (2007) 1835–1844.

[4] Glauber R., *Coherent and Incoherent States of the Radiation Field*, Phys. Rev. **131** (1963) 2766–2788.

[5] Mostafazadeh A., *Pseudo-Hermiticity Versus PT Symmetry: The Necessary Condition for the Reality of the Spectrum of a Non-Hermitian Hamiltonian*, J. Math. Phys. **43** (2002) 205–214.

[6] Mostafazadeh A., *Statistical Origin of Pseudo-Hermitian Supersymmetry and Pseudo-Hermitian Fermions*, J. Phys. A: Math. Gen. **37** (2004) 10193–10207.

[7] Nikolov B. and Trifonov D., *Dynamics of Generalized Coherent States I. Exact and Stable Evolution*, Commun. JINR E2-81-797, Dubna, 1981.

[8] Ohnuki J. and Kashiwa T., *Coherent States of Fermi Operators and the Path Integral*, Prog. Theor. Phys. **60** (1978) 548–564.

[9] Perelomov A., *Generalized Coherent States and their Applications*, Springer, Berlin 1986.

[10] Roy B. and Roy P., *Coherent States of Non-Hermitian Quantum Systems*, Phys. Lett. A **359** (2006) 110–113.

[11] Trifonov D., *Generalized Intelligent States and Squeezing*, J. Math. Phys. **35** (1994) 2297–2308.

[12] Zhang W., Feng D. and Gilmore R., *Coherent States: Theory and Some Applications*, Rev. Mod. Phys. **62** (1990) 867–927.