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A NOTE ON THE PASSAGE FROM THE FREE TO THE ELASTICA WITH A TENSION

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Abstract. Solutions of the free elastica are deformed in order to provide the analytical part of the corresponding mechanical construction which produce the shapes of Euler elastica by bending appropriately the rectangular one.

1. Introduction

The formal mathematical definition of the elastic line or elastica will be presented quite soon but one should easily imagine the equilibrium configurations of an inextensible wire which potential energy is due to the deviation of the wire from the straight line, i.e., the so called bending energy.

Elsewhere [3], we have presented a short historical review on this fascinating subject and here we will outline only some conceptional developments which we hope will be interesting and helpful in its understanding.

Assuming that the deflection is governed by the Bernoulli-Euler law, i.e., that the bending moment M at any point of the elastica is proportional to its curvature κ leads to the following sequence of equations

$$M = EI\kappa = EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = EI\frac{\dfrac{\mathrm{d}^2z}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{3/2}} \tag{1}$$

where EI is the flexural rigidity, θ is the slope of the tangent of the elastica, s is the so called arclength parameter and $z = z(x)$ is the transversal deflection.

By continuity of the bending moment M and the axial force N acting along the curve one easily proves that they are related by the equation

$$\frac{M^2}{2EI} = N. \tag{2}$$

The meaning of this equation is that in mechanical equilibrium the sum of forces at all points of the elastica is zero. Rewritten in differential-geometric terms it states that the curvature $\kappa(s)$ obeys to the equation

$$\ddot{\kappa}(s) + \frac{1}{2}\kappa^3(s) - \frac{\sigma}{2}\kappa(s) = 0 \tag{3}$$

in which the parameter σ denotes the tension. One can arrive (following Euler) at the same equation by considering the variational problem of minimizing the so called bending energy $\mathscr{E}$ given by the integral

$$\mathscr{E} = \int \kappa^2(s)\,\mathrm{d}s \tag{4}$$

under a constraint of the fixed length

$$\mathscr{L} = \int \mathrm{d}s \tag{5}$$

and an appropriately chosen Lagrangian multiplier.

2. The Free Elastica

In the case when the tension is not present, the equation (3) reduces to

$$\ddot{\kappa}(s) + \frac{1}{2}\kappa^3(s) = 0 \tag{6}$$

which is known as the equation of the free elastica. For the first time in this form it appears in the paper by Birkhoff and de Boor [1] but has been used implicitly by Bernoulli for describing the so called rectangular elastica.

Note that this equation is scale invariant and therefore up to dilation, there is only a one-parameter family of possible profile curves. Another way to realize why the portion of the undulating elastica deformed into the specific form shown in Fig. 1 generates (when scaled) the rest of known forms is to look again to the fundamental relation (2). It is apparent that in the case under consideration the applied forces are normal at the elastica ends which means that equation (2) holds since the bending moment and axial force are just zero there.

In what follows we will present an analytical proof of the above statement.

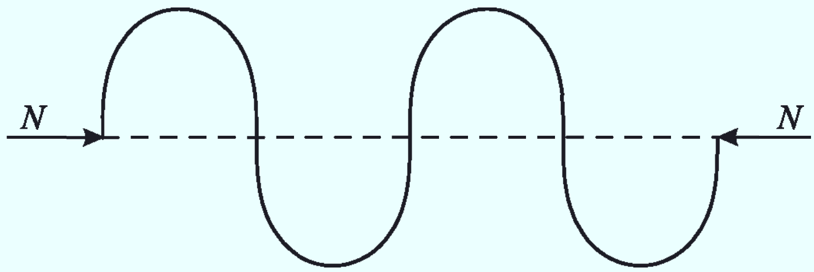

Figure 1. Bernoulli's elastica with normal applied forces.

3. Solutions in Terms of Elliptic Functions

It is an easy task to integrate once the second order equation (6) and this gives the first order equation

$$\left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 = \frac{a^4}{4} - \frac{\kappa^4}{4} \tag{7}$$

in which a is an integration constant. The later equation can be rewritten into the form

$$\int \frac{\mathrm{d}\kappa}{\sqrt{a^4 - \kappa^4}} = \frac{s}{2} \tag{8}$$

and can be integrated eventually via Jacobian elliptic functions.

Various basic facts about these functions and their integrals are collected in the Appendix.

Relying on it, one can easily check that any of the functions

$$\kappa(s) = a \ \mathrm{sn}\left(\frac{a\,s}{2}, \mathrm{i}\right) \tag{9}$$

$$\kappa(s) = a \ \mathrm{cn}\left(\frac{a\,s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{10}$$

$$\kappa(s) = a \ \mathrm{dn}\left(\frac{a\,s}{2}, \sqrt{2}\right) \tag{11}$$

satisfies both of equations (6) and (8). Combining this with the theorem in the classical differential geometry (see e.g. [6]) which states that any plane curves is determined uniquely (up to an Euclidean motion in the plane) by its curvature, we conclude that actually we have solved the free elastica problem.

4. Elastica with Tension

Elsewhere (see [7]) we have proved that the "energy" which appears in the integral of the equation (3) describing the elastica with tension, i.e.,

$$\left(\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right)^2 = 2E - \frac{\kappa^4}{4} + \frac{1}{2}\sigma\kappa^2 \tag{12}$$

and the "tension" itself can be written in the forms

Case (I) $\quad E = \frac{1}{8}a^2c^2, \qquad \sigma = \frac{1}{2}(a^2 - c^2), \qquad a, c \in \mathbb{R}$

Case (II) $\quad E = -\frac{1}{8}a^2c^2, \qquad \sigma = \frac{1}{2}(a^2 + c^2).$

Let us now "deform" the solutions (9), (10) and (11) and write them as

$$\kappa(s) = n\ \mathrm{sn}(\lambda s, k) \tag{13}$$

$$\kappa(s) = n\ \mathrm{cn}(\lambda s, k) \tag{14}$$

$$\kappa(s) = n\ \mathrm{dn}(\lambda s, k) \tag{15}$$

in which n, λ and k are some parameters that have to be determined so that the equations (3) and (12) would be satisfied simultaneously. Entering with (13) in any of these equations gives us for the first case (I) that we have among several others the solution

$$n = \pm\, a, \qquad \lambda = \pm\,\frac{c}{2}, \qquad k = \pm\,\frac{a\mathrm{i}}{c}\cdot \tag{16}$$

Similarly, (14) and (15) produce

$$n = \pm\, a, \qquad \lambda = \pm\,\frac{\sqrt{a^2 + c^2}}{2}, \qquad k = \pm\,\frac{a}{\sqrt{a^2 + c^2}} \tag{17}$$

$$n = \pm\, a, \qquad \lambda = \pm\,\frac{a}{2}, \qquad k = \pm\,\frac{\sqrt{a^2 + c^2}}{a}\cdot \tag{18}$$

Following the same idea, we have in the second Case (II) the solutions with parameters specified in Table 1. To the above two cases we should add also the case

Table 1. Parameters for the solutions of the elastica with tension in Case (II).

Type (II)	n	λ	k
$\kappa(s) = n\,\mathrm{sn}(\lambda s, k)$	$\pm a$	$\pm\frac{c\mathrm{i}}{2}$	$\pm\frac{a}{c}$
	$\pm c$	$\pm\frac{a\mathrm{i}}{2}$	$\pm\frac{c}{a}$
$\kappa(s) = n\,\mathrm{cn}(\lambda s, k)$	$\pm a$	$\pm\frac{\sqrt{a^2-c^2}}{2}$	$\pm\frac{a}{\sqrt{a^2-c^2}}$
	$\pm c$	$\pm\frac{\sqrt{c^2-a^2}}{2}$	$\pm\frac{c}{\sqrt{c^2-a^2}}$
$\kappa(s) = n\,\mathrm{dn}(\lambda s, k)$	$\pm a$	$\pm\frac{a}{2}$	$\pm\frac{\sqrt{a^2-c^2}}{a}$
	$\pm c$	$\pm\frac{c}{2}$	$\pm\frac{\sqrt{c^2-a^2}}{c}$

when the "energy" is just zero. In these circumstances the relevant solution is

$$\kappa(s) = \frac{n}{\cosh(\lambda s)}, \qquad n = \pm\sqrt{2\sigma}, \qquad \lambda = \pm\sqrt{\frac{\sigma}{2}}, \qquad \sigma > 0. \tag{19}$$

5. Elastica Shapes

As we have mentioned before, after having an explicit expression for the curvature, one is able to find the respective curve as well. Formally, this can be achieved by solving the fundamental geometrical equations

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos(\theta(s)), \qquad \frac{\mathrm{d}z(s)}{\mathrm{d}s} = \sin(\theta(s)) \tag{20}$$

but we have described more direct way to furnish this (see [3]). More precisely, there it has been proven that the explicit parameterizations of the elastica curves with tension can be found by taking into account the intrinsic equation of the curve and in this way one could bypass one of the integrations, i.e.,

$$\cos\theta(s) = \frac{\kappa^2(s) - \sigma}{\kappa^2(0) - \sigma}, \qquad \sin\theta(s) = -\frac{2}{\kappa^2(0) - \sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\,. \tag{21}$$

Combining these expressions with (20) we obtain

$$x(s) = \frac{1}{\kappa^2(0) - \sigma}\int \kappa^2(s)\,\mathrm{d}s - \frac{\sigma s}{\kappa^2(0) - \sigma}, \qquad z(s) = -\frac{2\kappa(s)}{\kappa^2(0) - \sigma} \tag{22}$$

which leads to the conclusion that we actually end with the problem to evaluate the bending energy, i.e., the integral of squared curvature. In the cases when κ is expressed via the Jacobian elliptic functions use has to be made of the following formulas

$$\int \mathrm{sn}^2(u,k)\mathrm{d}u = (u - E(\mathrm{am}(u,k),k))/k^2 \tag{23}$$

$$\int \mathrm{cn}^2(u,k)\mathrm{d}u = (E(\mathrm{am}(u,k),k) - (1-k^2)\,u)/k^2 \tag{24}$$

$$\int \mathrm{dn}^2(u,k)\mathrm{d}u = E(\mathrm{am}(u,k),k). \tag{25}$$

Finally, when κ is expressed via the hyperbolic function $\mathrm{sech}(u)$ (see (19)) the bending energy can be easily found by taking into account that

$$\int \mathrm{sech}^2(u)\,\mathrm{d}u = \tanh(u). \tag{26}$$

The complete list of the elastic curves (drawn by different parameterizations) can be found in [2], [3] and [5].

Appendix

The easiest way to introduce Jacobian elliptic functions is to consider them as formal analogues of the ordinary trigonometric functions. From any calculus course, we know that

$$\arcsin(x) = \int_0^x \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

Of course, if $x = \sin(t)$ $(-\pi/2 \le t \le \pi/2)$, then we have

$$t = \arcsin(\sin(t)) = \int_0^{\sin(t)} \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

In this way, we may view $\sin(t)$ as an inverse function for the integral. Now, fixing some k with $0 \le k \le 1$ (called the **modulus**), we define the **Jacobi sine function** $\mathrm{sn}(u,k)$ as the inverse function of the following integral. Namely,

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}. \tag{27}$$

More generally, we write

$$F(z,k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{28}$$

and call $F(z,k)$ an **elliptic integral of the first kind**. An **elliptic integral of the second kind** is defined by

$$E(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

When $z = 1$ in $F(z,k)$ and $E(z,k)$, then these integrals are respectively denoted by $K(k)$ and $E(k)$ and called the **complete elliptic integrals** of the first and second kind. The **Jacobi cosine function** $\mathrm{cn}(u,k)$ may be defined in terms of $\mathrm{sn}(u,k)$

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1. \tag{29}$$

A third Jacobi elliptic function $\mathrm{dn}(u,k)$ is defined by the equation

$$\mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1. \tag{30}$$

The integral definition of $\mathrm{sn}(u,k)$ makes it clear that, $\mathrm{sn}(u,0) = \sin(u)$ and of course, $\mathrm{cn}(u,0) = \cos(u)$ as well.

The derivatives of the elliptic functions can be found from the definitions. For instance, let us compute the derivative of $\mathrm{sn}(u,k)$. For that purpose let us suppose in (28) that $z = z(u)$. Then

$$\frac{\mathrm{d}F}{\mathrm{d}u} = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{1}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}\frac{\mathrm{d}z}{\mathrm{d}u}.$$

But, from (27), we know also that, when $z = \operatorname{sn}(u,k)$, we have $F(z,k) = u$. Hence, replacing z by $\operatorname{sn}(u,k)$ and using $\mathrm{d}u/\mathrm{d}u = 1$, we obtain

$$\begin{aligned}
1 &= \frac{1}{\sqrt{1-\operatorname{sn}(u,k)^2}\sqrt{1-k^2\operatorname{sn}(u,k)^2}}\frac{\mathrm{d}\operatorname{sn}(u,k)}{\mathrm{d}u}\\
\frac{\mathrm{d}\operatorname{sn}(u,k)}{\mathrm{d}u} &= \sqrt{1-\operatorname{sn}(u,k)^2}\sqrt{1-k^2\operatorname{sn}(u,k)^2}\\
\frac{\mathrm{d}\operatorname{sn}(u,k)}{\mathrm{d}u} &= \operatorname{cn}(u,k)\ \operatorname{dn}(u,k).
\end{aligned}$$

In a similar way we get also

$$\frac{\mathrm{d}\operatorname{cn}(u,k)}{\mathrm{d}u} = -\operatorname{sn}(u,k)\ \operatorname{dn}(u,k) \quad \text{and} \quad \frac{\mathrm{d}\operatorname{dn}(u,k)}{\mathrm{d}u} = -k^2\operatorname{sn}(u,k)\ \operatorname{cn}(u,k).$$

Finally, the **Jacobi amplitude** $\operatorname{am}(u,k)$ is defined by writing

$$\operatorname{sn}(u,k) = \sin(\operatorname{am}(u,k))$$

with analogous relations for $\operatorname{cn}(u,k)$ and $\operatorname{dn}(u,k)$. More details about elliptic functions and integrals can be found in [4].

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